1. Introduction to Control problems.

Control System:-

In a System When the output quantity is Controlled by varying the input quantity, the System is Called Control System.

The output quantity is Called 'Controlled Variable' or 'Response' and input quantity is called 'Command' or 'Excitation'.

When a Number of elements or Components are Connected in a Sequence to perform a specific function is Called a 'System'.

Control System theory extended to various fields like economy, Sociology, biology, Medicine ... etc.

The Automatic Control has become an integral part of Modern manufacturing and industrial processes.

Ex: Numerical Control of Machine tools in manufacturing industries, controlling pressure, temperature, humidity, viscosity and flow in process industry.

Types of Control System:

According to Construction of a System there are two types of Control Systems are possible.

1. Openloop 2. Closed loop Control Systems.

Open loop Control System :-

A physical system which does not automatically Correct the Variation in its output, is Called an open loop system.

Control System in which the output quantity has no effect upon the input quantity are Called open loop Control System. (the output is not feedback to the input for Correction).

In this System the output Can be Varied by varying the input. Due to the external disturbances the system output may change. The output Can be Corrected by changing the input manually.

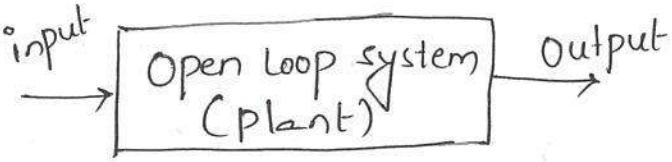


fig: Open loop System.

Closed loop System :-

Control systems in which the output has an effect upon the input quantity in which such a manner as to maintain the desired output value are Called Closed loop Systems.

The open loop system Can be modified as Closed loop System by providing a feedback. The provision of feedback automatically Corrects the Changes in output due to disturbances. Hence the

Closed loop system also called "Automatic Control System".

The block diagram of Closed loop Control System Consists of an Error detector, a Controller, plant (open loop system) and feedback path elements.

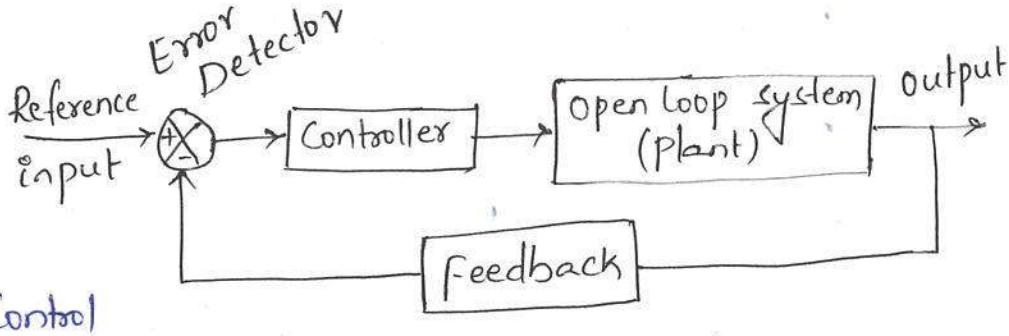


Fig: closed loop system.

The Reference signal (input signal) Corresponds to the desired output. The feedback path elements Samples the output and Converts it to a signal of same type as that of reference signal.

The feedback signal is proportional to the Output signal and it is fed to the error detector.

"The error signal generated by the error dector is the difference between reference signal and feedback signal."

The Controller modifies and amplifies the error signal to produce Control action. The modified error signal is fed to the plant to Correct it's output.

Advantages of open loop system:-

1. The open loop systems are simple and economical
2. The open loop systems are easier to Construct
3. Generally the open loop systems are stable.

Disadvantages of open loop system:-

1. The open loop systems are inaccurate & unreliable
2. The changes in the output due to external disturbances are not Corrected automatically.

Advantages of Closed loop system:-

1. Accurate system even in the presense of Non-linearities
2. The closed systems are less effected by the Noise.
3. more stable.

Disadvantages of Closed loop system:-

1. Complex and Costlier
2. The feedback in close loop system may lead to oscillatory response.
3. The feedback reduces the overall gain of the system.
4. Stability is major problem, more Care is needed to design a stable Closed loop system.

Industrial Control Examples :-

1. Temperature Control System :-

Openloop System :-

The output in the system is the desired temperature The temperature of the system is raised by heat

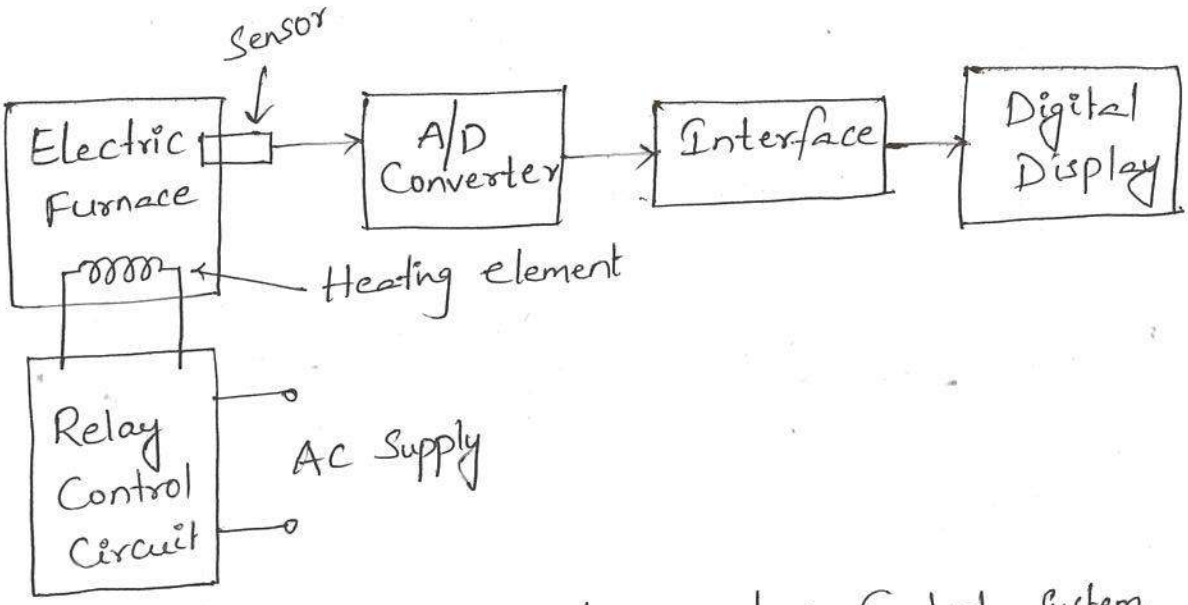


Fig: Open loop temperature Control system.

generated by the heating element. The output temperature depends on the time during which the supply to heater remains ON.

The ON and OFF of the supply is governed by the time setting of the relay. The temperature is measured by a sensor, which gives an analog voltage Corresponding to the temperature of the furnace. The analog signal is Converted to digital signal by an analog to digital Converter (A/D Converter). The

digital signal is given to the digital display device to display the temperature. If there is any change in output temperature then the time setting of the relay is not altered automatically.

Closed Loop System :-

The output of the system is the desired temperature and it depends on the time during which the supply to heater remains ON.

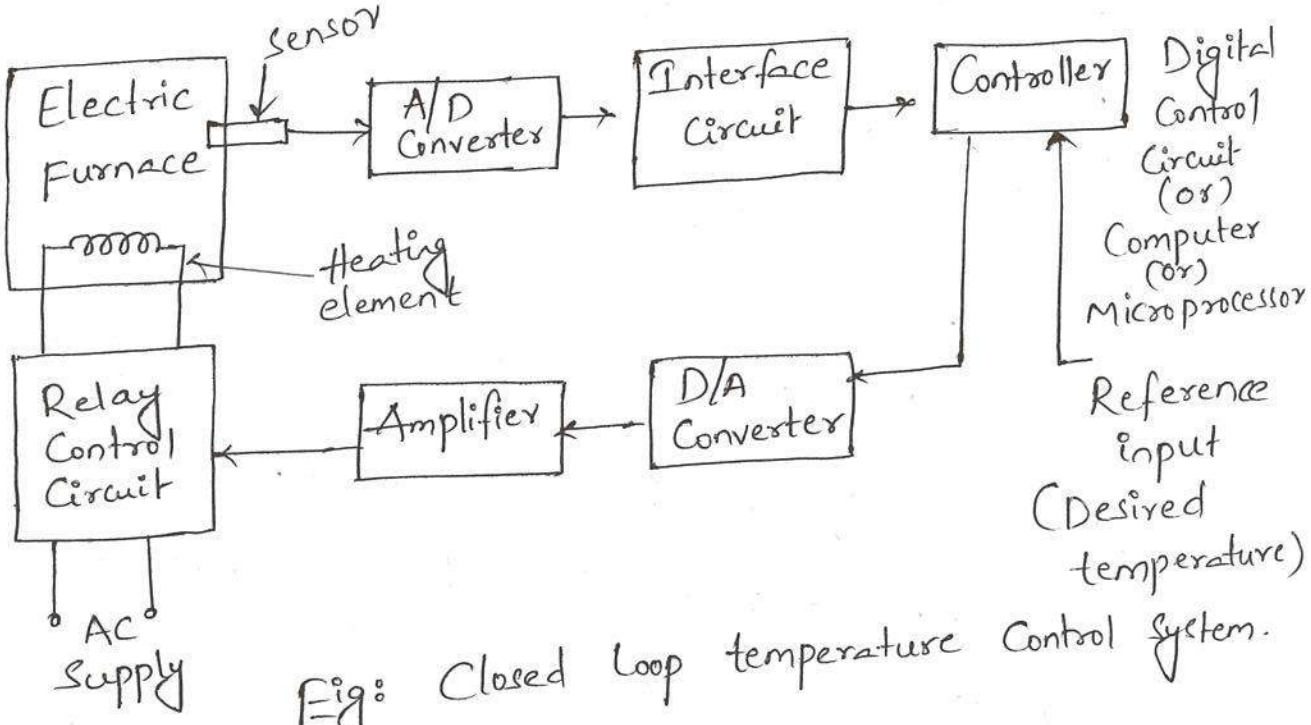


Fig: Closed loop temperature Control system.

The switching ON and OFF of the relay is Controlled by a Controller which is a digital system or Computer. The desired temperatur is input to the system through keyboard or as a signal Corresponding to the desired temperature via ports. The actual temperature sensed by sensor and converted to digital

Signal by the A/D Converter. The Computer reads the actual temperature and Compares with desired temperature. If it finds any difference then it sends signal to switch ON or OFF the relay through D/A Converter and amplifier. Thus the system automatically Corrects any changes in output.

2. Traffic Control System:

Openloop System:-

Traffic Control by means of traffic signals operated on a time basis Constitutes an openloop system. The sequence of Control signals are based on a time slot given for each signal. The time slots are decided based on a traffic study. The system will not measure the delay density of the traffic before giving the signals. Since the time slot does not changes according to traffic density.

Closed Loop System:-

Traffic Control system Can be made as a Closed loop system if the time slots of the signals are based on the density of the traffic. In closed loop traffic Control system, the density of the traffic is measured on all the sides and the information

is fed to a Computer. The timings of the Control signals are decided by the Computer based on the density of traffic. Since the Closed loop System dynamicaly changes the timings the flow of vehicles will be better than open loop system.

3. Numerical Control system:

Openloop System:

Numerical Control is made a method of Controlling the motion of machine Components by use

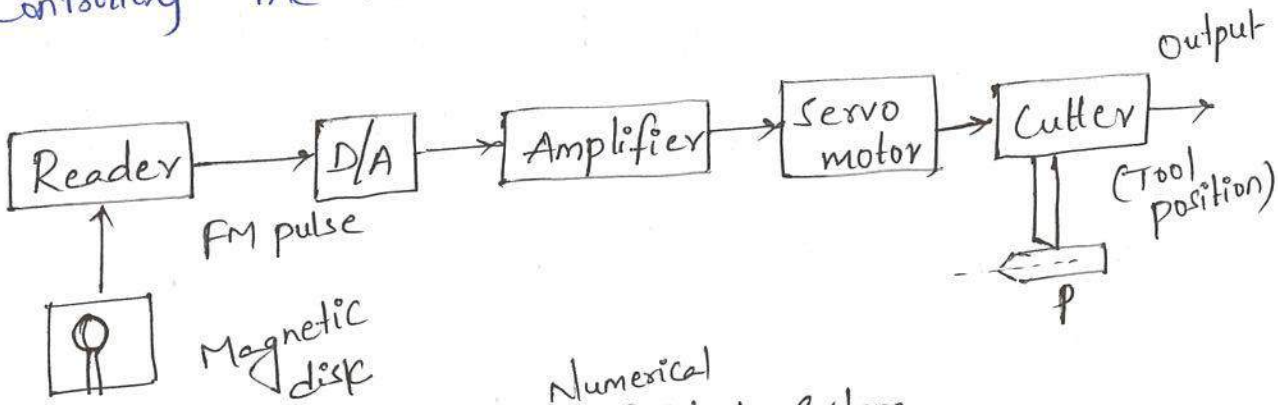


fig: open loop Numerical Control system.

Of Numbers. Here the position of work head tool is Controlled by the binary information Contained on a disk.

A magnetic disk is prepared in binary form representing the desired part P (metal part to be machined). The tool will operate on the desired part P. To start the system, the disk is fed through the reader to the D/A Converter. The D/A Converter

Converts the FM (frequency Modulated) output of the reader to a analog signal. It is amplified and fed to servomotor which positions the cutter on the desired part p. The position of the cutter head is Controlled by the angular motion of the servomotor. This is an open loop system since no feedback path exist between the output and the input. The system positions the tool for a given input Command. Any deviation in the desired position is not checked and Corrected automatically.

Closed loop System:

A Magnetic disk is prepared in binary form

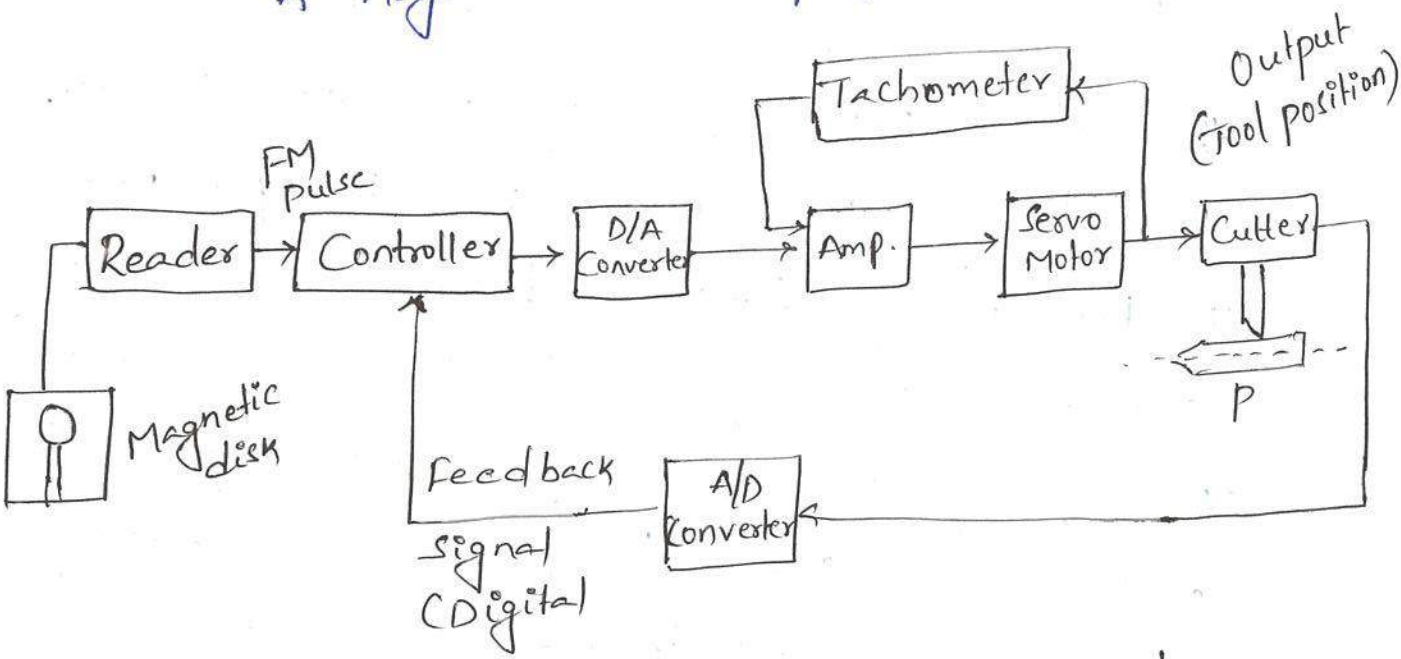


Fig: Closed loop Numerical Control system.

representing the desired part p. To start the system, the disk is loaded in the reader. The Controller Compares

the frequency modulated input pulse signal with the feedback pulse signal. The Controller is Computer or microprocessor system. The Controller Carries out Mathematical operations on the difference in the pulse signals and generates an error signal The D/A Converter Converts the Controller output pulse (error signal) into an analog signal. The amplified analog signal rotates the servomotor to position the tool on the job. The position of the Cutterhead is Controlled according to the input of the servomotor. The Transducer attached to the Cutterhead Converts the motion into an electrical signal.

The analog electrical signal is Converted to the digital pulse signal by the A/D Converter. Then the signal is Compared with the input pulse signal. If there is any difference between these two, the Controller sends a signal to the servomotor to reduce it. Thus the system automatically Corrects any deviation in the desired output tool position. An advantage of numerical Control is that Complex parts Can be produced with uniform tolerances at the maximum milling speed.

4. Position Control System using servomotor.

The position Control system is closed loop system. The system consists of a servomotor powered by a generator. The load whose position has to be controlled is connected to motor shaft through gear wheels. potentiometers are used to convert the mechanical motion to electrical signals. The desired load position (θ_R) is set on the input potentiometer and the actual

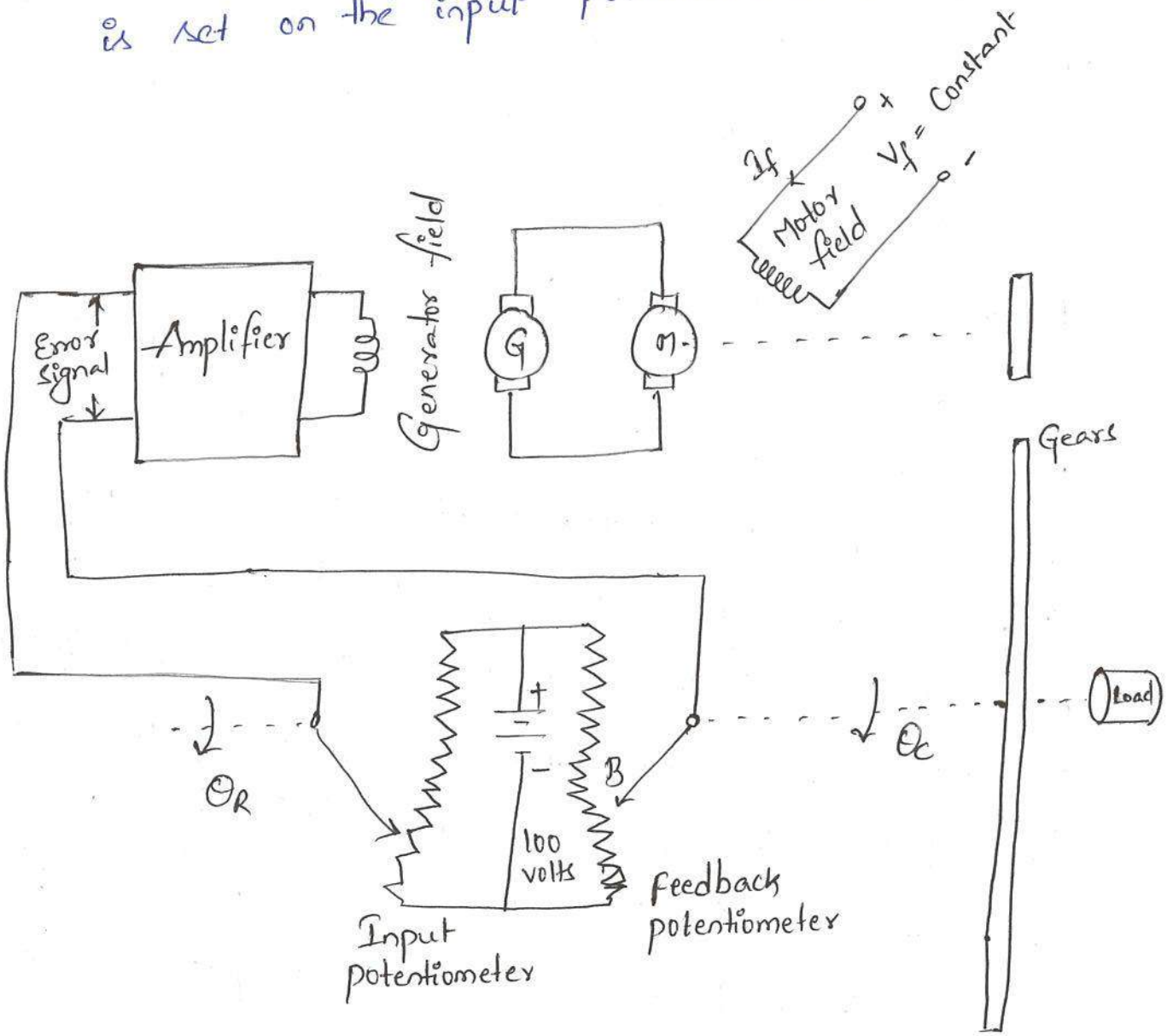


Fig: A position Control system (servomechanism).

load position (θ_c) is fed to the feedback potentiometer. The difference between the two angular positions generates an error signal. which is amplified and fed to generator field Circuit. The induced emf of the generator drives the motor. The rotation of the motor stops when the error signal is zero i.e., when the desired load position is reached.

This type of Control systems are Called servomechanisms. The servo (or) servomechanisms are feedback Control systems in which the output is mechanical position (or time derivatives of position e.g. velocity and acceleration).

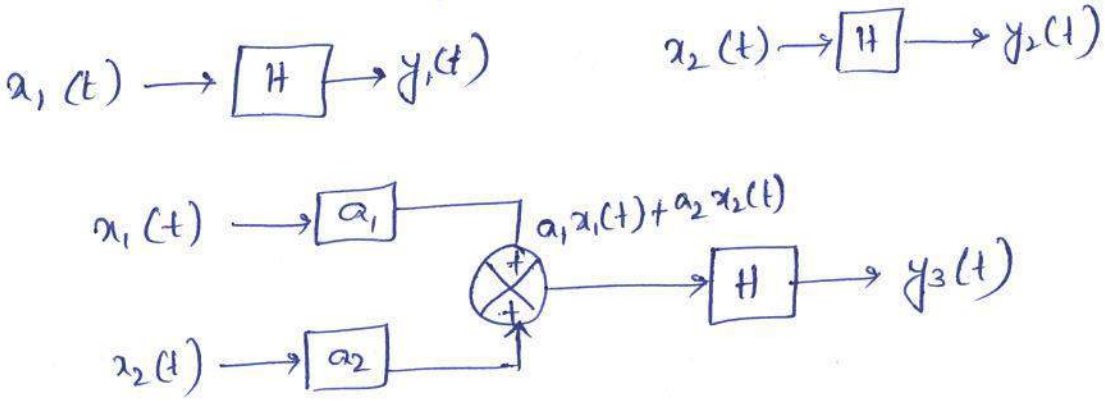
Methematical Models of Control Systems :-

A Control system is a Collection of physical Objects (components) Connected together to serve an objective. The input output relations of various physical Components of a system are governed by differential equations. The mathematical model of a Control system Constitutes a set of differential equations. The response or output of the system Can be studied by solving the differential equations for various input Conditions.

The mathematical model of a system is linear if it obeys the principal of superposition and homogenity. The principal implies that if a system model has reponses

$y_1(t)$ and $y_2(t)$ to any input $x_1(t)$ and $x_2(t)$ respectively then the system response to the linear Combination of these inputs $a_1 x_1(t) + a_2 x_2(t)$ is given by linear Combination of the individual outputs $a_1 y_1(t) + a_2 y_2(t)$ where a_1 and a_2 Constants.

Let H be a linear system



$$y_3(t) = a_1 y_1(t) + a_2 y_2(t).$$

A Mathematical model will be linear if the differential equations describing the system has Constant Coefficients (Coefficients may be functions of independent variables).

If the coefficients of the differential equation describing the system are Constant then the model is "linear time invariant".

If the Coefficient of differential equations governing the system are functions of time then the model is "linear time varying".

For single input and single output system analysis

is transfer function of the system. The Transfer function of a system is defined as the ratio of Laplace transform of output to the laplace transform of input with zero intial Conditions.

$$\text{Transfer function} = \left.\frac{\text{Laplace Transform of o/p}}{\text{Laplace Transform of i/p}}\right|_{\text{with zero initial Conditions.}}$$

The transfer function Can be obtained by taking Laplace transform of the differential equations governing the system with zero initial Conditions and rearranging the resulting algebraic equations to get the ratio of output to input.

Mechanical Translational Systems :-

The model of mechanical translational systems Can be obtained by using three basic elements mass, spring, dashpot. These three elements represents three essential phenonmena which occur in various ways in mechanical systems.

✓ The 'weight' of the mechanical system is represented by the element mass and it is Concentrated at the Centre of the body

✓ The 'spring' of the mechanical system is represented the elastic deformation of the body.

✓ The 'Dash-pot' can be represented as the friction existing in mechanical system. The dashpot is a piston moving inside a cylinder filled with viscus fluid.

When a force is applied to translational mechanical system, it is opposed by opposing forces due to mass, friction and elastic of the system. The force acting on a mechanical body are governed by Newton's second Law of motion. for translational systems it states that the the sum of forces acting on a body is zero. (Newton's second law states that the sum of applied forces is equal to the sum of opposing forces on a body).

Symbols Used in mechanical Translational system:

x — Displacement, m

v - $\frac{dx}{dt}$ - velocity, m/sec

a - $\frac{dv}{dt} = \frac{d^2x}{dt^2}$ - Acceleration, m/sec^2

f - Applied force, N (Newtons)

f_m - Opposing force offered by mass of the body, N

f_k - Opposing force offered by the elasticity of the body (spring, N)

f_b - opposing force offered by the friction of the body (Dash-pot), N.

M - Mass, Kg

K - Stiffness Spring, N/m

B - Viscous friction Co-efficient, N-sec/m.

(Lower Case letters are functions of time).

Force balancing equations of idealized elements:

1. Consider an ideal mass element, which has negligible friction and elasticity.

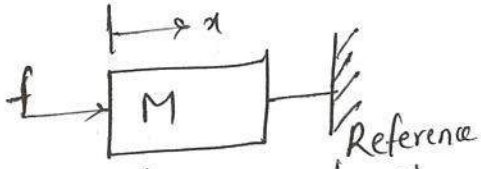


fig: Ideal mass element.

Let a force be applied on it. The mass will offer an opposing force which is proportiona to the acceleration of the body.

Let f - Applied force

f_m - opposing force due to mass

Here $f_m \propto \frac{d^2x}{dt^2}$ (or) $f_m = M\frac{d^2x}{dt^2}$

By Newton's Second Law, $f = f_m = M\frac{d^2x}{dt^2}$

2. Consider an ideal friction element dashpot, which has negligible mass and elasticity.

Let a force be applied on it. The dashpot will offer an opposite force which is proportional to velocity of the body.

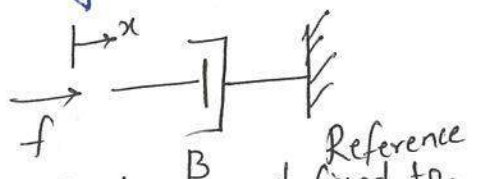


fig: Dashpot one end fixed to refere

Let f - Applied force

f_b - opposing force due to friction

Here, $f_b \propto \frac{dx}{dt}$ (or) $f_b = B\frac{dx}{dt}$

By Newton's second law $f = f_b = B\frac{dx}{dt}$

When the dashpot has displacement at both ends the opposing force is proportional to differential velocity.

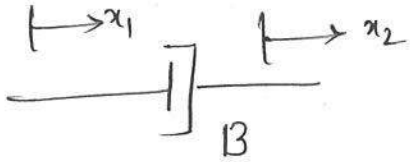


fig: Ideal dashpot with displacement at both ends.

$$f_b \propto \frac{d}{dt}(x_1 - x_2)$$

$$f_b = B\frac{d}{dt}(x_1 - x_2)$$

$$\therefore f = f_b = B\cdot\frac{d}{dt}(x_1 - x_2)$$

3. Consider an ideal elastic element spring, which has negligible mass and friction.

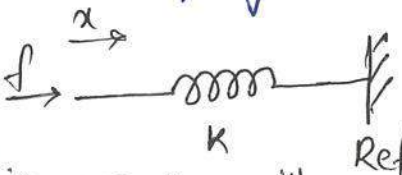


fig: Spring with on end fixed to reference.

Let a force be applied on it. The spring will offer an opposing force which is proportional to displacement of the body.

Let f = Applied force

f_k = opposing force due to elasticity

Here $f_k \propto x$ or $f_k = kx$

By Newton's second law, $f = f_k = kx$

When the spring has displacement at both ends. the opposing force is proportional to differential displacement.

$$f_k \propto (x_1 - x_2)$$

$$f_k = k(x_1 - x_2)$$

$$\therefore f = f_k = k(x_1 - x_2).$$

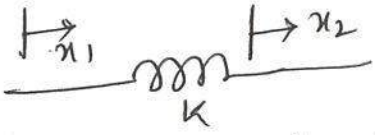


fig: Ideal spring with displacement at both sides.

Guideline to determine the Transfer function of Mechanical Translational System :-

1. In mechanical Translational System, the differential equations governing the system are obtained by writing force balancing equations at nodes in the system. The nodes are meeting point of elements. Generally the nodes are mass elements in the system. In some cases the nodes may be without mass element.

2. The linear displacement of the masses (nodes) are assumed as x_1, x_2, x_3 -- etc. and assign a displacement to each mass (node). The first derivative of the displacement is velocity and the second derivative of the displacement is acceleration.

3. Draw the free body diagram of the system. The free body diagram is obtained by drawing each mass seperately and then marking all the forces acting on that mass (node). Always opposing force acts in a direction of the applied force. Hence the displacement velocity and acceleration of the mass will be in the direction of the applied force. If there is no applied force then the displacement, velocity and acceleration of the mass is in a direction opposite to that of opposing force.

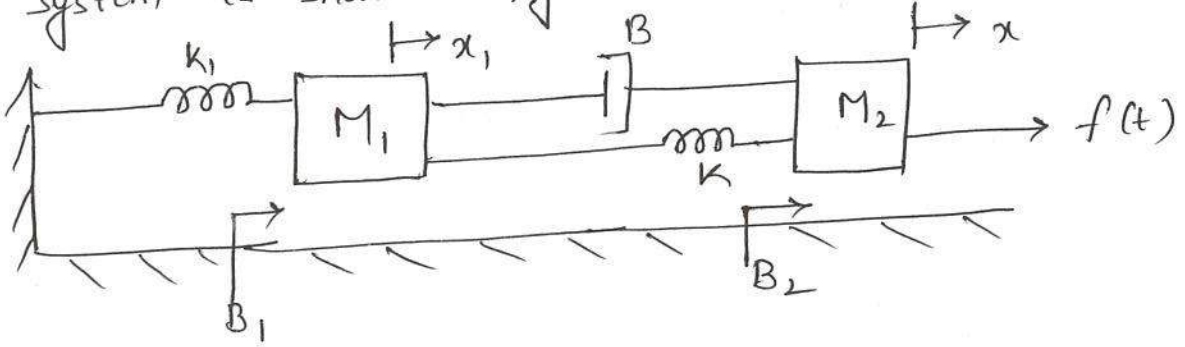
4. For each free body diagram write one differential equation by equating the sum of applied forces to the sum of opposing forces.

5. Take Laplace transform of differential equation to convert them to algebraic equations. Then rearrange the s-domain equations to eliminate the unwanted variables and obtain the ratio between output variable and input variable. The ratio of the output variable and the input variable is called the "Transfer function" of the system.

- ✓ Laplace Transform of $x(t) = L[x(t)] = X(s)$
- ✓ Laplace Transform of $\frac{dx(t)}{dt} = L\left[\frac{d}{dt}x(t)\right] = sX(s)$ with zero initial conditions
- ✓ Laplace transform of $\frac{d^2x(t)}{dt^2} = L\left[\frac{d^2}{dt}x(t)\right] = s^2X(s)$ with zero initial conditions.

Problems:

1. Write the differential equations governing the mechanical system is shown in fig. and determine the Transfer function.



Sol:- In the given system applied force $f(t)$ is input and displacement x is the output.

Let Laplace transform of $f(t) = L[(f(t))] = F(s)$

and Laplace transform of $x = L[x] = X(s)$

Hence Transfer function is $\frac{X(s)}{F(s)}$

The system has two nodes i.e., Mass M_1 and M_2.

The differential equations governing the system are given by force balance equations at these nodes.

Let the displacement of mass M_1 be x_1

The free diagram of mass M_1 is shown in fig 1.

The opposing forces acting on mass M_1 are marked as $f_{m_1}, f_{b_1}, f_b, f_{k_1}, f_k$.

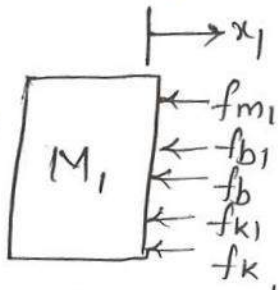


fig 1:- freebody diagram of Mass M_1 (Node 1)

$$f_{m_1} = M_1 \frac{d^2x_1}{dt^2}$$

$$f_{b_1} = B_1 \frac{dx_1}{dt}$$

$$f_{k_1} = k_1 x_1$$

$$f_b = B \frac{d}{dt}(x_1 - x)$$

$$f_k = k(x_1 - x)$$

By Newton's Second Law

$$f_{m_1} + f_{b_1} + f_{k_1} + f_b + f_k = 0$$

$$\therefore M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + k_1 x_1 + B \frac{d}{dt}(x_1 - x) + k(x_1 - x) = 0 \quad \text{---(1)}$$

On taking the Laplace Transform with zero initial Condition

Equation ① ⇒

$$M_1 s^2 X_1(s) + B_1 s X_1(s) + Bs[X_1(s) - X(s)] + k_1 X_1(s) + k[X_1(s) - X(s)] = 0.$$

$$X_1(s)[M_1 s^2 + (B_1+B)s + (k_1+k)] - X(s)[Bs+k] = 0$$

$$X_1(s)[M_1 s^2 + (B_1+B)s + (k_1+k)] = X(s)[Bs+k]$$

$$\therefore X_1(s) = X(s) \frac{(Bs+k)}{M_1 s^2 + (B_1+B)s + (k_1+k)} \quad \text{——— (2)}$$

The free body diagram of mass M_2 is shown in fig2. The opposing forces acting on M_2 are marked as f_{m_2}, f_{b_2}, f_b and f_k

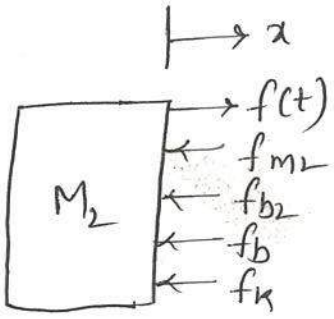


fig2: Free body diagram of mass M_2 (Node 2).

$$f_{m_2} = M_2 \frac{d^2x}{dt^2}$$

$$f_{b_2} = B_2 \frac{dx}{dt}$$

$$f_b = B \frac{d}{dt}(x - x_1)$$

$$f_k = k(x - x_1)$$

By Newton's second law

$$f_{m_2} + f_{b_2} + f_b + f_k = f(t)$$

$$M_2 \frac{d^2x}{dt^2} + B_2 \frac{dx}{dt} + B \frac{d}{dt}(x - x_1) + k(x - x_1) = f(t) \quad \text{——— (3)}$$

On taking Laplace Transform with zero initial Conditions

Equation (3) ⇒

$$M_2 s^2 X(s) + B_2 s X(s) + Bs[X(s) - X_1(s)] + k[X(s) - X_1(s)] = F(s)$$

$$X(s)\left[M_2 s^2 + (B_2+B)s + k\right] - X_1(s)\left[Bs + k\right] = F(s) \quad \text{——(4)}$$

Substitute equation (2) in equation (4) we get.

$$X(s)\left[M_2 s^2 + (B_2+B)s + k\right] - X(s)\frac{(Bs+k)^2}{M_1 s^2 + (B_1+B)s + (k_1+k)} = F(s)$$

$$X(s)\left[\frac{[M_1 s^2 + (B_1+B)s + (k_1+k)][M_2 s^2 + (B_2+B)s + k] - (Bs+k)^2}{M_1 s^2 + (B_1+B)s + (k_1+k)}\right] = F(s)$$

$$\therefore \frac{X(s)}{F(s)} = \frac{M_1 s^2 + (B_1+B)s + (k_1+k)}{[M_1 s^2 + (B_1+B)s + (k_1+k)][M_2 s^2 + (B_2+B)s + k] - (Bs+k)^2} \quad \text{——(5)}$$

Result:

① Equations (1) and (3) are the differential Equations governing the system.

② Equation (5) is the transfer function of the system

Problem 2:

Determine the Transfer function $\frac{Y_2(s)}{F(s)}$ of the system shown in the fig.

Sol:- Laplace transform of

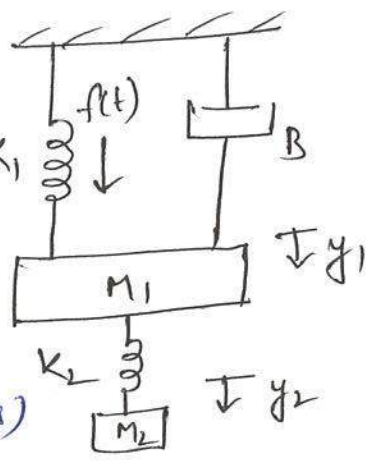
$f(t) = L[f(t)] = F(s)$

Laplace transform of

$y_1(t) = L[y_1(t)] = Y_1(s)$

Laplace transform of

$y_2(t) = L[y_2(t)] = Y_2(s)$



The System has two nodes and they are mass M_1 and M_2. The differential equations governing the System are the force balance equation at these nodes.

The free body diagram of mass M_1 is shown in fig1. The opposing forces are

f_b, f_{k_1}, f_{m_1}, f_{k_2}

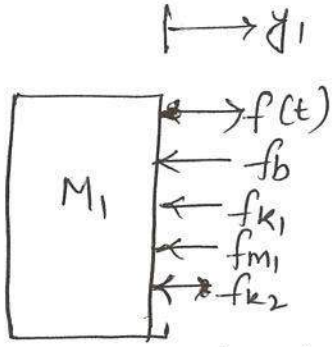


fig1: Freebody diagram of Mass M_1.

$$f_{m_1} = M_1 \frac{d^2 y_1}{dt^2}$$

$$f_b = B \frac{dy_1}{dt}$$

$$f_{k_1} = k_1 y_1$$

$$f_{k_2} = k_2 (y_1 - y_2)$$

By Newton's Second Law $f_{m_1} + f_b + f_{k_1} + f_{k_2} = f(t)$

$$\therefore M_1 \frac{d^2 y_1}{dt^2} + B \frac{dy_1}{dt} + k_1 y_1 + k_2 (y_1 - y_2) = f(t) \quad \text{——(1)}$$

On taking Laplace transform with zero initial conditions

$$M_1 s^2 Y_1(s) + BsY_1(s) + k_1 Y_1(s) + k_2 [Y_1(s) - Y_2(s)] = F(s)$$

$$Y_1(s) [M_1 s^2 + Bs + (k_1 + k_2)] - k_2 Y_2(s) = F(s) \quad \text{——(2)}$$

The free body diagram of mass M_2 is shown in fig2. The opposing forces acting on M_2 are f_{m_2} & f_{k_2}

$$f_{m_2} = M_2 \frac{d^2 y_2}{dt^2}$$

$$f_{k_2} = k_2 (y_2 - y_1)$$

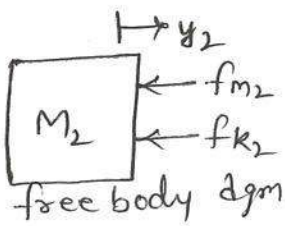


fig2: free body dgm of M_2

By Newton's second Law $-f_{m_2} + f_{k_2} = 0$

$$\therefore M_2 \frac{d^2y_2}{dt^2} + K_2(y_2 - y_1) = 0 \quad \text{——— (3)}$$

on taking Laplace Transform

$$M_2 s^2 Y_2(s) + K_2 [Y_2(s) - Y_1(s)] = 0$$

$$Y_2(s)[M_2 s^2 + K_2] - Y_1(s) K_2 = 0$$

$$Y_1(s) = \left[\frac{M_1 s^2 + K_2}{K_2}\right] Y_2(s) \quad \text{——— (4)}$$

Substitute eq (4) in eq (2) we get.

$$Y_2(s)\left[\frac{M_2 s^2 + K_2}{K_2}\right][M_1 s^2 + BS + (K_1 + K_2)] - Y_2(s) K_2 = F(s)$$

$$Y_2(s)\left[\frac{(M_2 s^2 + K_2)(M_1 s^2 + BS + (K_1 + K_2)) - K_2^2}{K_2}\right] = F(s)$$

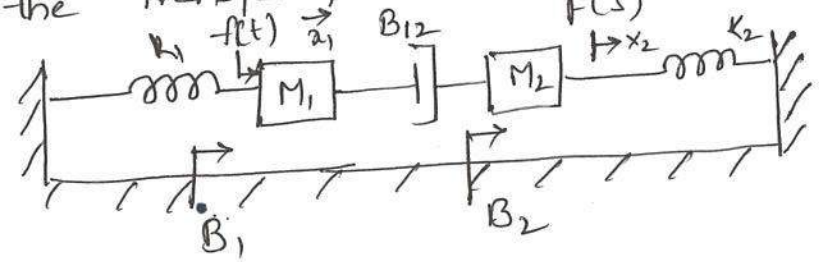
$$\therefore \frac{Y_2(s)}{F(s)} = \frac{K_2}{[M_1 s^2 + BS + (K_1 + K_2)][M_2 s^2 + K_2] - K_2^2} \quad \text{——— (5)}$$

Result:

- ✓ Equation (1) and (3) are the differential Equations of the system
- ✓ equation (5) is the Transfer function of the system.

Problem : 3

Determine the Transfer function $\frac{X_1(s)}{F(s)}$ and $\frac{X_2(s)}{F(s)}$ for the fig



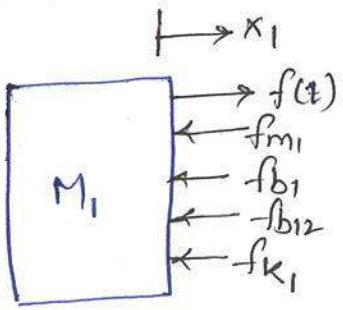
Sol:- Let Laplace Transform of $f(t) = F(s)$

Laplacetransform of $x_1 = X_1(s)$

Laplace transform of $x_2 = X_2(s)$

The System has two nodes and they are mass M_1 and M_2. The differential equations governing the system are the force balance equation at these nodes.

The free body diagram of mass M_1 is shown in fig. The opposing forces are :



$-f_{m_1}, -f_{b_1}, -f_{b_{12}}, -f_{k_1}$

$$f_{m_1} = M_1 \frac{d^2x_1}{dt^2}$$

$$f_{b_1} = B_1 \frac{dx_1}{dt}$$

$$f_{b_{12}} = B_{12} \frac{d}{dt}(x_1 - x_2)$$

$$f_{k_1} = K_1 x_1$$

By Newton's Second Law, $f_{m_1} + f_{b_1} + f_{b_{12}} + f_{k_1} = f(t)$

$$M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B_{12} \frac{d}{dt}(x_1 - x_2) + k_1 x_1 = f(t) \quad \text{---(1)}$$

On taking Laplace Transform with zero initial Condition

$$F(s) = M_1 s^2 X_1(s) + B_1 s X_1(s) + B_{12} s (X_1(s) - X_2(s)) + k_1 X_1(s)$$

$$= X_1(s)(M_1 s^2 + B_1 s + B_{12} s + K_1) \ominus X_2(s) \cdot B_{12} s$$

$$= X_1(s)(M_1 s^2 + s(B_1 + B_{12}) + K_1) - X_2(s) B_{12} s \quad \text{---(2)}$$

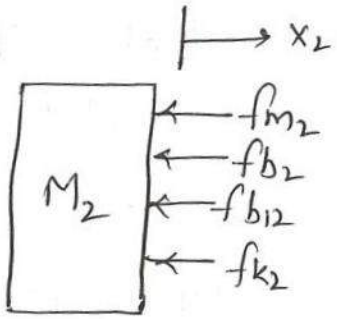
The free body diagram of mass M_2 is shown in fig. The opposing forces are marked as f_{m_2}, f_{b_2}, $f_{b_{12}}$ and f_{k_2}.

$$f_{m_2} = M_2 \frac{d^2x_2}{dt^2}$$

$$f_{b_2} = B_2 \frac{dx_2}{dt}$$

$$f_{b_{12}} = B_{12} \frac{d}{dt}(x_2 - x_1)$$

$$f_{k_2} = k_2 x_2$$



By Newton's second law $f_{m_2} + f_{b_2} + f_{b_{12}} + f_{k_2} = 0$

$$M_2 \frac{d^2x_2}{dt^2} + B_2 \frac{dx_2}{dt} + B_{12} \frac{d}{dt}(x_2 - x_1) + k_2 x_2 = 0 \quad \text{---(3)}$$

By taking Laplace Transform

$$M_2 s^2 X_2(s) + B_2 s X_2(s) + B_{12} s (X_2(s) - X_1(s)) + k_2 X_2(s) = 0$$

$$X_2(s)\left[s^2 M_2 + s(B_2 + B_{12}) + k_2\right] - s B_{12} X_1(s) = 0$$

$$X_2(s)\left[s^2 M_2 + s(B_2 + B_{12}) + k_2\right] = s B_{12} X_1(s)$$

$$\therefore X_2(s) = \frac{B_{12} X_1(s)}{s^2 M_2 + s(B_2 + B_{12}) + k_2} \quad \text{---(4)}$$

Subtitute (4) eq in eq (2).

$$F(s) = X_1(s)\left[M_1 s^2 + (B_1 + B_{12}) s + k_1\right] - \frac{(B_{12} s)^2 X_1(s)}{M_2 s^2 + (B_2 + B_{12}) s + k_2}$$

$$= X_1(s)\left[M_1 s^2 + (B_1 + B_{12}) s + k\right]\left[M_2 s^2 + (B_2 + B_{12}) s + k_2\right] - (B_{12} s)^2 X_1(s)$$

$$M_2 s^2 + (B_2 + B_{12}) s + k_2$$

$$\therefore \frac{X_1(s)}{F(s)} = \frac{M_2 s^2 + (B_2 + B_{12}) s + k_2}{X_1(s)\left[\left[M_1 s^2 + (B_1 + B_{12}) s + k\right]\left[M_2 s^2 + (B_2 + B_{12}) s + k_2\right] - B_{12} s^2\right] X_1(s)} \quad \text{---(5)}$$

From equation (4) $X_1(s) = \dfrac{\left[M_2 s^2 + (B_2 + B_{12})s + k_2\right] X_2(s)}{B_{12} s}$ — (6)

Substitute eq(5) in eq (2)

$$X_2(s)\left[\frac{M_2 s^2 + (B_2 + B_{12})s + k_2}{B_{12} s}\right]\left[M_1 s^2 + (B_1 + B_{12})s + k_1\right] - B_{12} s X_2(s) = F(s)$$

$$F(s) = X_2(s)\left[\frac{\left[M_2 s^2 + (B_2 + B_{12})s + k_2\right]\left[M_1 s^2 + (B_1 + B_{12})s + k_1\right]}{B_{12} s}\right]$$

$$\therefore \frac{X_2(s)}{F(s)} = \frac{B_{12} s}{\left[M_2 s^2 + (B_2 + B_{12})s + k_2\right]\left[M_1 s^2 + (B_1 + B_{12})s + k_1\right] - (B_{12} s)^2} \quad \text{— (7)}$$

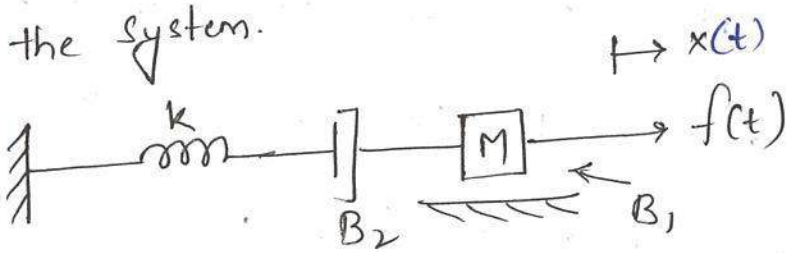
Result:

The differential equation governing system are eq (1) + (3)

The Transfer function of the system are eq (5) + (7)

Problem 4:

Write the equations of motion in s-domain for the System shown in fig. Determine the transfer function of the system.



Sol:

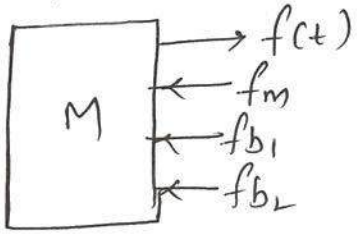
Let Laplace Transform of $x(t) = X(s)$

Laplace Transform of $f(t) = F(s)$

Let x_1 be the displacement at the meeting point of spring and dashpot. Laplace transform of x_1 is $X_1(s)$

The system has two nodes and they are mass M and the meeting point of spring and dashpot. The differential equations governing the system are the force balancing equations at these nodes. The equations of motion in the s-domain are obtained by taking Laplace Transform of the differential equations.

The free body diagram of mass M is shown in fig. The opposing forces are : f_m, f_{b_1} and f_{b_2}



$$f_m = M \frac{d^2x}{dt^2}$$

$$f_{b_1} = B_1 \frac{dx}{dt}$$

$$f_{b_2} = B_2 \frac{d}{dt}(x - x_1)$$

By Newton's second law the force balance equation is $f_m + f_{b_1} + f_{b_2} = f(t)$

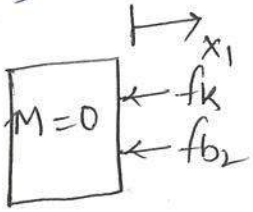
$$M \frac{d^2x}{dt^2} + B_1 \frac{dx}{dt} + B_2 \frac{d}{dt}(x - x_1) = f(t) \quad \text{---(1)}$$

On taking Laplace Transform on both sides

$$M s^2 X(s) + B_1 s[X(s) + B_2 s (X(s) - X_1(s)) = F(s)$$

$$X(s)\left[M s^2 + B_1 s + B_2 s\right] - B_2 s X_1(s) = F(s) \quad \text{---(2)}$$

The free body diagram at the meeting point of spring and dashpot is shown in fig. The opposing forces are f_k, f_{b_2}.



$$f_{b_2} = B_2 \frac{d}{dt}(x_1 - x)$$

$$f_k = k x_1$$

By Newton's Second Law, $f_{b_2} + f_k = 0$

$$\therefore B_2 \frac{d}{dt}(x_1 - x) + k x_1 = 0 \quad \text{——(3)}$$

On taking Laplace Transform

$$B_2 s [X_1(s) - X(s)] + k X_1(s) = 0$$

$$(B_2 s + k) X_1(s) - B_2 s X(s) = 0$$

$$\therefore X_1(s) = \frac{B_2 s}{B_2 s + k} X(s) \quad \text{——(4)}$$

Substituting for $X_1(s)$ from equation (4) in (2)

$$\left[M s^2 + (B_1 + B_2) s\right] X(s) - B_2 s \left[\frac{B_2 s}{B_2 s + k}\right] X(s) = F(s)$$

$$X(s) \left[\frac{\left[M s^2 + (B_1 + B_2) s\right](B_2 s + k) - (B_2 s)^2}{B_2 s + k} \right] = F(s)$$

$$\therefore \frac{X(s)}{F(s)} = \frac{B_2 s + k}{\left[M s^2 + (B_1 + B_2) s\right](B_2 s + k) - (B_2 s)^2} \quad \text{——(5)}$$

Result:-

- ✓ The differential equations governing the system are: equation (2) and (3)
- ✓ Transfer function of the system is eq (5).

Mechanical Rotational System:

The model of rotational Mechanical systems can be obtained by using three elements.

1. moment of Inertia [J] of mass.
2. Dash pot with rotational frictional coefficient [B]
3. Tortional spring with stiffness [k].

The weight of the rotational Mechanical system is represented by the moment of inertia of the mass. The moment of inertia of the system or body is considered at the centre of gravity of the body. The elastic deformation of the body can be represented by a spring (tortional spring). The frictional existing in rotational mechanical system can be represented by the dashpot. The dashpot is a piston rotating inside a cylinder filled with viscous fluid.

When a torque is applied to a rotational mechanical system, it is opposed by opposing torques due to moment of inertia, friction and elasticity of the system. The torques acting on a rotational mechanical body are governed by Newton's second law of motion for rotational systems. It states that the sum of torques acting on a body is zero. (Newton's law states that the sum of applied torques is equal to the sum of opposing torques on a body).

List of Symbols used in Mechanical Rotational System:

θ = Angular displacement, rad

$\frac{d\theta}{dt}$ = Angular velocity, rad/sec

$\frac{d^2\theta}{dt^2}$ = Angular Acceleration, rad/sec^2

T = Applied Torque N-m.

J = moment of inertia, Kg-m^2/rad.

B = Rotational frictional Coefficient, N-m/(rad/sec)

K - Stiffness of the spring, N-m/rad.

Torque Balance Equations of Idealised Elements:

Consider an ideal mass element shown in fig which has negligible friction and elasticity. The opposing torque due to moment of inertia is proportional to the angular acceleration.

Let T = Applied torque

T_j = opposing torque due to moment of inertia of the body.

Here $T_j \propto \frac{d^2\theta}{dt^2}$

$T_j = J\frac{d^2\theta}{dt^2}$

By Newton's Second Law

$T = T_j = J\frac{d^2\theta}{dt^2}$

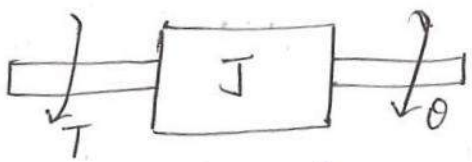


Fig: Ideal rotational mass element.

Consider an ideal frictional element dashpot shown in fig. which has negligible moment of inertia and elasticity. Let a torque be applied on it. The dashpot will offer an opposing torque which is proportional to the angular velocity of the body.

Let T - Applied Torque

T_b - opposing torque due to friction.

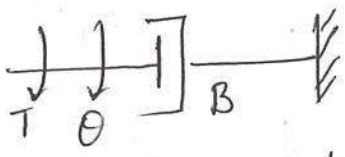


Fig: Ideal rotational dash-pot with one-end fixed to reference.

$$T_b \propto \frac{d\theta}{dt}$$

$$T_b = B \cdot \frac{d\theta}{dt}$$

By Newton's second Law

$$T = T_b = B \cdot \frac{d\theta}{dt}$$

When the dashpot has angular displacement at both ends the opposing torque is proportional to the differential angular velocity.

$$T_b \propto \frac{d}{dt}(\theta_1 - \theta_2)$$

$$T_b = B \frac{d}{dt}(\theta_1 - \theta_2)$$

$$\therefore T = T_b = B \cdot \frac{d}{dt}(\theta_1 - \theta_2)$$

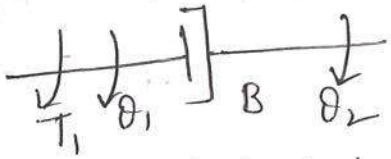


Fig: Ideal dashpot with angular displacement at both ends.

Consider an ideal elastic element, tortional spring as shown in fig, which has negligible moment of inertia and friction.

Let a torque be applied on it. The torsional Spring well offer an opposing torque which is proportional to angular displacement of the body.

Let T = Applied torque.

T_k = opposing torque to the elasticity

$T_k \propto \theta$

$T_k = k\theta$

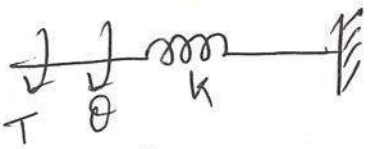


fig: Ideal spring with one end fixed to reference.

By Newton's Second Law

$T = T_k = k\theta$

When the Spring has angular displacement at both ends as shown in fig2, the opposing ~~force~~ torque is proportional to the differential angular displacement.

$T_k \propto (\theta_1 - \theta_2)$

$T_k = k(\theta_1 - \theta_2)$

$T = T_k = k(\theta_1 - \theta_2)$

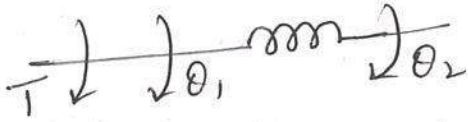


Fig2: Ideal spring with angular displacement at both ends.

Guidelines to determine the Transfer function of Mechanical Rotational System:-

1. In Mechanical rotational System, the differential equations governing the system are obtained by writing torque balance equations at nodes in the system. The nodes are meeting point of elements.

Generally the nodes are mass elements with moment of inertia in the system. In some cases the nodes may be without mass element.

2. The angular displacement of the moment of inertia of the masses (nodes) are assumed as $\theta_1, \theta_2, \theta_3$ etc and assign a displacement to each mass (node). The first derivative of angular displacement is angular velocity and the second derivative of the angular displacement is angular acceleration.

3. Draw the free body diagram of the system. The free body diagram is obtained by drawing each moment of inertia of mass seperately and the marking all the torques acting on that body. Always the opposing torques acts in a direction opposite to applied torque.

4. The mass has to rotate in the direction of the applied torque. Hence the angular displacement, velocity and acceleration of the mass will be in the direction of the applied torque. If there is no applied torque then the angular displacement, velocity and acceleration of the mass is in a direction opposite to that of opposing torque.

5. For each free body diagram write one differential equation by equating the sum of applied

torques to the sum of opposing torques.

6. Take Laplace transform of differential equation to Convert them to algebraic equations. Then rearrange the s-domain equations to eliminate the unwanted Variables and obtain the relation between output Variable and input variable. This ratio is the "Tranfer function of the system.

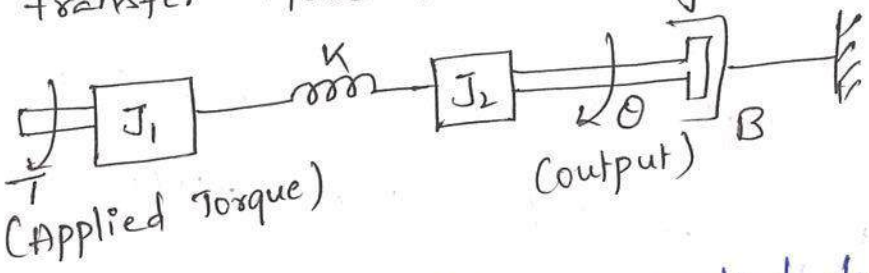
Note: Laplace transform of $\theta = L[\theta] = \theta(s)$

Laplace transform of $\frac{d\theta}{dt}$ with zero initial Conditions $= L\left[\frac{d\theta}{dt}\right] = s\theta(s)$

Laplace transform of $\frac{d^2\theta}{dt^2}$ with zero initial Conditions $L\left[\frac{d^2\theta}{dt^2}\right] = s^2\theta(s)$.

Problem 1:

Write the differential equations governing the mechanical rotational system shown in fig. obtain the transfer function of the system.



Sol:- In the given system, applied torque T is the input and angular displacement θ is the output.

Let Laplace Transform of T is $L[T] = T(s)$

Laplace Transform of $\theta \Rightarrow L[\theta] = \theta(s)$

Hence the required Transfer function is $\frac{\theta(s)}{T(s)}$

The system has two nodes: masses with moment of inertia J_1 and the differential equations governing the system are given by torque balance equations at these nodes.

Let the angular displacement of mass with moment of inertia J_1 be θ_1. The Laplace transform of θ_1 is $L[\theta_1] = \theta_1(s)$. The freebody diagram of J_1 is shown in fig. The opposing torques acting on J_1 are marked as T_{j1} and T_k.

$$T_{j1} = J_1 \frac{d^2\theta}{dt^2}$$

$$T_k = K(\theta_1 - \theta_2)$$

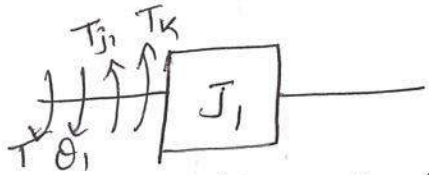


fig: free body diagram of mass with moment of inertia J_1.

By Newton's second law

$$T_{j1} + T_k = T$$

$$J_1 \frac{d^2\theta_1}{dt^2} + K(\theta_1 - \theta) = T$$

$$J_1 \frac{d^2\theta}{dt^2} + K\theta_1 - K\theta = T \quad \text{——} (1)$$

On Taking Laplace Transform on both sides.

$$J_1 s^2 \theta_1(s) + K\theta_1(s) - K\theta(s) = T(s)$$

$$(J_1 s^2 + K)\theta_1(s) - K\theta(s) = T(s) \quad \text{——} (2)$$

The free body diagram of mass with moment of Inertia J_2 is shown in fig. The opposing torques acting on J_2 are marked as T_{j2}, T_b and T_k.

$T_{j2} = J_2 \frac{d^2\theta}{dt^2}$; $T_b = B \cdot \frac{d\theta}{dt}$; $T_k = K(\theta_2 - \theta_1)$

By Newton's second Law

$T_{j2} + T_b + T_k = 0$

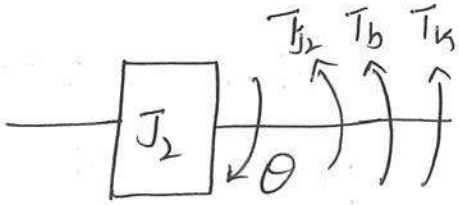


fig:- Free body diagram of mass with moment of inertia J_2.

$\therefore J_2 \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K(\theta - \theta_1) = 0$

$J_2 \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta - K\theta_1 = 0$ ——— (3)

On taking Laplace transform with zero initial Conditions..

$J_2 s^2\theta(s) + Bs\,\theta(s) + K\theta(s) - K\theta_1(s) = 0$

$(J_2 s^2 + Bs + K)\theta(s) - K\theta_1(s) = 0.$

$\theta_1(s) = \frac{(J_2 s^2 + Bs + K)}{K}\theta(s)$ ——— (4)

Substituting for $\theta_1(s)$ into eq (2).

$(J_1 s^2 + K)\frac{(J_2 s^2 + Bs + K)}{K}\theta(s) - K\theta(s) = T(s)$

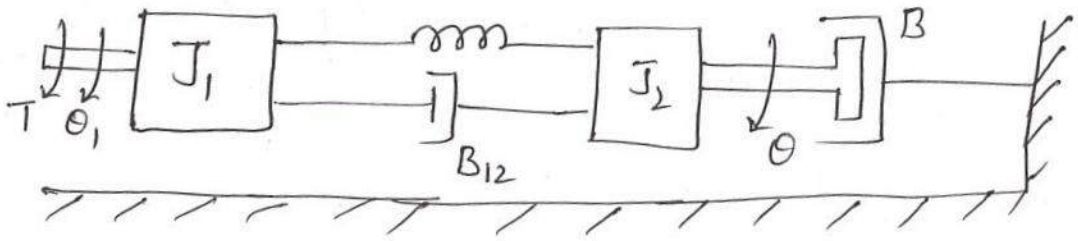
$\left[\frac{(J_1 s^2 + K)(J_2 s^2 + Bs + K) - K^2}{K}\right]\theta(s) = T(s)$

$\therefore \frac{\theta(s)}{T(s)} = \frac{K}{(J_1 s^2 + K)(J_2 s^2 + Bs + K) - K^2}$ ——— (5)

Result:- The differential equations governing the System are eq (2) + (3) and the Transfer function of the system eq (5).

Problem : 2.

Write the differential equations governing the mechanical rotational system shown in fig, and determine the transfer function ~~$\theta(s)$~~ $\theta(s)/T(s)$.



Sol: In the given system the torque T is the output and the angular displacement θ is the output.

Let the Laplace Transform of $T = L[T] = T(s)$

the Laplace transform of $\theta = L[\theta] = \theta(s)$

Hence the required Transfer function is $\theta(s)/T(s)$.

The system has two nodes and they are masses with moment of inertia J_1 and J_2. The differential equations governing the system are given by torque balance equation at these nodes.

Let the angular displacement of mass with moment of inertia J_1 be θ_1. The Laplace Transform of $\theta_1 = L[\theta_1] = \theta_1(s)$.

The free body diagram of J_1 is shown in fig. The opposing torques acting on J_1 are marked

as T_{j1}, T_{b12} and T_k.

$$T_{j1} = J_1 \frac{d^2\theta}{dt^2}$$

$$T_{b12} = B_{12} \frac{d}{dt}(\theta_1 - \theta)$$

$$T_k = k(\theta_1 - \theta)$$

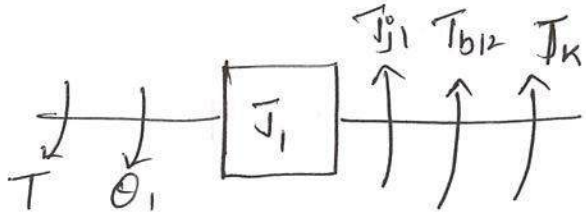


fig: free body diagram of mass with moment of inertia J_1.

By Newton's Second Law

$$T_{j1} + T_{b12} + T_k = T$$

$$J_1 \frac{d^2\theta_1}{dt^2} + B_{12} \frac{d}{dt}(\theta_1 - \theta) + k(\theta_1 - \theta) = T \quad \text{——} (1)$$

On taking Laplace Transform with zero initial conditions.

$$J_1 s^2 \theta_1(s) + B_{12} s[\theta_1(s) - \theta(s)] + k\theta_1(s) - k\theta(s) = T(s)$$

$$\theta_1(s)[J_1 s^2 + s B_{12} + k] - \theta(s)[sB_{12} + k] = T(s) \quad \text{——} (2)$$

The free body diagram of mass with moment of inertia J_2 is shown in fig. The opposing torques are marked as T_{j2}, T_{b12}, T_b and T_k.

$$T_{j2} = J_2 \frac{d^2\theta}{dt^2}$$

$$T_{b12} = B_{12} \frac{d}{dt}(\theta - \theta_1)$$

$$T_b = B \frac{d\theta}{dt}$$

$$T_k = k(\theta - \theta_1)$$

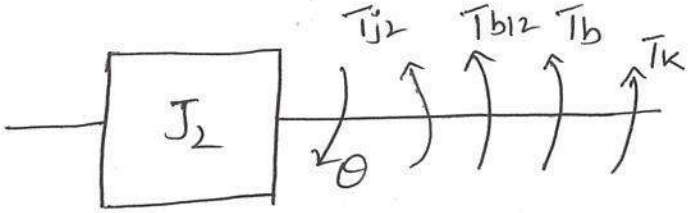


fig:- free body diagram of mass with moment of inertia J_2.

By Newton's Second Law

$$T_{j2} + T_{b12} + T_b + T_k = 0.$$

$$J_2 \frac{d^2\theta}{dt^2} + B_{12} \frac{d}{dt}(\theta - \theta_1) + B \frac{d\theta}{dt} + k(\theta - \theta_1) = 0 \quad \text{---(3)}$$

On taking Laplace Transform with zero initial Conditions.

$$J_2 s^2 \theta(s) - B_{12} s \theta_1(s) + s\theta(s)[B_{12} + B] + k\theta(s) - k\theta_1(s) = 0$$

$$\theta(s)\left[s^2 J_2 + s(B_{12} + B) + k\right] - \theta_1(s)[sB_{12} + k] = 0$$

$$\theta_1(s) = \frac{\left[s^2 J_2 + s(B_{12} + B) + k\right]}{[sB_{12} + k]} \theta(s) \quad \text{---(4)}$$

Substitute $\theta_1(s)$ in eq (2).

$$[J_1 s^2 + sB_{12} + k] \frac{[J_2 s^2 + s(B_{12} + B) + k]\theta(s)}{(sB_{12} + k)} - (sB_{12} + B)\theta(s) = T(s)$$

$$\left[\frac{(J_1 s^2 + sB_{12} + k)(J_2 s^2 + s(B_{12} + B) + k) - (sB_{12} + k)^2}{(sB_{12} + k)}\right] \theta(s) = T(s)$$

$$\therefore \frac{\theta(s)}{T(s)} = \frac{(sB_{12} + k)}{(J_1 s^2 + sB_{12} + k)(J_2 s^2 + s(B_{12} + B) + k) - (sB_{12} + k)^2} \quad \text{---(5)}$$

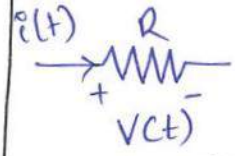
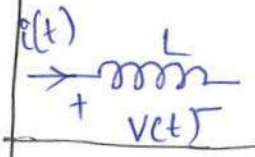
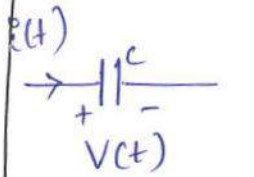
Result :- 1. The differential equations governing the system are : eq (2) + (3).

2. The Transfer function of the system is eq (5).

Electrical Systems :

The models of electrical systems can be obtained by using resistor, capacitor and inductor.

Current - voltage relation of R, L and C

Element	Voltage across the element	Current through the element.
	$V(t) = Ri(t)$	$i(t) = \frac{V(t)}{R}$
	$V(t) = L\frac{d}{dt} i(t)$	$i(t) = \frac{1}{L}\int v(t)\,dt$
	$V(t) = \frac{1}{C}\int i(t)\,dt$	$i(t) = C\frac{dv(t)}{dt}$

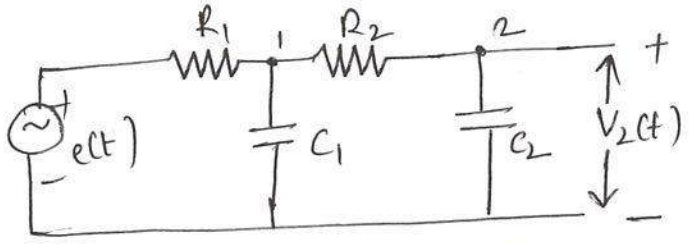
For modelling the electrical systems the electrical network or equivalent circuit is formed by using R, L and C and voltage (or) current source.

The differential equations governing the electrical system can be formed by writing kirchoff's current law (KCL) equations by choosing various nodes in the network.

The transfer function can be obtained by taking laplace transform of the differential equations, and rearranging them as a ratio of o/p to i/p.

Problem 1:

Obtain the transfer function of the electrical Network shown in the fig.



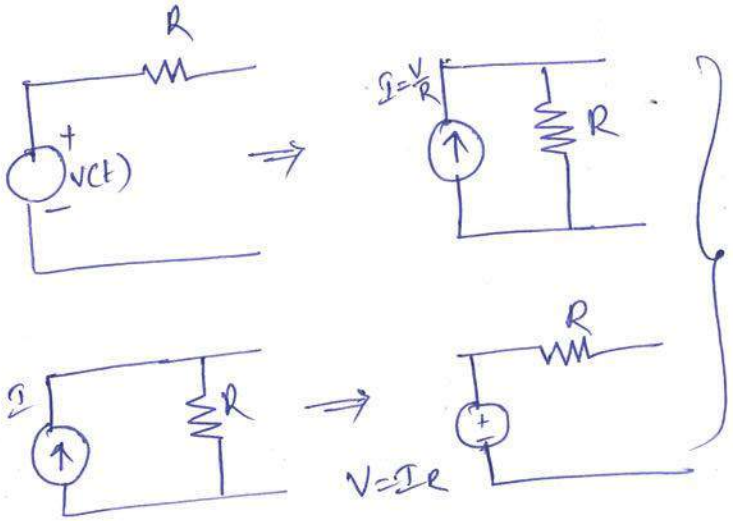
Sol: In the given network input is e(t) and output is $V_2(t)$.

Let Laplace Transform of $e(t) = L[e(t)] = E(s)$

Laplace Transform of $V_2(t) = L[V_2(t)] = V_2(s)$

∴ The transfer function of the Network is $\frac{V_2(s)}{E(s)}$.

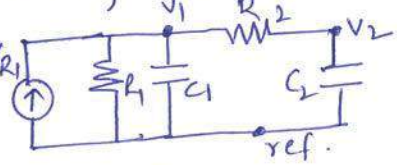
Transform of the voltage source in series with Resistance R_1 into equivalent current source as shown in fig.



Source Transformation (Model).

The network has two nodes. Let the node voltage be V_1 and V_2. The Laplace transform of node voltages V_1 and V_2 are $V_1(s)$ and $V_2(s)$.

The differential equation's governing the network are given by KCL at these nodes.



At node 1 from fig ①

$$\frac{V_1}{R_1} + C_1\frac{dv_1}{dt} + \frac{V_1 - V_2}{R_2} = \frac{e}{R_1} \quad \text{—— ①}$$

On taking Laplace Transform with zero initial Conditions

$$\frac{V_1(s)}{R_1} + C_1 s V_1(s) + \frac{V_1(s)}{R_2} - \frac{V_2(s)}{R_2} = \frac{E(s)}{R_1}$$

$$V_1(s)\left[\frac{1}{R_1} + sC_1 + \frac{1}{R_2}\right] - \frac{V_2(s)}{R_2} = \frac{E(s)}{R_1} \quad \text{——(2)}$$

At node 2 by KCL from fig (2)

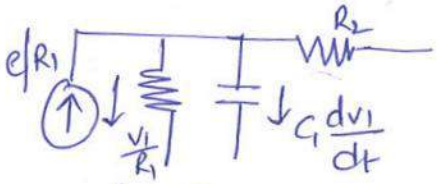


fig (1)

$$\frac{V_2 - V_1}{R_2} + C_2 \frac{dv_2}{dt} = 0. \quad \text{——(3)}$$

On taking Laplace transform of equation (3) with zero initial Conditions

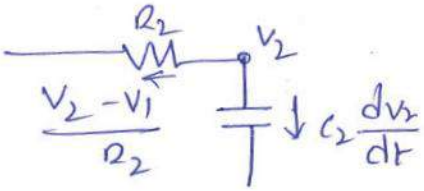


fig (2).

$$\frac{V_2(s)}{R_2} - \frac{V_1(s)}{R_2} + C_2 s V_2(s) = 0$$

$$\frac{V_1(s)}{R_2} = \frac{V_2(s)}{R_2} + C_2 s V_2(s) \Rightarrow \left[\frac{1}{R_2} + sC_2\right] V_2(s).$$

$$\therefore V_1(s) = \left[1 + sC_2R_2\right] V_2(s) \quad \text{——(4)}.$$

Substitute $V_1(s)$ in eq (4) and (2) we get.

$$(1 + sR_2C_2)\, V_2(s)\left[\frac{1}{R_2} + sC_1 + \frac{1}{R_2}\right] - \frac{V_2(s)}{R_2} = \frac{E(s)}{R_1}$$

$$\left[\frac{(1+sR_2C_2)(R_2 + R_1 + sC_1R_1R_2) - R_1}{R_1R_2}\right] V_2(s) = \frac{E(s)}{R_1}$$

$$\therefore \frac{V_2(s)}{E(s)} = \frac{R_2}{\left[(1+sR_2C_2)(R_1+R_2+sC_1R_1R_2) - R_1\right]}. \quad \text{——(5)}$$

Result: ✓ Differential Equations eq (1) & (3).

✓ T.f — eq (5).

Electrical Analogous of Mechanical Translational Systems:

The three basic element mass, dashpot and spring that are used in modelling mechanical translational systems are analogous to resistance and capacitance of electrical system.

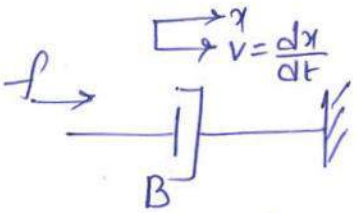
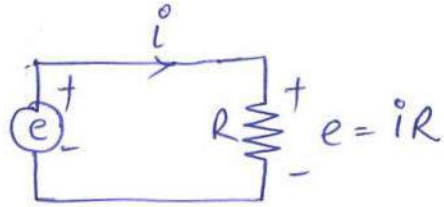
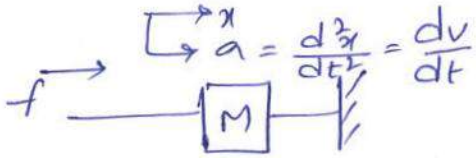
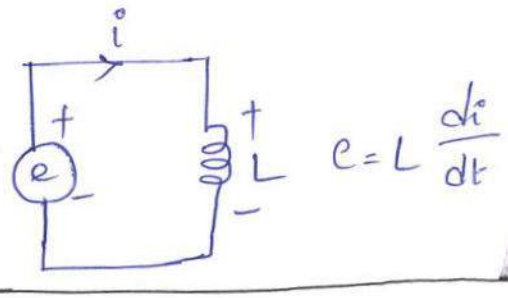
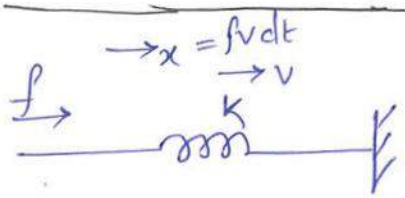
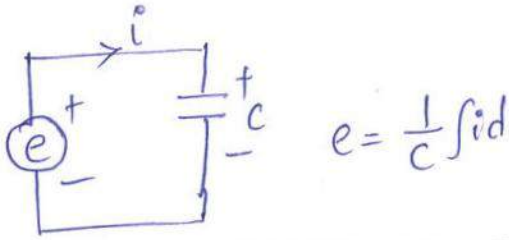
The input force in mechanical system is analogous to either voltage source or current source in electrical systems.

The output velocity (first derivative of displacement) in mechanical system is analogous to either current or voltage in an element in electrical system.

Since the electrical systems has two types of inputs either voltage or current source, there are two types of analogies: force - voltage analogy & force - current analogy.

Force - Voltage Analogy:-

The force Balancing equations of Mechanical elements and their analogous electrical elements in force - voltage are shown in Table.

Mechanical System	Electrical System
Input: Force Output: Velocity	Input: Voltage source Output: Current through the element.
 $f = B\frac{dx}{dt}$ $= Bv$	 $e = iR$
 $f = M\frac{d^2x}{dt^2} = M\frac{dv}{dt}$	 $e = L\frac{di}{dt}$
 $f = kx = k\int v\,dt$.	 $e = \frac{1}{C}\int i\,dt$

Guidelines to obtain electrical analogous of mechanical systems based on force - voltage analogy.

1. In electrical systems the elements in series will have same current Likewise in mechanical systems, the elements having same velocity are said to be in series.

2. The element having same velocity in mechanical system should have analogous same current in electrical analogous system.

3. Each node (meeting point of the elements) in the mechanical system corresponds to a closed loop in electrical system. A mass is considered as a Node.

4. The number of meshes in electrical analogous is same as that of the number of nodes (masses) in mechanical system. Hence the number of mesh currents and system equations will be same as that of the number of velocities of nodes (masses) in mechanical system.

5. The Mechanical driving sources (force) and passive elements connected to the node (mass) in mechanical system should be represented by analogous elements in a closed loop in analogous electrical system.

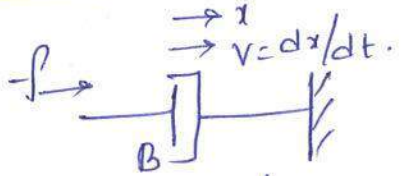
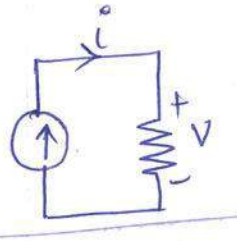
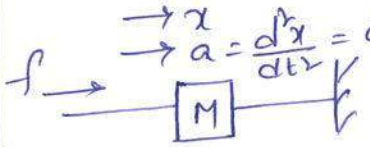
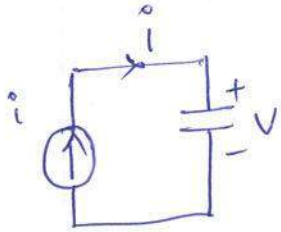
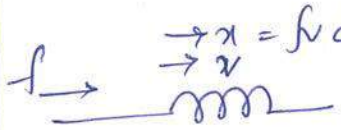
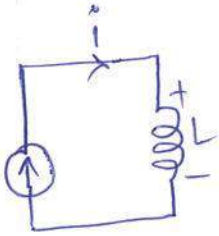
6. The element connected between two (nodes) masses in mechanical system is represented as a common element between two meshes in electrical analogous system.

List of analogous quantities in force-voltage analogy.

Item	Mechanical system	Electrical system
Independent Variable (Input)	Force, f	Voltage, e
Dependent Variable (Output)	velocity, v Displacement, x	Current, i Charge, q
Dissipative element.	Frictional Coefficient of dashpot, B	Resistance, R
Storage Element	Mass, M stiffness of spring, K	Inductance L Inverse of Capacitance, $1/c$
physical Law	Newton's second Law $\Sigma F=0$	KVL $\Sigma V=0$.
Changing the Level of independent Variable	Lever $\frac{f_1}{f_2}=\frac{l_1}{l_2}$	Transformer $\frac{e_1}{e_2}=\frac{N_1}{N_2}$

Force - Current Analogy:

The force balance equations of mechanical elements and their analogous electrical elements in force - current analogy are shown in the table.

Mechanical System	Electrical System.
Input : Force Output : velocity	Input : Current source Output : Voltage across the element
 $f = B\frac{dx}{dt} = BV$	 $i = \frac{1}{R}V$
 $f = M\frac{d^2x}{dt^2}$ $= M\frac{dv}{dt}$	 $i = C\frac{dv}{dt}$
 $f = kx = k\int v\,dt$	 $i = \frac{1}{L}\int v\,dt$

Guidelines to obtain electrical analogous of mechanical system based on force - current analogy.

1. In electrical systems element in parallel we have

Same voltage, Likewise in Mechanical systems, the elements having same force are said to be in parallel.

2. The elements having same velocity in mechanical system should have analogous same voltage in electrical analogous system.

3 Each node (meeting point of elements) in the mechanical system corresponds to a node in electrical system. ~~the Elements having~~ A mass is considered as a node.

4. The number of nodes in electrical analogy is same that of the number of nodes (masses) in mechanical system. Hence the number of node voltages and system equations will be same as that of the number of velocities of (masses) nodes in mechanical system.

5. The mechanical driving sources (forces) and passive elements connected to the node (mass) in mechanical system should be represented by analogy elements connected to a node in electrical system.

6. The element connected between two nodes (masses) in mechanical system is represented as a common element between two nodes in electrical analgous system.

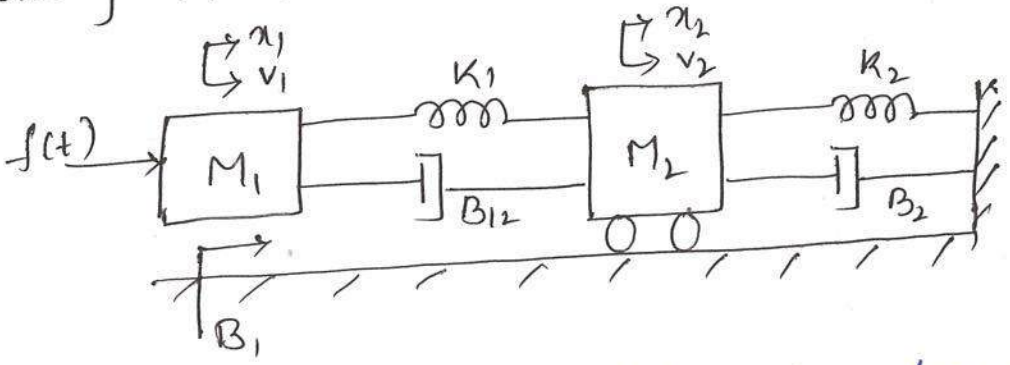
7.

The List of analogous quantities in force - current analogy:

Item	Mechanical System	Electrical System (Node basis system)
Independent Variable (Input)	force, f	Current, i
Dependent Variable (Output)	Velocity, v Displacement, x	Voltag, v flux, ϕ
Dissipative element	Frictional Coefficient of Dashpot, B	Conductance $G = 1/R$
Storage element	Mass, M Stiffness of spring, k	Capacitance, C Inverse of inductance $1/L$
Physical Law	Newton's second Law $\Sigma f = 0$	Kirchoff's current Law $\Sigma i = 0$.
Changing the Level of independent Variable	Lever $\frac{f_1}{f_2} = \frac{l_1}{l_2}$	Transformer $\frac{i_1}{i_2} = \frac{N_2}{N_1}$

Problem 1:

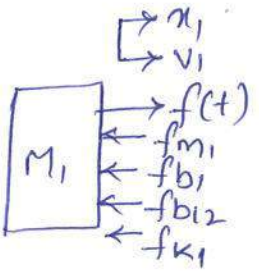
Write the differential equations governing the mechanical system shown in fig. Draw the force - voltage and force current electrical analogous circuits and verify by writing mesh and node equations.



Sol: The given mechanical system has two nodes (masses) The differential equations governing the mechanical system are given by force balance equations at these nodes.

Let the displacements of masses M_1 and M_2 be x_1 and x_2 respectively. The corresponding velocities be v_1 and v_2.

The free body diagram of M_1 as shown in fig.



The opposing forces are:

$$f_{M_1} = M_1 \frac{d^2x_1}{dt^2}$$

$$f_{b_1} = B_1 \frac{dx_1}{dt}$$

$$f_{b_{12}} = B_{12} \frac{d}{dt}(x_1 - x_2)$$

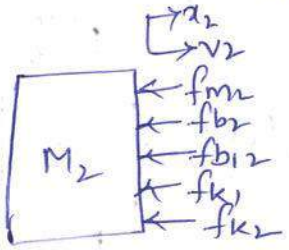
$$f_{k_1} = K_1 (x_1 - x_2).$$

By Newton's Second Law

$$f_{m_1} + f_{b_1} + f_{b_{12}} + f_{k_1} = f(t)$$

$$M_1 \frac{d^2x_1}{dt^2} + B \frac{dx_1}{dt} + B_{12} \frac{d}{dt}(x_1 - x_2) + k_1 (x_1 - x_2) = f(t) \quad \text{—— (1)}$$

The freebody diagram of M_2 is shown in fig.



The opposing forces are

$$f_{m_2} = M_2 \frac{d^2x_2}{dt^2}$$

$$f_{b_2} = B_2 \frac{dx_2}{dt}$$

$$f_{b_{12}} = B_{12} \frac{d}{dt}(x_2 - x_1)$$

$$f_{k_1} = k_1 (x_2 - x_1)$$

$$f_{k_2} = k_2 x_2$$

By Newton's Second Law.

$$f_{m_2} + f_{b_2} + f_{b_{12}} + f_{k_1} + f_{k_2} = 0.$$

$$M_2 \frac{d^2x_2}{dt^2} + B_2 \frac{dx_2}{dt} + B_{12} \frac{d}{dt}(x_2 - x_1) + k_1 (x_2 - x_1) + k_2 x_2 = 0. \quad \text{—— (2)}$$

On replacing the displacements by velocity in the differential equations (1) & (2) of the mechanical system we get, i.e., $\frac{d^2x}{dt^2} = \frac{dv}{dt}$, $\frac{dx}{dt} = v$, $x = \int v\,dt$.

$$M_1 \frac{dv_1}{dt} + B_1 v_1 + B_{12}(v_1 - v_2) + k_1 \int (v_1 - v_2)\,dt = f(t)$$

$$M_2 \frac{dv_2}{dt} + B_2 v_2 + k_2 \int v_2\,dt + B_{12}(v_2 - v_1) + k_1 \int (v_2 - v_1)\,dt = 0$$

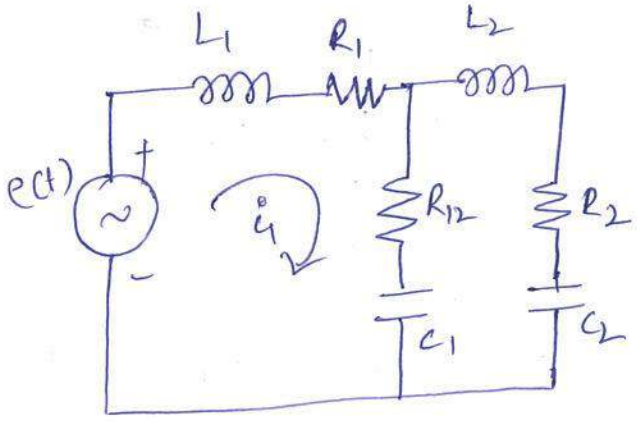
Force - Voltage Analogous Circuit:

The given mechanical system has two nodes (masses) Hence the force voltage analogous electrical circuit will have two meshes.

The force applied to M_1 is represented by a voltage source in first mesh. The elements M_1, B_1, K_1 and B_{12} are Connected to first node. Hence they are represented by analogous element in mesh 1 forming closed path. The elements k_1, B_{12}, M_2, K_2 and B_2 are Connected to second node. Hence they are represented by analogous element in mesh 2 forming a closed path.

The elements K_1 and B_{12} are Common between Node 1 and 2 and so they are represented by analogous element as Common elements between two meshes. The force - voltage electrical analogous circuit is shown in fig.

$f(t) \rightarrow e(t)$	$B_1 \Rightarrow R_1$
$M_1 \rightarrow L_1$	$B_2 \rightarrow R_2$
$M_2 \rightarrow L_2$	$B_{12} \rightarrow R_{12}$
$V_1 \rightarrow i_1$	$k_1 \rightarrow 1/C_1$
$V_2 \rightarrow i_2$	$K_2 \rightarrow 1/C_2$



The mesh basis equations using kirchoff's voltage law for the circuit shown in fig.

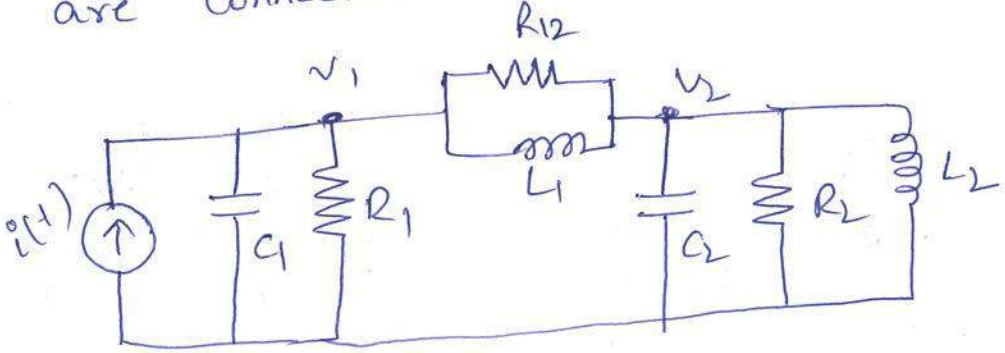
$$L_1 \frac{di_1}{dt} + i_1 R_1 + R_{12}(i_1 - i_2) + \frac{1}{C_1}\int (i_1 - i_2)\,dt = e(t)$$

$$L_2 \frac{di_2}{dt} + i_2 R_2 + \frac{1}{C_2}\int i_2\,dt + R_{12}(i_2 - i_1) + \frac{1}{C_1}\int (i_2 - i_1)\,dt = 0.$$

Force Current Analogous Circuit.

The given mechanical system has two nodes (masses). Hence the force current analogous electrical circuit will have two nodes.

The force applied to mass M_1 is represented as a current source connected to node1 in analogous electrical circuit. The elements M_1, B_1, K_1 and B_{12} are connected to first node. Hence they are represented by analogous elements connected to node1 in analogous electrical circuit. The elements K_1, B_{12}, M_2, K_2 and B_2 are connected to second node. Hence they are



$$C_1 \frac{dv_1}{dt} + \frac{1}{R} v_1 + \frac{1}{R_{12}}(v_1 - v_2) + \frac{1}{L_1}\int (v_1 - v_2)\,dt = i(t)$$

$$C_2 \frac{dv_2}{dt} + \frac{1}{R_2} v_2 + \frac{1}{L_2}\int v_2\,dt + \frac{1}{R_{12}}(v_2 - v_1) + \frac{1}{L_1}\int (v_2 - v_1)\,dt = 0.$$

represented by analogous elements as elements connected to node 2 in analogous electrical circuit.

The elements K_1 and B_{12} are common between node 1 and 2 and so they are represented by analogous elements as common element between two nodes in analogous circuit.

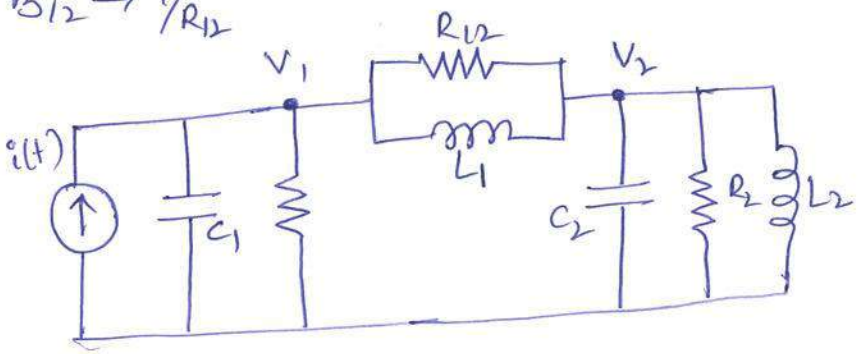
The electrical analogous elements for the elements of mechanical system are given below.

$f(t) \rightarrow i(t)$ $\quad M_1 \rightarrow C_1$ $\quad B_1 \rightarrow 1/R_1$ $\quad K_1 \rightarrow 1/L_1$

$V_1 \rightarrow V_1$ $\quad M_2 \rightarrow C_2$ $\quad B_2 \rightarrow 1/R_2$ $\quad K_2 \rightarrow 1/L_2$

$V_2 \rightarrow V_2$ $\quad B_{12} \rightarrow 1/R_{12}$

The node basis equations using kirchoff's current law for the circuit



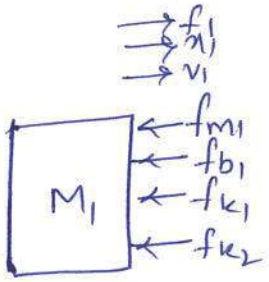
$$C_1 \frac{dv_1}{dt} + \frac{1}{R_1} v_1 + \frac{1}{R_{12}} (v_1 - v_2) + \frac{1}{L_1} \int (v_1 - v_2)\, dt = i(t)$$

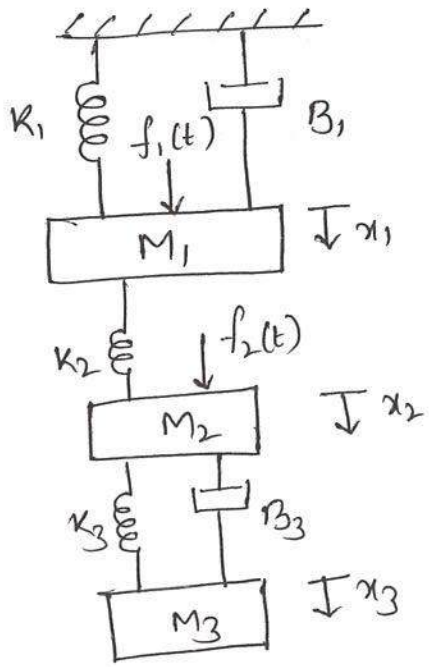
$$C_2 \frac{dv_2}{dt} + \frac{1}{R_2} v_2 + \frac{1}{L_2} \int v_2\, dt + \frac{1}{R_{12}} (v_2 - v_1) + \frac{1}{L_1} \int (v_2 - v_1)\, dt = 0.$$

Problem 2:

Write the differential equations governing the mechanical system shown in fig. Draw the force voltage and force current electrical analogous circuit and verify by writing mesh and node equations.

The free body diagram of M_1





$$f_{m_1} = M_1 \frac{d^2x_1}{dt^2}$$

$$f_{b_1} = B_1 \frac{dx_1}{dt}$$

$$f_{k_2} = k_2 (x_1 - x_2)$$

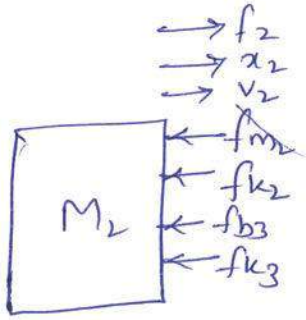
$$f_{k_1} = k_1 x_1$$

By Newton's second Law

$$f_{m_1} + f_{b_1} + f_{k_2} + f_{k_1} = f(t)$$

$$M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + k_1 x_1 + k_2 (x_1 - x_2) = f_1(t) \quad \text{—— (1)}$$

The free body diagram of M_2



$$-f_{m_2} = M_2 \frac{d^2x_2}{dt^2}$$

$$f_{b_3} = B_3 \frac{d}{dt}(x_2 - x_3)$$

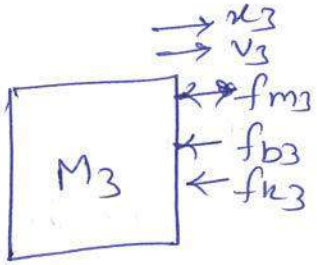
$$f_{k_2} = k_2 (x_2 - x_1)$$

$$f_{k_3} = k_3 (x_2 - x_3)$$

By Newton's second Law $f_{m_2} + f_{b_3} + f_{k_2} + f_{k_3} = f_2(t)$

$$f_2(t) = \frac{M_2 d^2x_2}{dt^2} + B_3 \frac{d}{dt}(x_2 - x_3) + k_2 (x_2 - x_1) + k_3 (x_2 - x_3) \quad \text{—— (2)}$$

The free body diagram of M_3



$$f_{m_3} = M_3 \frac{d^2x_3}{dt^2}$$

$$f_{b_3} = B_3 \frac{d}{dt}(x_3 - x_2)$$

$$f_{k_3} = k_3 (x_3 - x_2)$$

By Newton's second Law $-f_{m_3} + f_{b_3} + f_{k_3} = 0$.

$$M_3 \frac{d^2x_3}{dt^2} + B_3 \frac{d}{dt}(x_3 - x_2) + k_3 (x_3 - x_2) = 0. \quad \text{—— (3)}$$

Once replacing the displacements by velocity in the differential equations.

i.e., $\frac{d^2x}{dt^2} = \frac{dv}{dt}$; $\frac{dx}{dt} = v$ and $x = \int v\,dt$.

$$M_1 \frac{dv_1}{dt} + B_1 v_1 + k_1 \int v_1\,dt + k_2 \int (v_1 - v_2)\,dt = f_1(t)$$

$$M_2 \frac{dv_2}{dt} + B_3 (v_2 - v_3) + k_2 \int (v_2 - v_1)\,dt + k_3 \int (v_2 - v_3)\,dt = f_2(t)$$

$$M_3 \frac{dv_3}{dt} + B_3 (v_3 - v_2) + k_3 \int (v_3 - v_2)\,dt = 0.$$

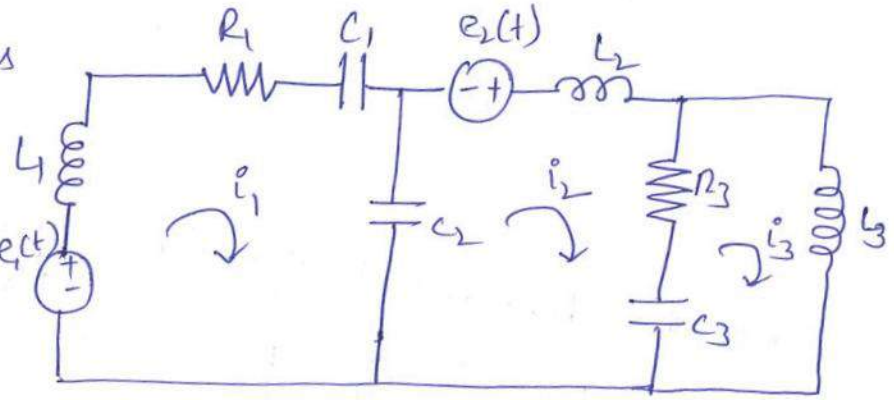
Force - Voltage Analogous Circuit:

$f_1(t) \rightarrow e_1(t)$
$f_2(t) \rightarrow e_2(t)$
$V_1 \rightarrow i_1$
$V_2 \rightarrow i_2$
$V_3 \rightarrow i_3$

$M_1 \rightarrow L_1$
$M_2 \rightarrow L_2$
$M_3 \rightarrow L_3$
$B_1 \rightarrow R_1$
$B_3 \rightarrow R_3$

$k_1 \rightarrow 1/C_1$
$k_2 \rightarrow 1/C_2$
$k_3 \rightarrow 1/C_3$.

The basis equations Using Kirchoff's voltage Law for the Circuit:-



$$L_1 \frac{di_1}{dt} + i_1 R_1 + \frac{1}{C_1} \int i_1\,dt + \frac{1}{C_2} \int (i_1 - i_2)\,dt = e_1(t)$$

$$L_2 \frac{di_2}{dt} + R_3 (i_2 - i_3) + \frac{1}{C_3} \int (i_2 - i_3)\,dt + \frac{1}{C_2} \int (i_2 - i_1)\,dt = e_2(t)$$

$$L_3 \frac{di_3}{dt} + R_3 (i_3 - i_2) + \frac{1}{C_3} \int (i_3 - i_2)\,dt = 0.$$

Force - Current Analogous Circuit:

$f_1(t) \rightarrow i_1(t)$

$f_2(t) \rightarrow i_2(t)$

$V_1 \rightarrow V_1$

$V_2 \rightarrow V_2$

$V_3 \rightarrow V_3$

$M_1 \rightarrow C_1$

$M_2 \rightarrow C_2$

$M_3 \rightarrow C_3$

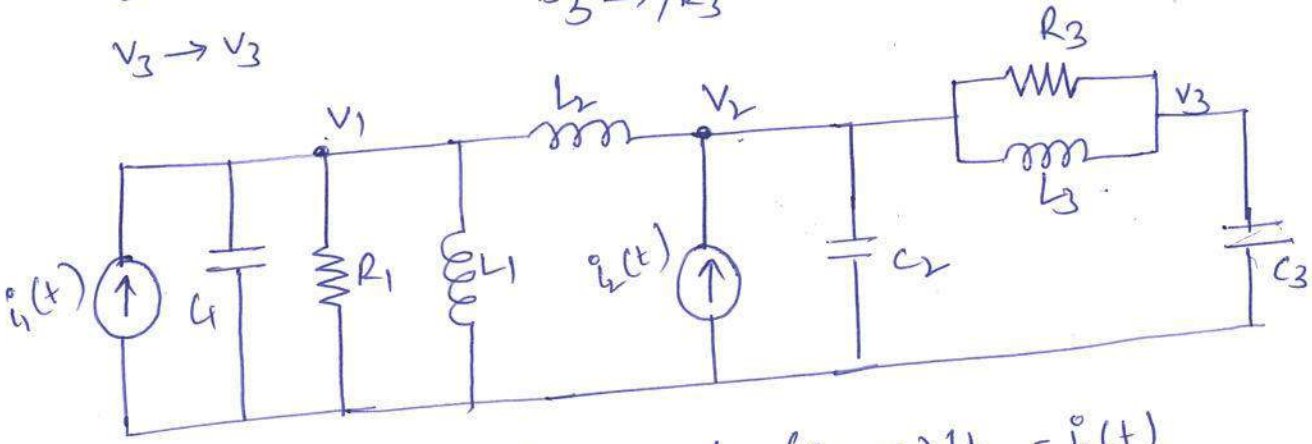
$B_1 \rightarrow 1/R_1$

$B_3 \rightarrow 1/R_3$

$K_1 \rightarrow 1/L_1$

$K_2 \rightarrow 1/L_2$

$K_3 \rightarrow 1/L_3$



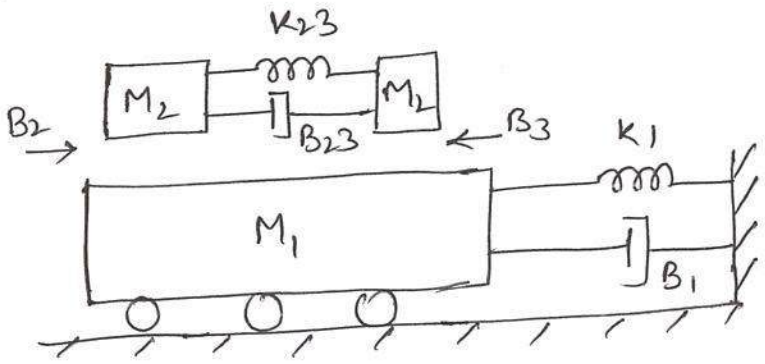
$$C_1\frac{dv_1}{dt} + \frac{1}{R_1}v_1 + \frac{1}{L_1}\int v_1\,dt + \frac{1}{L_2}\int (v_1 - v_2)\,dt = i_1(t)$$

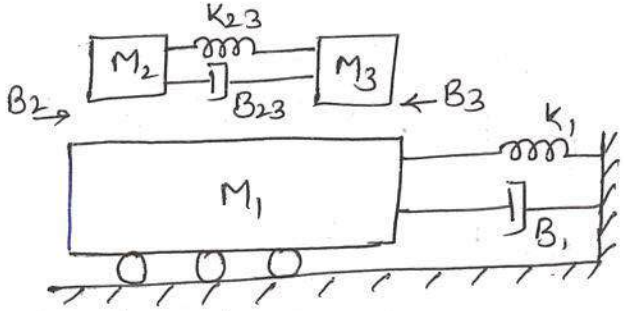
$$C_2\frac{dv_2}{dt} + \frac{1}{R_3}(v_2 - v_3) + \frac{1}{L_3}\int (v_2 - v_3)\,dt + \frac{1}{L_2}\int (v_2 - v_1)\,dt = i_2(t)$$

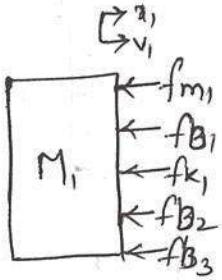
$$C_3\frac{dv_3}{dt} + \frac{1}{R_3}(v_3 - v_2) + \frac{1}{L_3}\int (v_3 - v_2)\,dt = 0.$$

Problem 3:

Write the differential equations governing the mechanical System shown in fig. Draw the force voltage and force Current electrical analogous Circuits and verify by writing mesh and node equations.



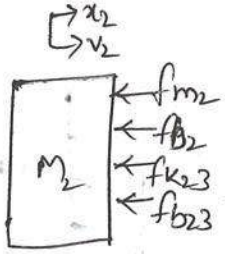




$$f_{m_1} + f_{b_1} + f_{b_2} + f_{b_3} + f_{k_1} = 0$$

$$M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B_2\left(\frac{d(x_1 + x_2)}{dt}\right) + B_3\left(\frac{d}{dt}(x_1 - x_3)\right) = 0$$

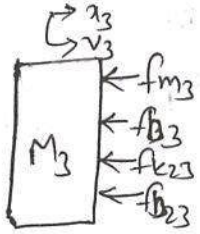
$$M_1 \frac{dv}{dt} + B_1 v_1 + B_2 (v_1 - v_2) + B_3 (v_1 - v_3) = 0 \quad \text{— (1)}$$



$$f_{m_2} + f_{b_2} + f_{k_{23}} + f_{b_{23}} = 0$$

$$M_2 \frac{d^2x_2}{dt^2} + B_2 \frac{d(x_2 - x_1)}{dt} + K_{23}(x_2 - x_3) + B_{23} \frac{d}{dt}(x_2 - x_3) = 0$$

$$M_2 \frac{dv_2}{dt} + B_2 (v_2 - v_1) + K_{23} \int (v_2 - v_3) + B_{23} (v_2 - v_3) = 0 \quad \text{— (2)}$$

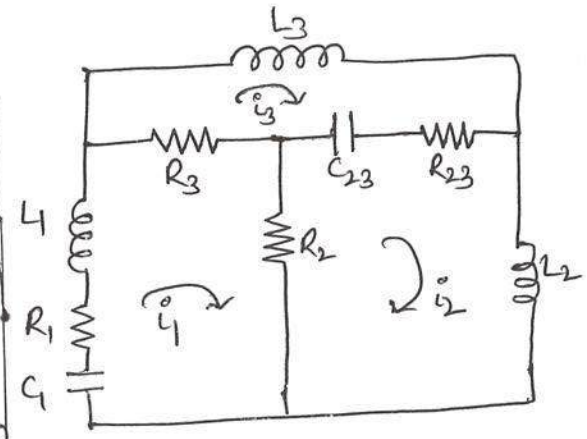


$$f_{m_3} + f_{B_3} + f_{k_{23}} + f_{b_{23}} = 0$$

$$M_3 \frac{d^2x_3}{dt^2} + B_3 \frac{d}{dt}(x_3 - x_1) + K_{23}(x_3 - x_2) + B_{23} \frac{d}{dt}(x_3 - x_2) = 0 \quad \text{— (3)}$$

$$M_3 \frac{dv_3}{dt} + B_3 \frac{d}{dt}(v_3 - v_1) + K_{23} \int (v_3 - v_2) + B_{23}(v_3 - v_2)$$

Force voltage

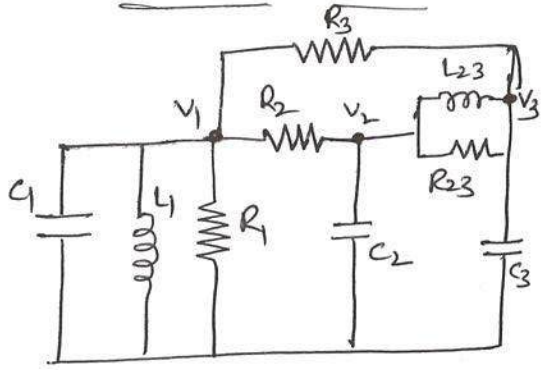


$$L_1 \frac{di_1}{dt} + i_1 R_1 + \frac{1}{C_1} \int i_1 dt + R_2 (i_1 - i_2) + R_3 (i_1 - i_3) = 0 \quad \text{— (1)}$$

$$R_2 (i_2 - i_1) + \frac{1}{C_{23}} \int (i_2 dt - i_3) dt + R_{23} (i_2 - i_3) + L_2 \frac{di_2}{dt} = 0 \quad \text{— (2)}$$

$$R_3 (i_3 - i_1) + L_3 \frac{di_3}{dt} + \frac{1}{C_{23}} \int (i_3 - i_2) dt + R_{23} (i_3 - i_2) = 0 \quad \text{— (3)}$$

Force - Current

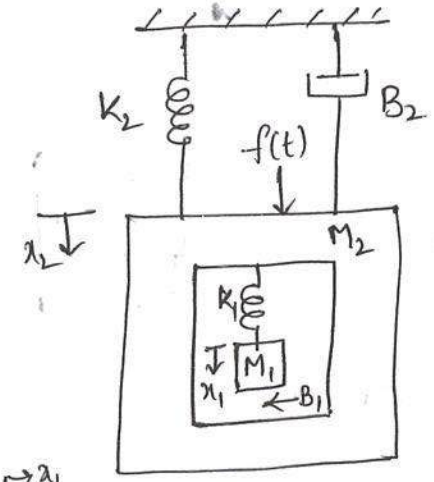


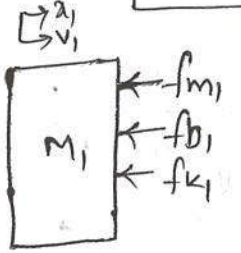
$$C_1 \frac{dv_1}{dt} + \frac{1}{L_1} \int v_1 dt + \frac{v_1}{R_1} + \frac{(v_1 - v_2)}{R_2} + \frac{(v_1 - v_3)}{R_3} = 0 \quad \text{— (1)}$$

$$C_2 \frac{dv_2}{dt} + \frac{(v_2 - v_1)}{R_2} + \frac{1}{L_{23}} \int v_2 - v_3 \, dt + \frac{(v_2 - v_3)}{R_{23}} = 0 \quad \text{— (2)}$$

$$\frac{1}{L_{23}} \int (v_3 - v_2) dt + \frac{(v_3 - v_2)}{R_{23}} + \frac{(v_3 - v_1)}{R_3} + C_3 \frac{dv_3}{dt} = 0 \quad \text{— (3)}$$

Problem-4

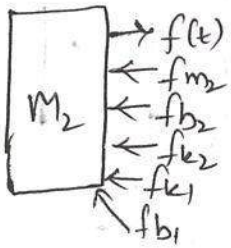




$$f_{m_1} + f_{b_1} + f_{k_1} = 0.$$

$$M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{d}{dt}(x_1 - x_2) + k_1 (x_1 - x_2) = 0 \quad \text{—— (1)}$$

$$M_1 \frac{dv_1}{dt} + B_1 (v_1 - v_2) + k_1 \int (v_1 - v_2) = 0$$

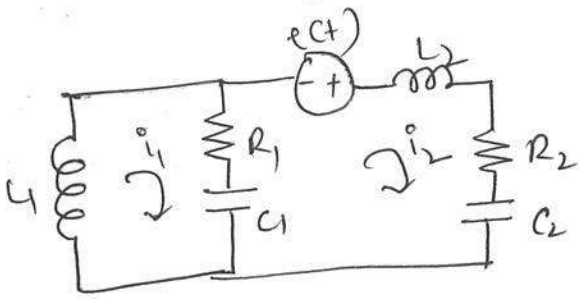


$$f_{m_2} + f_{b_2} + f_{k_2} + f_{k_1} + f_{b_1} = f(t)$$

$$= M_2 \frac{d^2x_2}{dt^2} + B_2 \frac{dx_2}{dt} + k_2 x_2 + k_1 (x_2 - x_1) + B_1 \frac{d}{dt}(x_2 - x_1)$$

$$f(t) = M_2 \frac{dv_2}{dt} + B_2 v_2 + k_2 \int v_2 dt + k_1 \int (v_2 - v_1) dt + B_1 (v_2 - v_1) = 0 \quad \text{—— (2)}$$

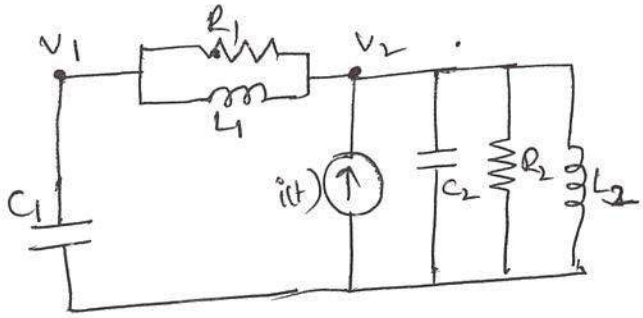
Force - voltage



$$L_1 \frac{di_1}{dt} + (i_1 - i_2) R_1 + \frac{1}{C_1} \int (i_1 - i_2) dt = 0 \quad \text{—— (1)}$$

$$e(t) = (i_2 - i_1) R_1 + \frac{1}{C_1} \int (i_2 - i_1) + L_2 \frac{di_2}{dt} + i_2 R_2 + \frac{1}{C_2} \int i_2 dt \quad \text{—— (2)}$$

Force - Current



$$C_1 \frac{dv_1}{dt} + \frac{(v_1 - v_2)}{R_1} + \frac{1}{L_1} \int (v_2 - v_1) dt = 0 \quad \text{—— (1)}$$

$$i(t) = C_2 \frac{dv_2}{dt} + \frac{v_2}{R_2} + \frac{1}{L_2} \int (v_2 - v_1) dt + \frac{1}{L} \int (v_2 - v_1) + \frac{(v_2 - v_1)}{R_1} \quad \text{—— (2)}$$

Electrical Analogous of Mechanical Rotational Systems.

Torque Voltage Analogy:

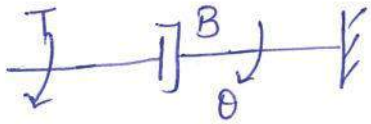
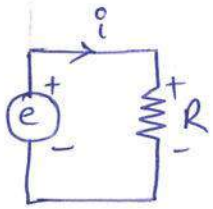
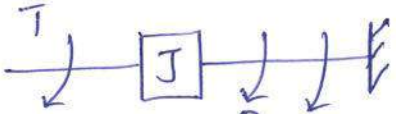
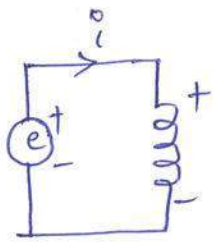
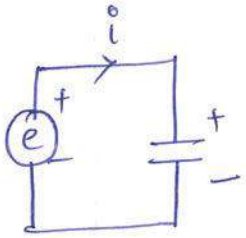
Mechanical Rotational System	Electrical System.
Input: Torque Output: Angular velocity	Input: voltage source Output: Current through the element
 $T = B\frac{d\theta}{dt} = B\omega$ $\omega = \frac{d\theta}{dt}$	 $e = iR$
 $T = J\frac{d^2\theta}{dt} = J\frac{d\omega}{dt}$	 $e = L\frac{di}{dt}$
  $T = K\theta$ $= K\int \omega\, dt$	 $e = \frac{1}{C}\int i\, dt$.

Table: The Torque balance equations of mechanical rotational elements and their analogous electrical elements in torque - voltage Analogy.

Item	Mechanical rotational System	Electrical System (mesh basis system)
Independent Variable (input)	Torque, T	Voltage, e
Dependent Variable (output)	Angular velocity, ω Angular displacement, θ	Current, i Current, q
Dissipative Element	Rotational Coefficient of dashpot, B	Resistance, R
Storage element	Moment of Inertia, J stiffness of Spring, K	Inductance, L Inverse of Capacitance, 1/c.
physical Law	Newton's Second Law $\Sigma T = 0$	Kirchoff's voltage Law $\Sigma V = 0$.
: Changing the Level of Independent Variable.	Gear $\frac{T_1}{T_2} = \frac{n_1}{n_2}$	Transformer $\frac{e_1}{e_2} = \frac{N_1}{N_2}$

Guidelines to obtain electrical analogous of mechanical rotational systems:

Torque voltage Analogy:

1. In electrical systems the elements in series will have same current, like wise in mechanical systems, the elements having same angular velocity are said to be in series.

2. The elements having same angular velocity in mechanical system should have analogous same current in electrical analogous system.

3. each node (meeting point of elements) in mechanical system corresponds to a closed loop in electrical system. The moment of Inertia of mass is considered as node.

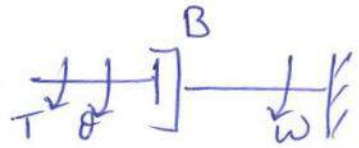
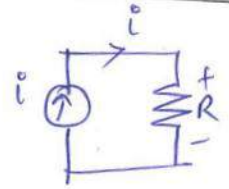
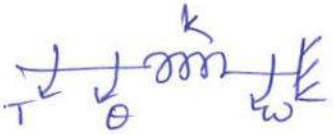
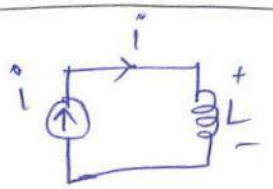
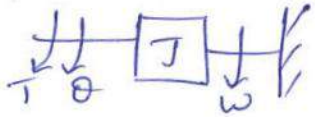
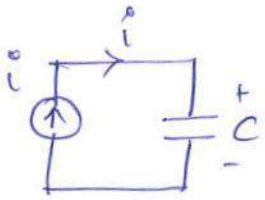
4. The number of meshes in electrical analogous is same as that of the number of nodes (moment of inertia of mass) in mechanical system. Hence the number of mesh currents and system equations will be same as that of the number of angular velocities of node (moment of inertia of mass) in mechanical system.

5. The Mechanical driving sources (Torque) and passive elements connected to the node (moment of inertia of mass) in mechanical system should be represented by

analogous element in a closed loop in analogous electrical System.

6. The element Connected between two nodes (moment of inertia) in mechanical System is represented as Common element between two meshes in electrical analogous System.

Torque Current Analogy

Mechanical Rotational System	Electrical System
I/P : Torque O/P : Angular velocity	I/P : current source O/P : voltage across the element.
 $T = B\frac{d\theta}{dt} = B\omega$	 $i = \frac{V}{R}$
	 $i = \frac{1}{L}\int v\,dt$.
 $T = J\frac{d^2\theta}{dt^2}$ $= J\frac{d\omega}{dt}$	 $i = C\frac{dv}{dt}$.

Guide lines to obtain electrical analogous of mechanical rotation system:

Torque - Current Analogy:

1. In electrical systems the elements in parallel will have same voltage, Like wise in mechanical systems the elements having same torque are said to be in parallel.

2. The elements having same angular velocity in mechanical system should have analogous same voltage in electrical analogous system.

3. Each node (meeting point of elements) in the mechanical system corresponds to a node in electrical system. The moment of inertia of mass is considered as a node.

4. The number of nodes in electrical analogous is same as that of the number of nodes (moment of inertia of mass) in mechanical system. Hence the number of node voltage and system equations will be same as that of the number of angular velocities of nodes (moment of inertia of mass) in mechanical system.

5. The mechanical driving sources (Torque) and passive elements connected to the node in mechanical system should be represented by analogous element connected

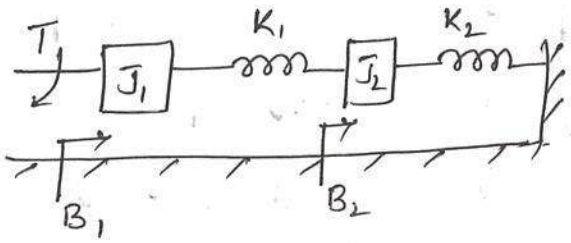
to a node in analogous electrical system.

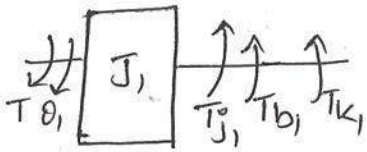
6. The element connected between two nodes in mechanical system is represented as common element b/w the two nodes in electrical analogous system.

Item	Mechanical rotation system	Electrical system (node basis)
Independent Variable (I/P)	Torque, T	Current, i
Dependent Variable (O/P)	Angular velocity, ω Angular displacement, θ	Voltage v Flux, ϕ
Dissipative element	Rotational frictional Coefficient of dashpot, B	Conductance $G=1/R$
Storage element	Moment of inertia, J Stiffness of spring, k	Capacitance, C Inverse of inductance, $1/L$.
Physical Law	Newton's Second Law $\Sigma T=0$	Kirchoff's current Law $\Sigma i=0$.
Changing the Level of independent Variable	Gear $\frac{T_1}{T_2}=\frac{n_1}{n_2}$	Transformer $\frac{i_1}{i_2}=\frac{N_2}{N_1}$

Table: List of the analogy quantities in torque-current analogy.

Mechanical Rotational System





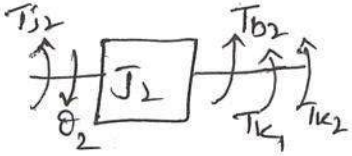
$$T = T_{j_1} + T_{b_1} + T_{k_1}$$

$$= J_1 \frac{d^2\theta_1}{dt^2} + B_1 \frac{d\theta}{dt} + k_1(\theta_1 - \theta_2)$$

$$= J_1 \frac{d\omega}{dt} + B_1 \omega + k \int (\omega_1 - \omega_2) dt \quad \text{—— (1)}$$

$\omega \rightarrow$ Angular velocity.

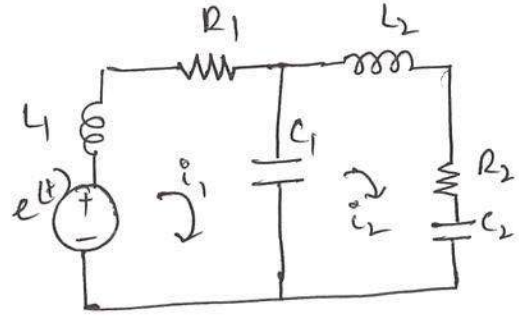
$$T_{j_2} + T_{b_2} + T_{k_1} + T_{k_2} = 0$$



$$J_2 \frac{d^2\theta_2}{dt^2} + B_2 \frac{d\theta_2}{dt} + k_1(\theta_2 - \theta_1) + k_2(\theta_2) = 0.$$

$$J_2 \frac{d\omega}{dt} + B_2 \frac{d\omega}{dt} + k \int (\omega_2 - \omega_1) dt + k_2 \omega_2 = 0. \quad \text{—— (2)}$$

Force-voltage Analog



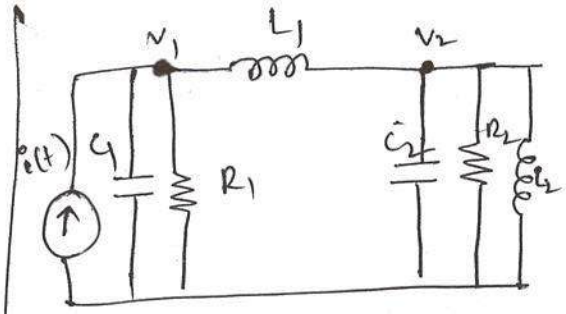
Loop 1

$$e(t) = L_1 \frac{di_1}{dt} + i_1 R_1 + \frac{1}{C_1} \int (i_1 - i_2) dt \quad \text{—— (1)}$$

Loop 2

$$\frac{1}{C_1} \int (i_2 - i_1) dt + L_2 \frac{di_2}{dt} + i_2 R_2 + \frac{1}{C_2} \int i_2 dt = 0 \quad \text{—— (2)}$$

Force current Analogy



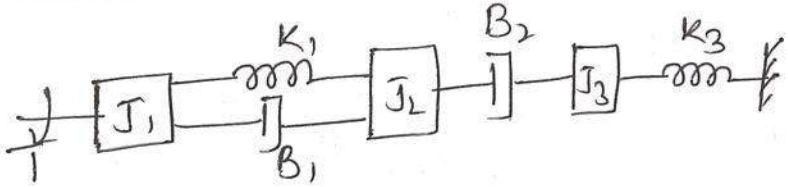
At Node V_1:

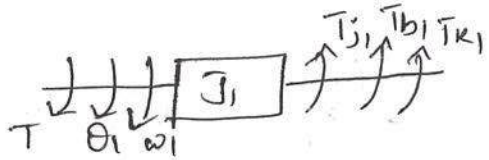
$$i(t) = C_1 \frac{dv_1}{dt} + \frac{v_1}{R_1} + \frac{1}{L_1} \int (v_1 - v_2) dt \quad \text{—— (1)}$$

At Node V_2

$$0 = C_2 \frac{dv_2}{dt} + \frac{1}{L_1} \int (v_2 - v_1) dt + \frac{v_2}{R_2} + \frac{1}{L_2} \int v_2 dt \quad \text{—— (2)}$$

Problem 2:

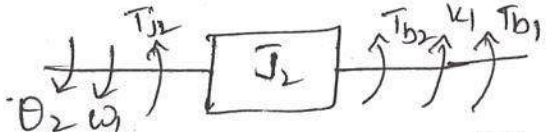




$$T = T_{J_1} + T_{b_1} + T_{K_1}$$

$$= J_1 \frac{d^2\theta_1}{dt^2} + B_1 \frac{d(\theta_1-\theta_2)}{dt} + k_1(\theta_1-\theta_2)$$

$$= T\frac{d\omega_1}{dt} + B_1(\omega_1-\omega_2) + k_1\int(\omega_1-\omega_2)dt \quad \text{—— (1)}$$

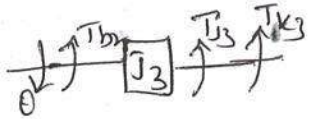


$$0 = T_{J_2} + T_{b_2} + T_{K_1} + T_{b_1}$$

$$= J_2 \frac{d^2\theta_2}{dt^2} + B_2\frac{d(\theta_2-\theta_3)}{dt} + B_1\frac{d}{dt}(\theta_2-\theta_1) + k_1(\theta_2-\theta_1)$$

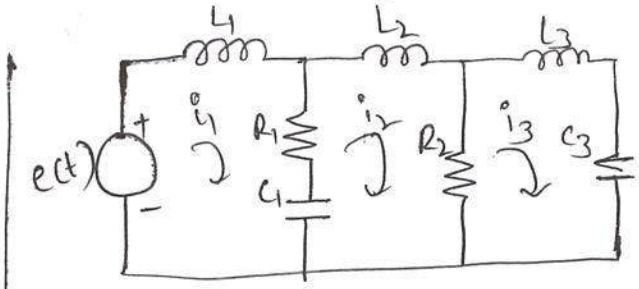
$$0 = J_2\frac{d^2\omega_2}{dt} + B_2\frac{d}{dt}(\omega_2-\omega_3) + B_1(\omega_2-\omega_1) + k_1\int(\omega_2-\omega_1) \quad \text{(2)}$$

$$T_{b_2} + T_{J_3} + T_{K_3} = 0$$



$$B_2(\omega_3-\omega_1) + J_3\frac{d\omega_3}{dt} + k_3\omega_3 = 0 \quad \text{—— (3)}$$

Torque - voltage Analogy



Mesh 1

$$e(t) = L_1\frac{di_1}{dt} + (i_1-i_2)R_1 + \frac{1}{C_1}\int(i_1-i_2)dt \quad \text{—— (1)}$$

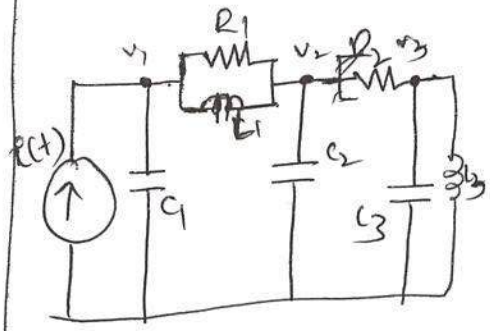
Mesh 2

$$0 = C_1\int(i_2-i_1)dt + (i_2-i_1)R_1 + L_2\frac{di_2}{dt} + (i_2-i_3)R_2 \quad \text{—— (2)}$$

Mesh 3

$$0 = (i_3-i_2)R_2 + L_3\frac{di_3}{dt} + C_3\int i_3\,dt \quad \text{—— (3)}$$

Torque current Analogy :



At node-1

$$e(t) = C_1\frac{dv_1}{dt} + \frac{(v_1-v_2)}{R_1} + \frac{1}{L}\int(v_1-v_2)dt \quad \text{—— (1)}$$

node 2

$$0 = C_2\frac{dv_2}{dt} + \frac{(v_2-v_3)}{R_2} + C_3\frac{dv_2}{dt}\frac{v_2-v_1}{R_1} + \frac{1}{L_1}\int(v_2-v_1)dt$$

node 3

$$0 = \frac{v_3-v_2}{R_2} + C_3\frac{dv_3}{dt} + \frac{1}{L_3}\int v_3\,dt$$

Transfer function of Armature Controlled DC Motor

The speed of DC motor is directly proportional to armature voltage and inversly proportional to flux in field winding. In armature controlled DC motor the desired speed is obtained by varying the armature voltage. This speed control system is an electromechanical control system. The electrical system consists of the armature and the field circuit but for analysis purpose, only the armature circuit considered because the field is excited by a constant voltage.

The mechanical system consists of the rotating part of the motor and load connected to the shaft the motor.

The armature controlled DC motor speed control system shown in fig.

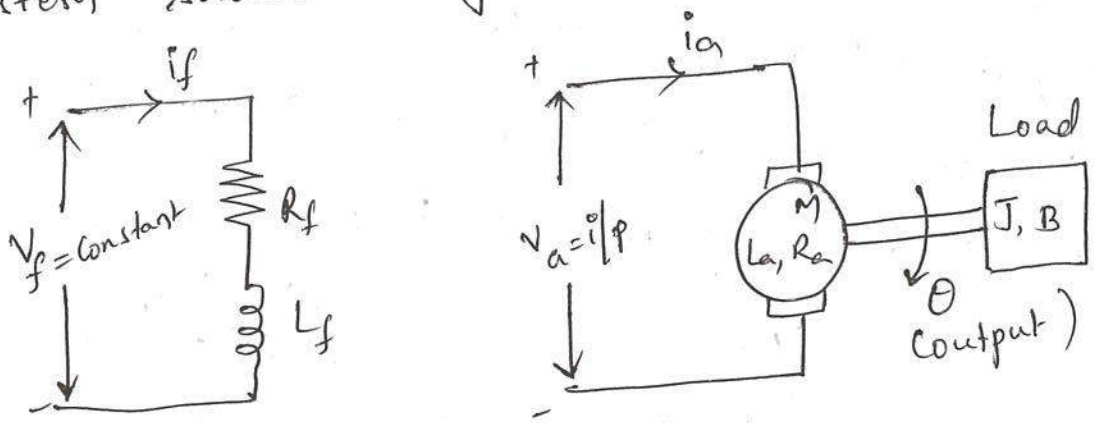


fig: Armature Controlled DC motor

Let R_a = Armature Resistance, Ω

L_a = Armature inductance, H

i_a = Armature Current, A

V_a = Armature voltage, V

e_b = Back emf, V

K_t = Torque constant, N-m/A

T = Torque developed by motor, N-m

θ = Angular displacement of shaft, rad

J = moment of inertia of motor and load kg-m²/rad

B = Frictional Coefficient of motor and load, N-m (rad/sec)

K_b = Back emf Constant, V (rad/sec).

The Equivalent Circuit of armature is shown in fig1

By Kirchoff's voltage law

$$i_a R_a + L_a \frac{di_a}{dt} + e_b = V_a$$

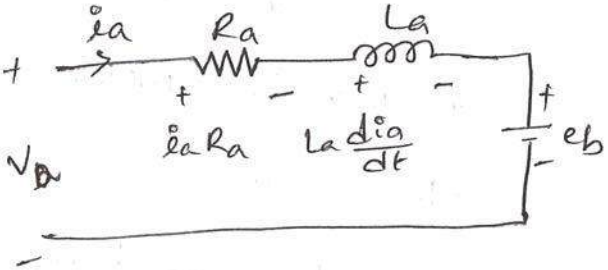


fig1. Equivalent circuit of armature.

Torque of DC motor is proportional to the product of flux and current. Since flux is Constant in this System, the torque is proportional to i_a alone.

$$T \propto i_a$$

$$\therefore T = K_t i_a. \quad \text{---(1)}$$

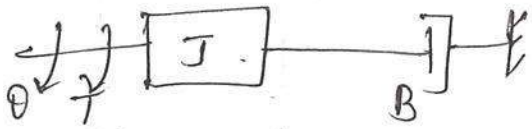


fig2:

The mechanical System of the motor shown in fig 2.

The differential equation governing the mechanical System of motor is given by

$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T \quad \text{——(2)}$$

The back emf of DC machine is proportional to speed (angular velocity) of shaft.

$$\therefore e_b \propto \frac{d\theta}{dt}$$

$$\text{Back emf } e_b = k_b \frac{d\theta}{dt}$$

The Laplace Transform of various time domain Signals involved in this System are.

$L[v_a] = V_a(s)$ $\qquad$ $L[i_a] = I_a(s)$

$L[e_b] = E_b(s)$ $\qquad$ $L[\theta] = \theta(s)$

$L[T] = T(s)$

The differential equations governing the armature Controlled DC motor Speed Control system and L.T of that are:

$$i_a R_a + L_a\frac{di_a}{dt} + e_b = v_a \Rightarrow I_a(s)R_a + L_a s I_a(s) + E_b(s) = V_a(s) \quad \text{——(3)}$$

$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T \Rightarrow Js^2\theta(s) + Bs\theta(s) = T(s) \quad \text{——(4)}$$

$$T = k_t i_a \Rightarrow T(s) = k_t I_a(s) \quad \text{——(5)}$$

$$e_b = k_b\frac{d\theta}{dt} \Rightarrow E_b(s) = k_b s\theta(s) \quad \text{——(6)}$$

On equating (4) (5) eq's. we get,

$$k_t I_a(s) = (Js^2 + Bs)\theta(s)$$

$$I_a(s) = \frac{(Js^2+Bs)}{k_t}\theta(s) \quad \text{---(7)}$$

Equation (3) can be written as

$$(R_a + sL_a)\, I_a(s) + E_b(s) = V_a(s) \quad \text{---(8)}$$

Substituting for $E_b(s)$ and $I_a(s)$ from eq (6) & (7) in eq (8).

$$(R_a+sL_a)\cdot \frac{Js^2+Bs}{k_t}\theta(s) + k_b s\theta(s) = V_a(s)$$

$$\left[\frac{(R_a+sL_a)(Js^2+Bs)+k_b k_t s}{k_t}\right]\theta(s) = V_a(s)$$

The Required Transfer function is $\theta(s)/V_a(s)$

$$\therefore \frac{\theta(s)}{V_a(s)} = \frac{k_t}{(R_a+sL_a)(Js^2+Bs)+k_b k_t s} \quad \text{---(9)}$$

$$= \frac{k_t}{R_aJs^2+R_aBs+L_aJs^3+L_aBs^2+k_b k_t s}$$

$$= \frac{k_t}{s[JL_as^2+(JR_a+BL_a)s+(BR_a+k_b k_t)]}$$

$$= \frac{k_t/JL_a}{s\left[s^2+\left(\frac{JR_a+BL_a}{JL_a}\right)s+\left(\frac{BR_a+k_b k_t}{JL_a}\right)\right]} \quad \text{---(10)}$$

The Transfer function of armature controlled DC motor can be expressed in another standard form as:

$$\frac{\theta(s)}{V_a(s)} = \frac{K_t}{(R_a + sL_a)(Js^2 + Bs) + K_b K_t s}$$

$$= \frac{K_t}{R_a\left(s\frac{L_a}{R_a} + 1\right) Bs\left(1 + \frac{Js^2}{Bs}\right) + K_b K_t s}$$

$$= \frac{K_t / R_a B}{s\left[(1 + sT_a)(1 + sT_m) + \frac{K_b K_t}{R_a B}\right]}$$

Where $L_a/R_a = T_a =$ Electrical Time Constant.

and $J/B = T_m =$ Mechanical Time Constant.

Transfer function of field Controlled DC motor :

The speed of a DC motor is directly proportional to armature voltage and inversly proportional to flux. In field Controlled DC motor the armature voltage is kept constant. and the speed is varied by varying the flux of the machine. Since the flux is directly proportional to the field current, the flux is varied by varying the field current.

The speed Control System is an electromechanical Control System. The electrical System consists of armature and field Circuit but for analysis purpose, only field circuit is considered because the armature is

excited by a constant voltage. The mechanical system consists of the rotating part of the motor and the load connected to the shaft of the motor. The field controlled DC motor speed control system shown in fig.

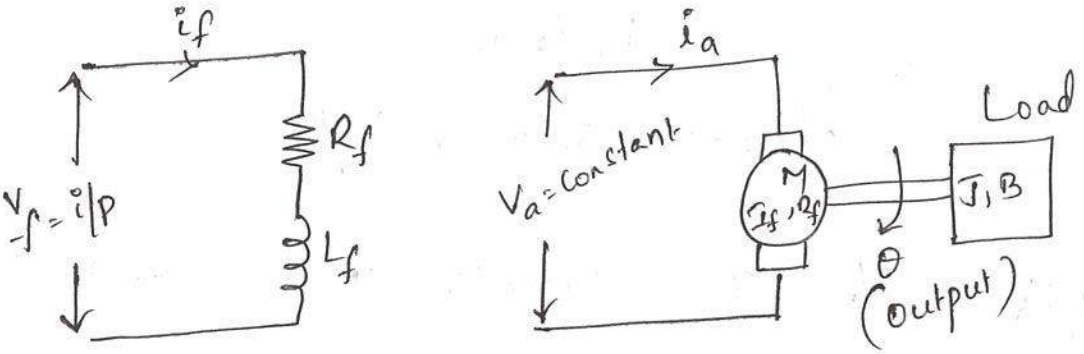


fig: field controlled DC motor.

Let R_f = field Resistance, Ω

L_f = field Inductance, H

i_f = field current, A

v_f = field voltage, V

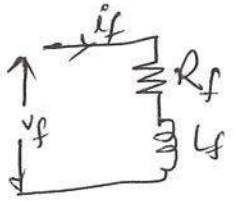
T = Torque developed by motor, N-m

k_{tf} = Torque constant, N-m/A

J = moment of inertia of motor and load kg-m²/rad.

B = frictional Coefficient of motor and load N-m/(rad/sec).

The equivalent circuit of field is shown in fig2.



By KVL.

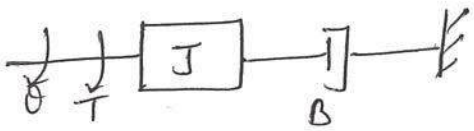
$$i_f R_f + L_f \frac{di_f}{dt} = v_f \quad \text{---} (1)$$

The torque of DC motor is proportional to product of flux and armature Current. Since armature Current is Constant in this system, the torque is proportional to flux alone, but flux is proportional to field Current.

$T \propto i_f$

$\therefore$ Torque $T = K_{tf} i_f$ —— (2)

The torque of DC motor is proportional to the product of flux and armature Current. Since armature Current is Constant in this system. The torque is proportional to the flux alone, but flux is proportional to the field current



The Mechanical System of the motor is shown in fig3. The differential equation governing the mechanical system of the motor is given by

$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T \quad \text{—— (3)}$$

The differential equations governing the field Controlled DC motor are: and L.T of that eq's are:

$R_f I_f + L_f \frac{di_f}{dt} = V_f \Rightarrow V_f(s) = R_f I_f(s) + L_f s I_f(s)$ —— (4)

$T = K_{tf} I_f \Rightarrow T(s) = K_{tf} I_f(s)$ —— (5)

$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T \Rightarrow Js^2\theta(s) + Bs\theta(s) = T(s)$ —— (6)

By equating eq's (5) & (6) we get.

$$k_{tf} \cdot I_f(s) = Js^2\theta(s) + Bs\theta(s)$$

$$I_f(s) = s\,\frac{(Js+B)}{k_{tf}}\,\theta(s) \quad \text{——— (7)}$$

The equation (4) can be written as

$$(R_f + sL_f)\,I_f(s) = V_f(s) \quad \text{——— (8)}$$

on substituting $I_f(s)$ from (7) eq in (8)

$$(R_f + sL_f)\,\frac{s(Js+B)}{k_{tf}}\,\theta(s) = V_f(s)$$

$$\frac{\theta(s)}{V_f(s)} = \frac{k_{tf}}{s(R_f+sL_f)(Js+B)}$$

$$= \frac{k_{tf}}{sR_f\left(1+\frac{sL_f}{R_f}\right)B\left(1+s\frac{J}{B}\right)}$$

$$= \frac{k_m}{s(1+sT_f)(1+sT_m)} \quad \text{——— (8)}$$

Where motor gain Constant $k_m = k_{tf}/R_f B$

field time Constant $T_f = L_f/R_f$

mechanical time Constant $T_m = J/B$.

Effect of feedback:

Transfer function:

Transfer function of a control system is the ratio of Laplace Transform of o/p to Laplace Transform of i/p.

$$\text{Transfer function} = \left.\frac{\text{LT of o/p}}{\text{LT of i/p}}\right|_{\text{with zero initial conditions.}}$$

$$T.F \Rightarrow \frac{C(s)}{R(s)}$$

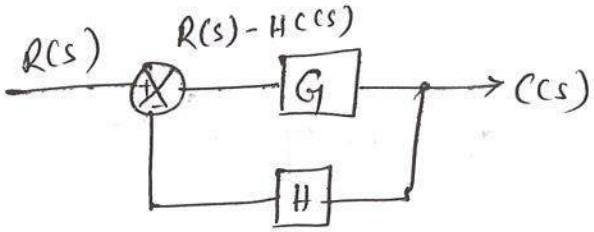
$$\Rightarrow C(s) = \left[R(s) - HC(s)\right] G$$

$$C(s) = R(s)G - HGC(s)$$

$$C(s) + GHC(s) = R(s)\,G$$

$$(1 + GH)\,C(s) = R(s)\,G$$

$$\frac{C(s)}{R(s)} = \frac{G}{1+GH} = T.F.$$



Where: G → open loop gain

H → feedback loop gain

R(s) → i/p

C(s) → o/p.

Effects can be seen due to feed back is:

1. On gain
2. On sensitivity
3. On Stability

1. Gain

The overall gain or Transfer function of the system is given by

$\frac{G}{1+GH}$ (Considered -ve feedback system)

✓ This overall gain can be controlled by the factor '$1+GH$'. When $1+GH$ is increased i.e., GH is more +ve the gain will reduced.

when $1+GH$ is decreased i.e., GH is less i.e., -ve. the gain of the system will increased.

Sensitivity: (S).

S = The ratio of percentage change in Transfer function with respect to percentage change in gain.

$$S = \frac{\%\ \text{Change in T.F}}{\%\ \text{change in G}}$$

$$= \frac{\frac{dT}{T}}{\frac{dG}{G}} \Rightarrow \frac{dT}{dG}\cdot\frac{G}{T}$$

$$\Rightarrow \frac{d}{dG}\left[\frac{G}{1+GH}\right]\cdot\frac{G}{T} \quad \left[\because T.F = \frac{G}{1+GH}\right]$$

$$\Rightarrow \frac{d}{dG}\left[\frac{(1+GH)\cdot 1 - G\cdot H}{(1+GH)^2}\right]\frac{G}{T} \quad \left[\because \frac{d}{dx}\frac{U}{V} \text{ rule} = \frac{V\,du - U\,dv}{V^2}\right]$$

$$\Rightarrow \left[\frac{1+GH-GH}{(1+GH)^2}\right]\frac{G}{T}.$$

$$\Rightarrow \left[\frac{1}{(1+GH)^2}\right]\frac{G}{T}$$

$$\left[\because T=\frac{G}{1+GH},\ \frac{G}{T}=1+GH\right]$$

$$\Rightarrow \left[\frac{1}{(1+GH)^2}\right]\left[1+GH\right]$$

$$\boxed{S_G^T = \frac{1}{1+GH}}$$

✓ When feedback path value increases (GH is +ve), Sensitivity of the system will decreases

✓ When feed back path value decreases (GH is -ve), Sensitivity of the system will increases.

Stability:

When ever o/p is in our Control then we can Say the system is Stable.

If the transfer function of the system

$$T = \frac{G}{1+GH}$$

✓ If GH is a "-1" value then the transfer function of the system is goes to "infinite".

So, overall Gain or T.F not in our control.

✓ So, stability also controlled by the feedback.

✓ Taking care of ~~stability~~ feedback so that we can get the stable system.

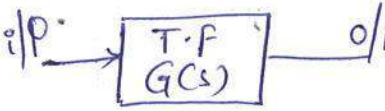
Block diagram Algebra:-

A Control System may Consists of a number of Components. A block diagram model is Commonly used to show function of an each Component. A block diagram of a system is a pictorial representation of the function performed by each Component, flow of the Signals.

The elements of a block diagram are:
Block, Branch point and Summing point.

Block:

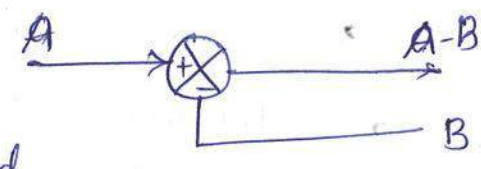
The functional block or simply block is a symbol for the mathematical operation on the i/p signal to the block that produces the o/p.



The arrow ahead pointing towards the block indicates the i/p and the arrow ahead leading away from the block represents the o/p. The o/p signal from the block is given by the product of i/p signal and transfer function in the block.

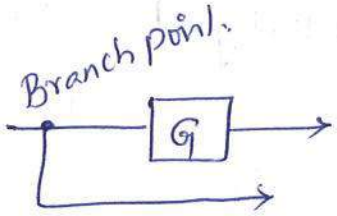
Summing point:

Summing points are used to add two or more signals in the system. (signal is added or subtracted).



Branch point:

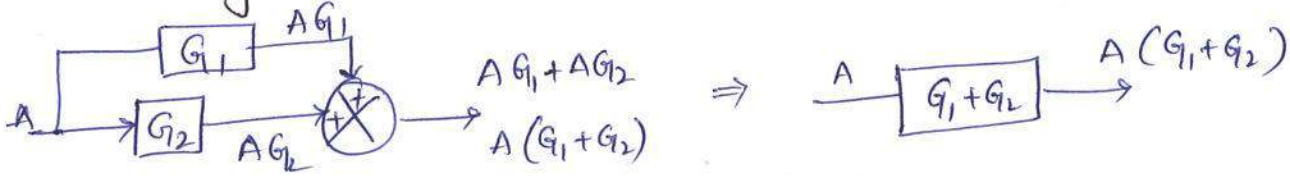
A branch point is a point from which the signal from a block goes concurrently to other blocks or summing points.



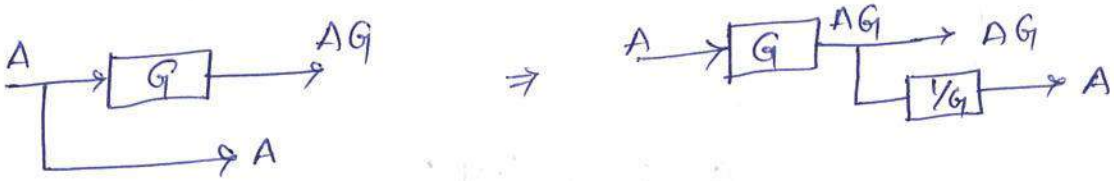
Rules of blocks diagram Algebra:

1. Combing the blocks in Cascade

2. Combing parallel blocks

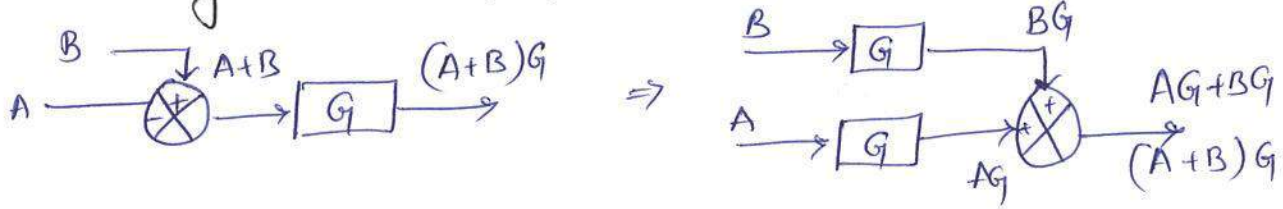


3. Moving the branch point ahead of the block

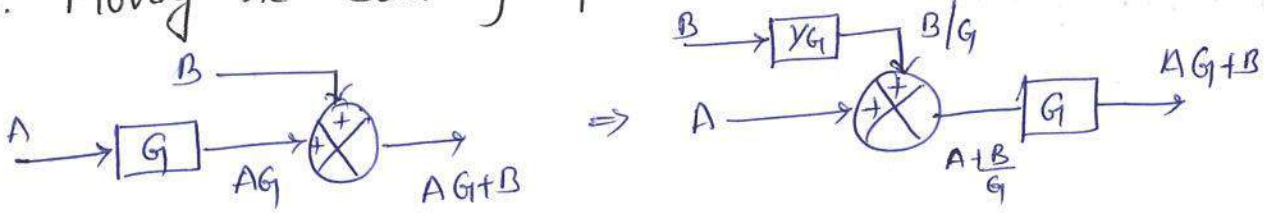


4. Moving the branch point before the block

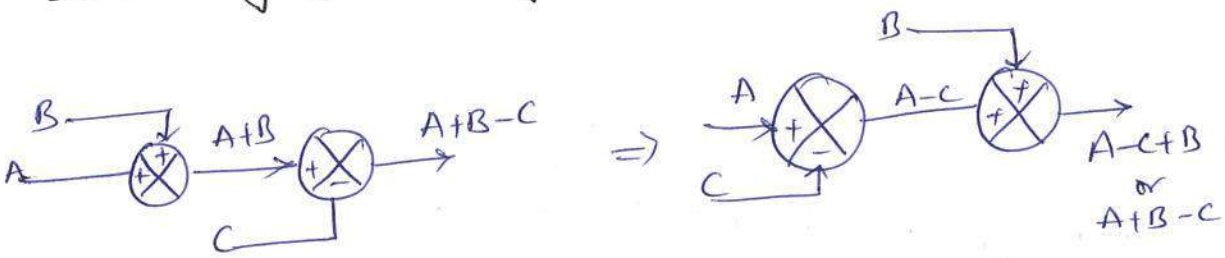
5. moving the Summing point ahead of the block



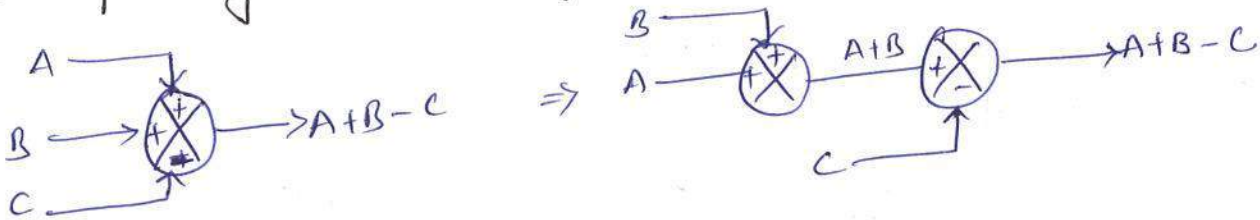
6. Moving the Summing point before the block:



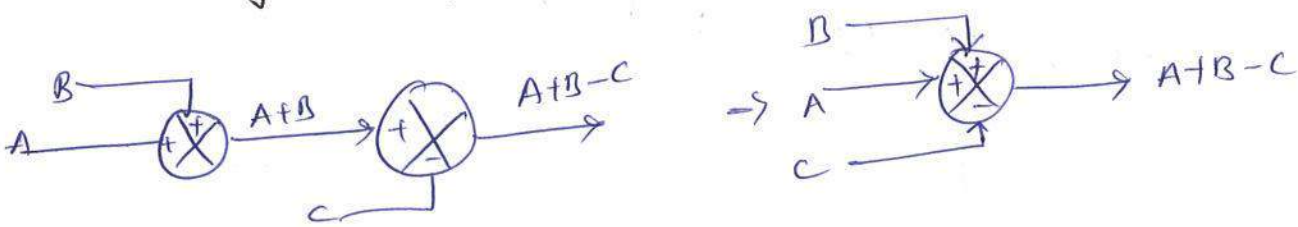
7. Interchanging Summing point:



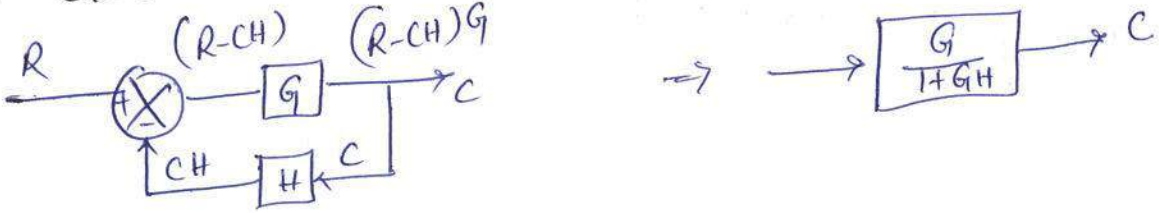
8. Splitting the Summing point:



9. Combing Summing points:



10. Elimination of feedback loop:



Proof $C = (R - CH)G$

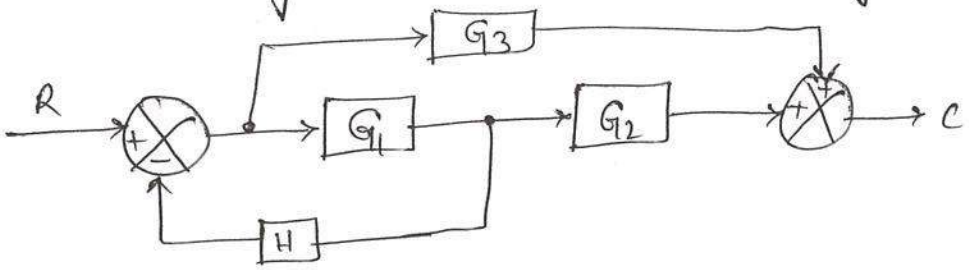
$= GR - CHG$

$C + CHG = GR$

$C(1+G) = GR \Rightarrow \boxed{\frac{C}{R} = \frac{G}{1+GH}}$

Problems :

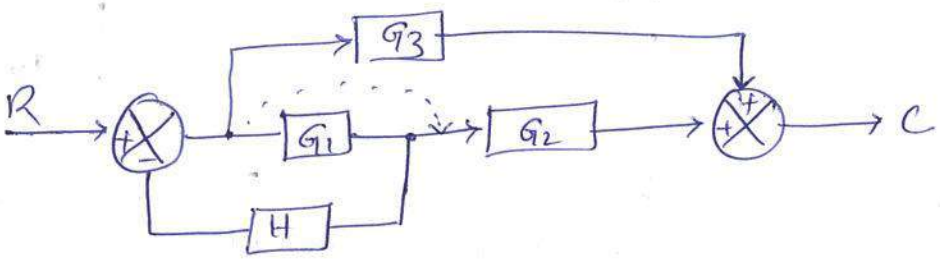
1. Reduce the block diagram shown in the fig and find C/R.



Sol:

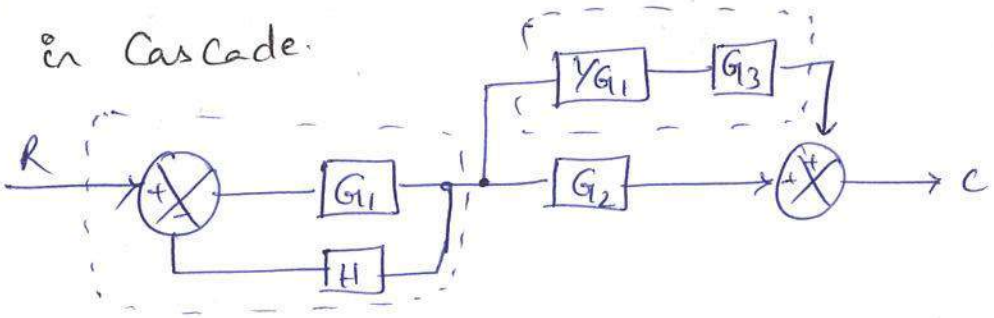
Step 1:

Move the branch point after the block



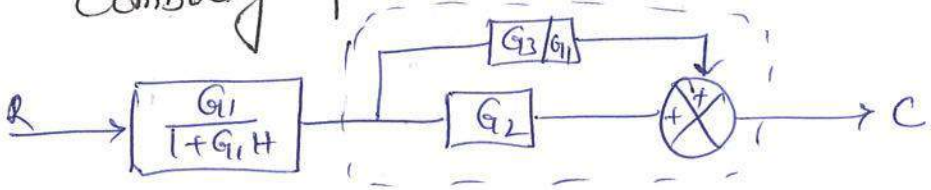
Step 2:

Eliminate the feedback path and Combining blocks in Cascade.



Step 3

Combing parallel blocks



Step 4

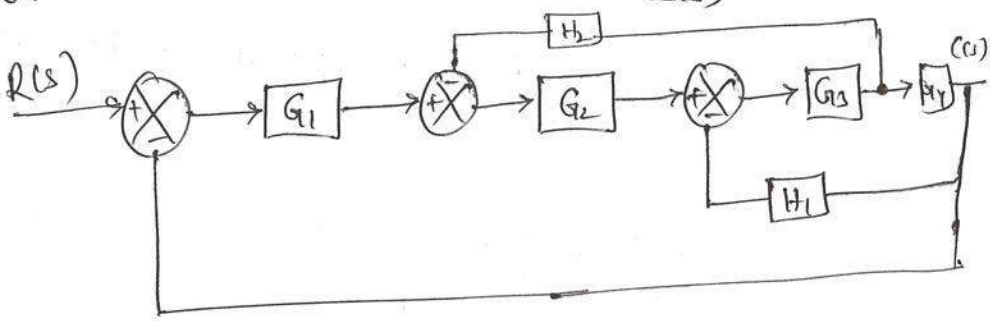
Combing the blocks in Cascade.

$$\frac{C}{R} = \left(1 + \frac{G_1}{G_1}\right)\left(\frac{G_1}{1+G_1H}\right)\left(G_2 + \frac{G_3}{G_1}\right)$$

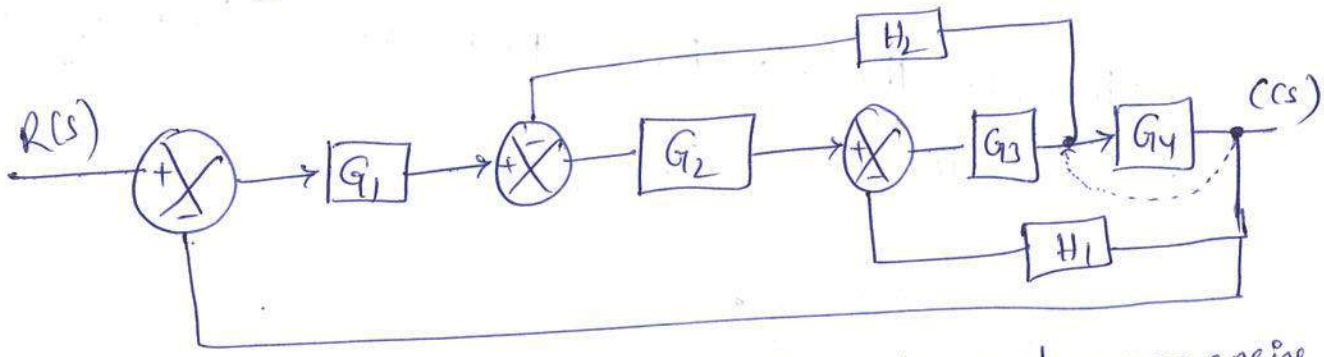
$$\boxed{\frac{C}{R} = \frac{G_1G_2 + G_3}{1+G_1H}}$$

Problem 2:

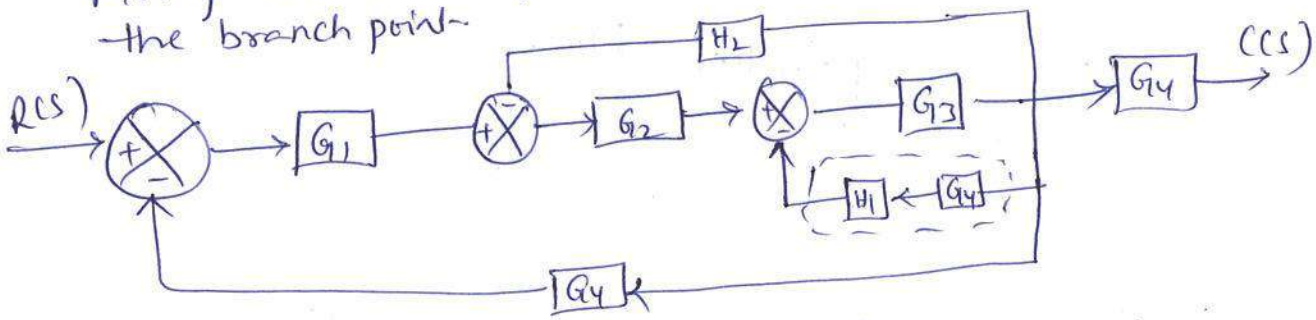
Determine the overall transfer function $\frac{C(s)}{R(s)}$ for the shown in fig.



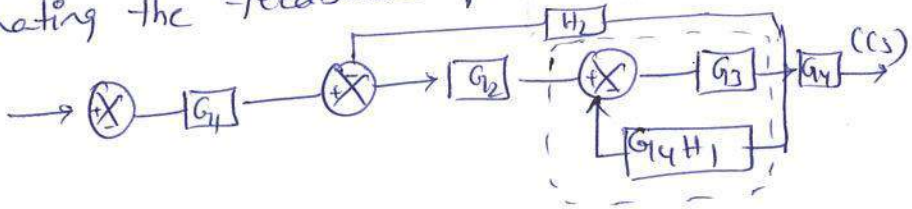
Sol: step 1 Moving the branch point before the block



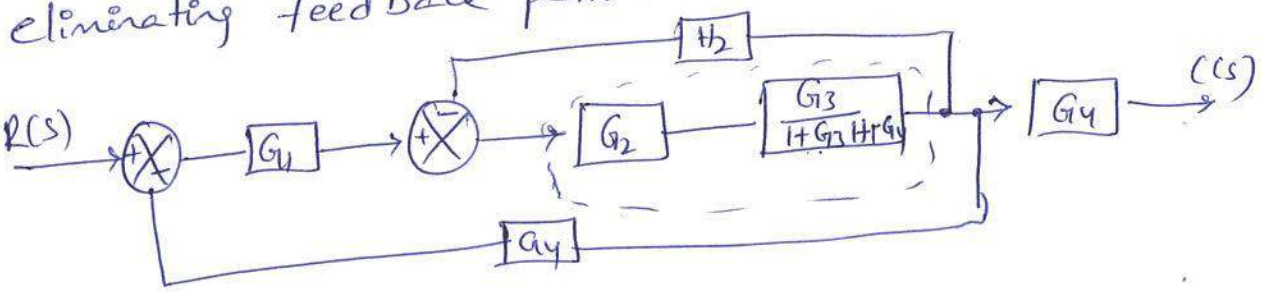
Step 2: Combining the blocks in Cascade and rearranging ~~Moving the branch point before the block~~ the branch point.



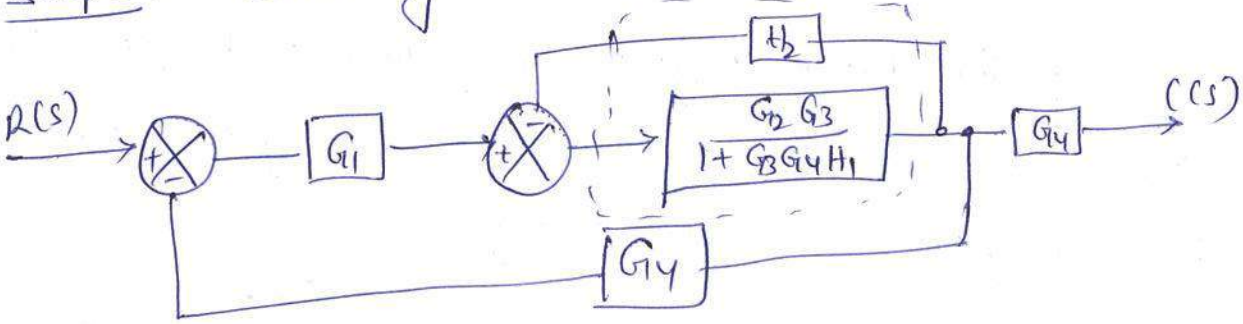
Step 3: Eliminating the feedback path.

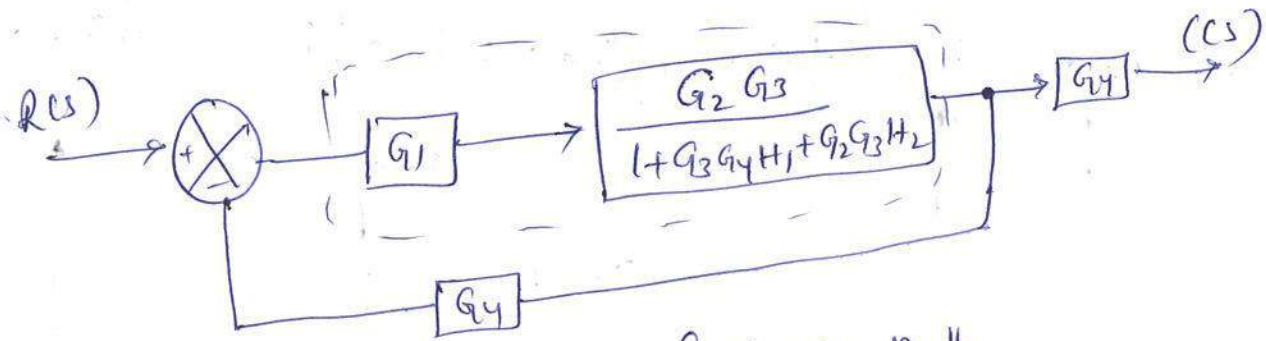


Step4: Combining the blocks in Cascade and eliminating feedback path.

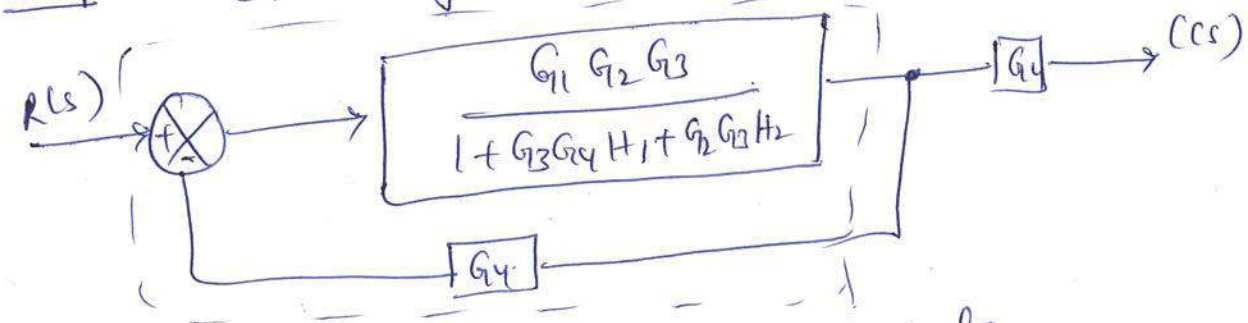


Step5: Combining the block in Cascade:

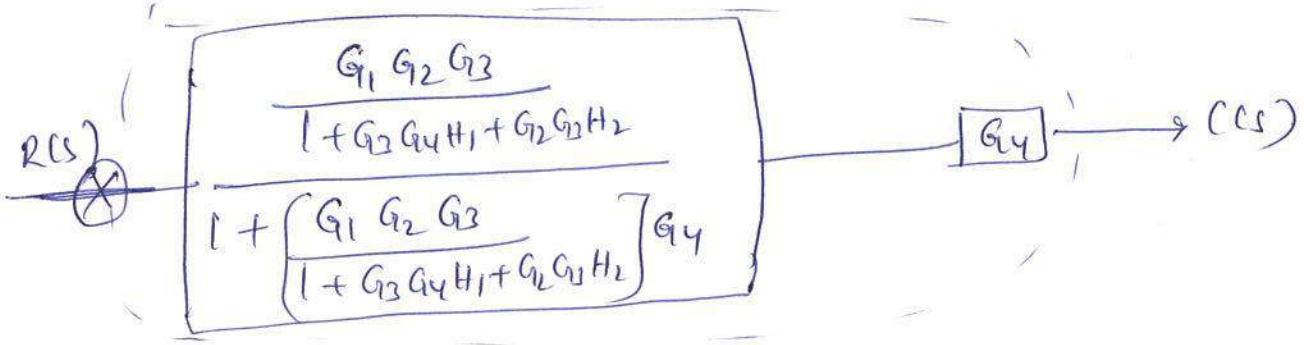


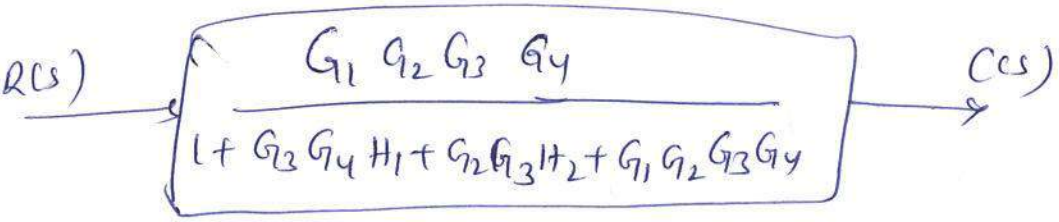


Step6: eliminating the feedback path



Step7: Combining the blocks in Cascade.

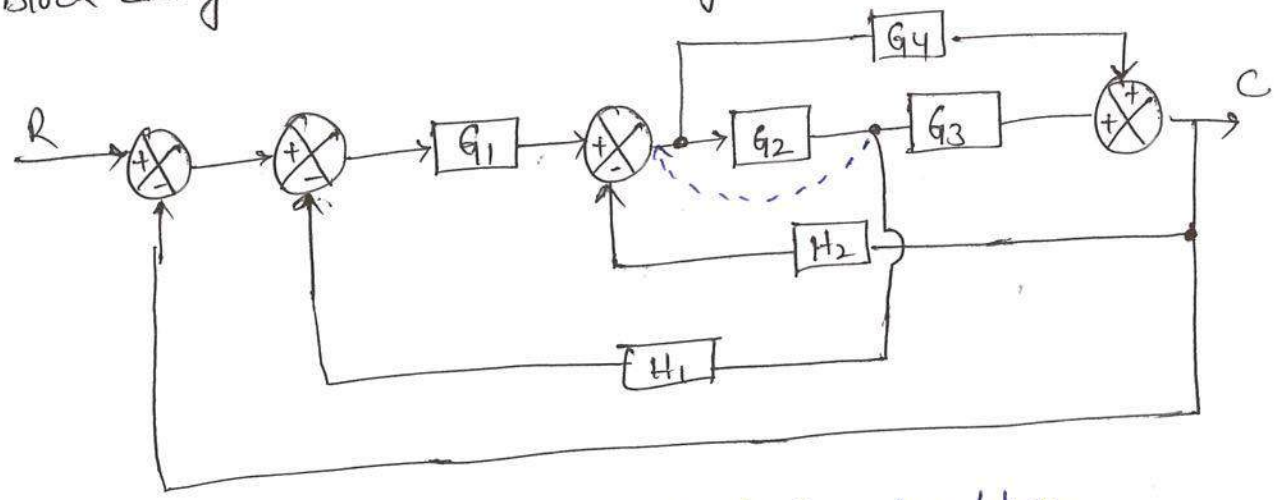




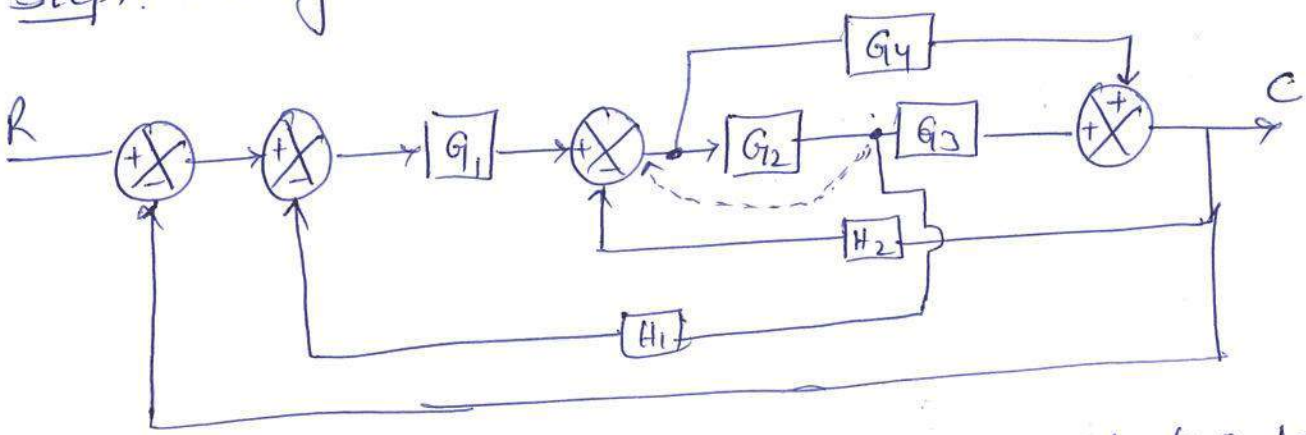
$$\text{T.F} = \frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3 G_4}{1 + G_3 G_4 H_1 + G_2 G_3 H_2 + G_1 G_2 G_3 G_4}$$

Problem 3:

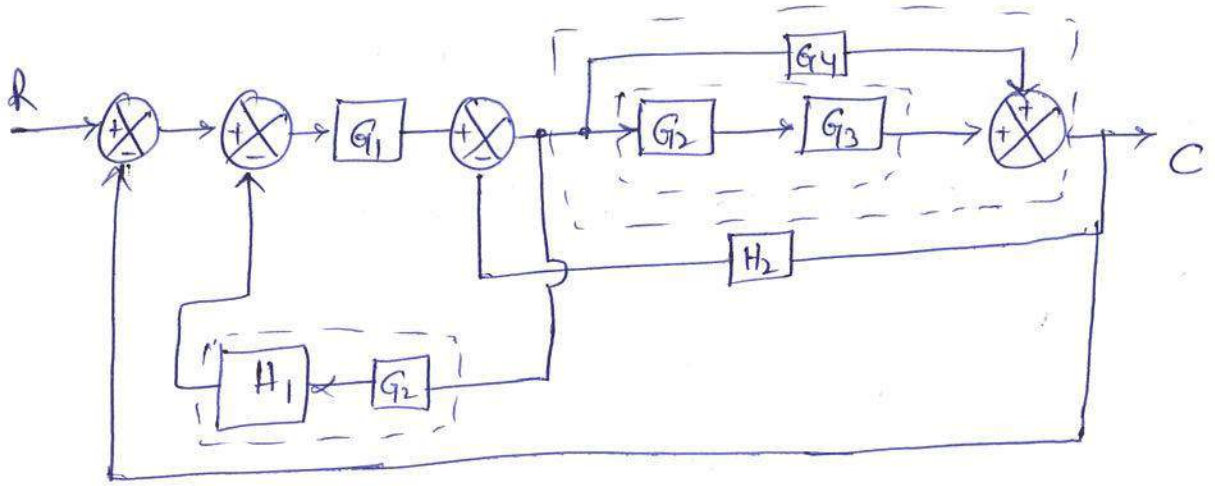
Using the block diagram reduction technique. find Closed loop transfer function of the system whose block diagram is shown in fig.



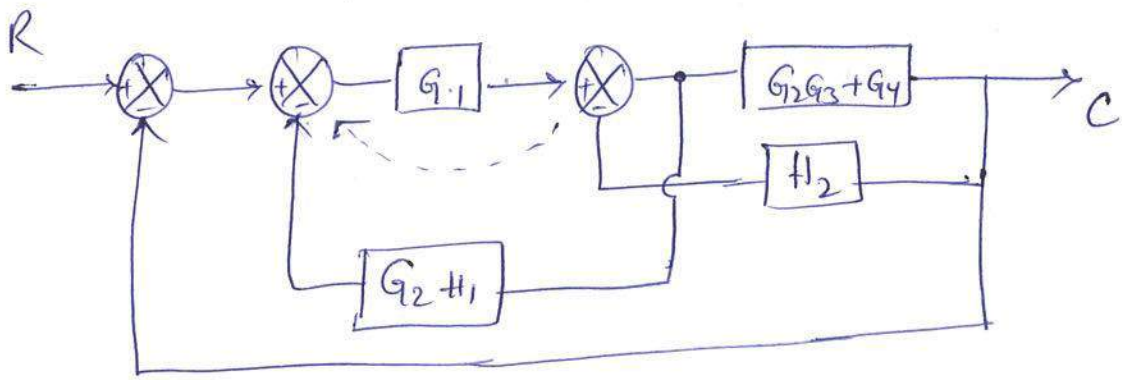
Step1: Moving the branch point before the block



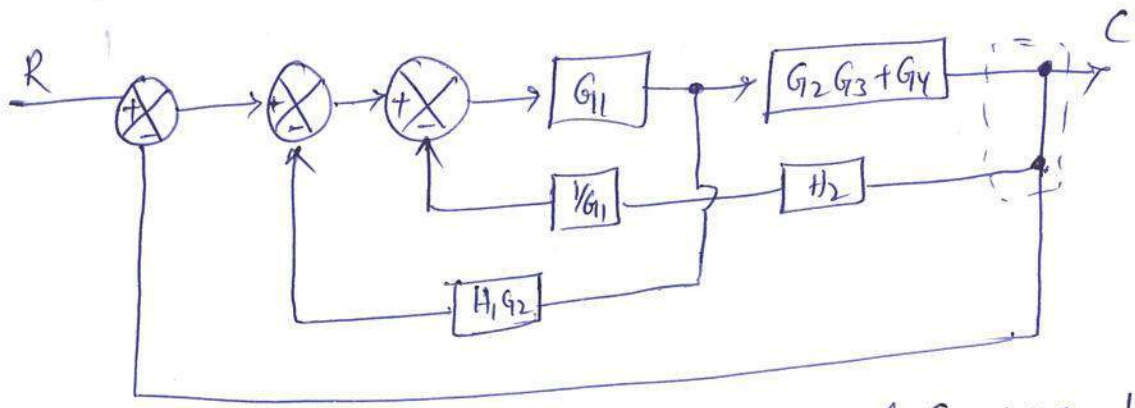
Step2:. Combining the blocks in Cascade and eliminating parallel blocks.



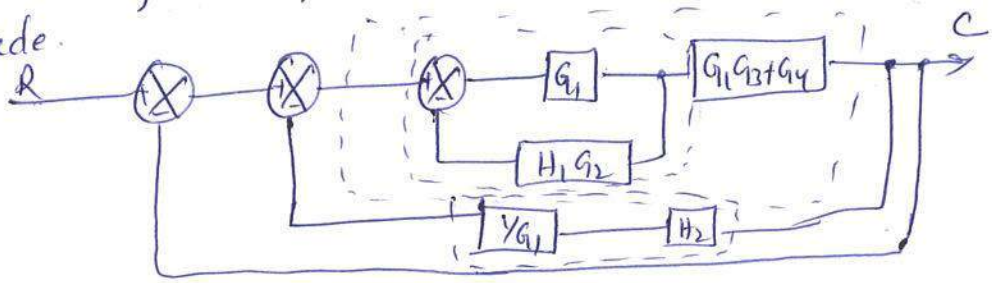
Step3 :- Moving summing point before the block



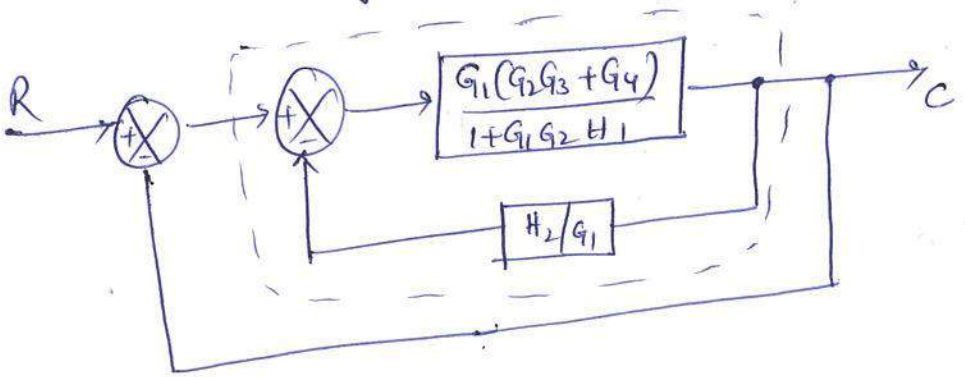
Step4 :- Interchanging summing points and modifying branch points.

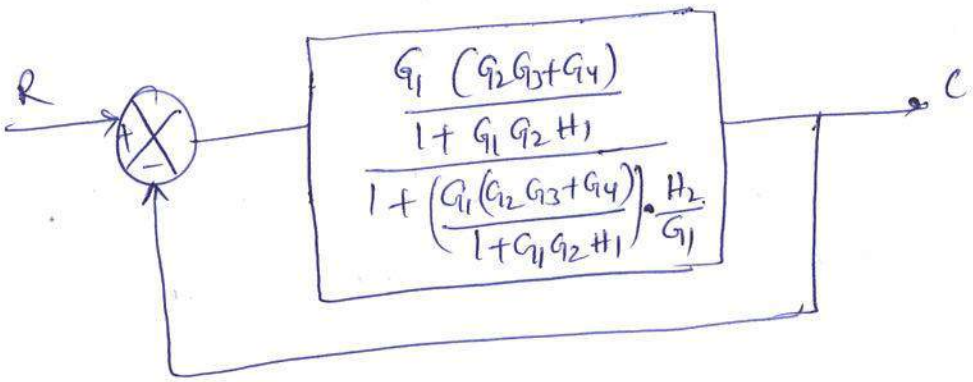


Step5 :- Eliminating the feedback path and Combining blocks in Cascade.



Step 6: Eliminating the feedback path





Step 7: Eliminating the feedback path.

$$\frac{C}{R} = \frac{\frac{G_1(G_2G_3+G_4)}{1+G_1G_2H_1}}{1+\frac{G_1G_2G_3H_2+G_4H_2}{1+G_1G_2H_1}}$$

$$1+\frac{G_1(G_2G_3+G_4)}{1+G_1G_2H_1}$$

$$\frac{G_1(G_2G_3+G_4)}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2}$$

$$\frac{(1+G_1G_2H_1)+G_1(G_2G_3+G_4)}{1+G_1G_2H_1}$$

$$\Rightarrow \frac{G_1(G_2G_3+G_4)}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2} \times \frac{1+G_1G_2H_1+G_2G_3H_2+G_4H_2}{1+G_1G_2H_1+G_1(G_2G_3+G_4)}$$

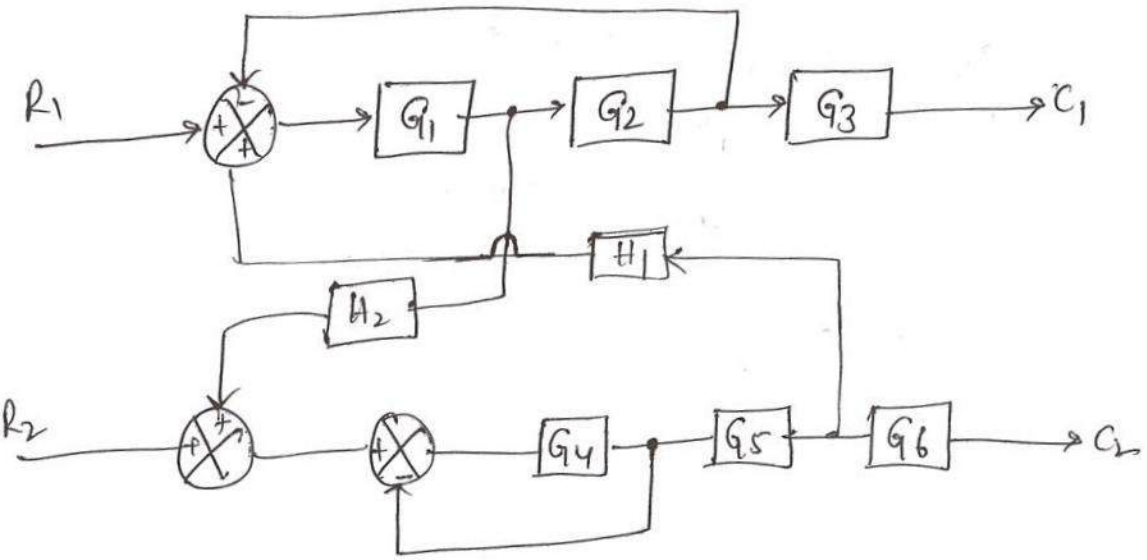
$\Rightarrow$

$$\frac{C}{R} = \frac{\dfrac{G_1(G_2G_3+G_4)}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2}}{1+\dfrac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_2+G_2G_3H_2+G_4H_2}}$$

$$\boxed{\frac{C}{R} = \frac{G_1(G_2G_3+G_4)}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2+G_1G_2G_3+G_1G_4}}$$

Problem 4:

For the system represented by the block diagram shown in fig. Determine C_1/R_1 and C_2/R_1.

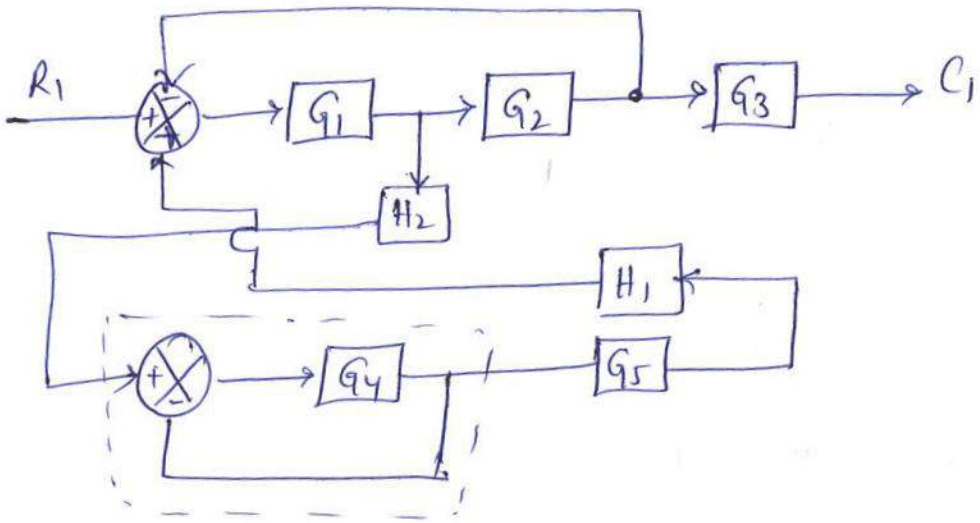


Sol: Cas 1: To find C_1/R_1

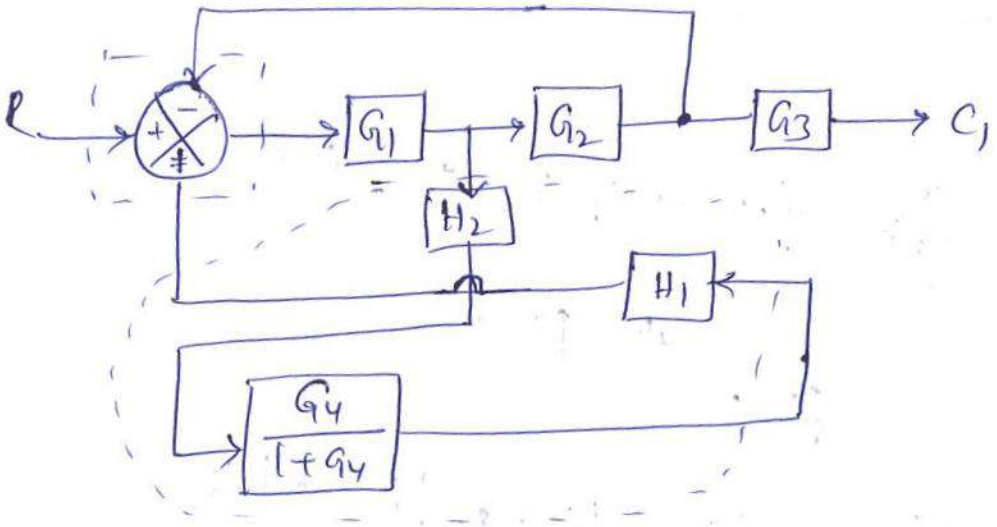
In this case set $R_2=0$ and consider only one output C_1, Hence we can remove the summing point

Which adds R_2 and need not Consider G_6, Since G_6 is on the open path.

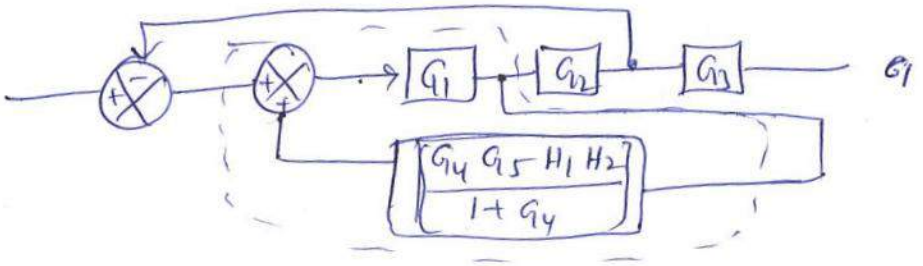
Step1: Eliminating the feedback path



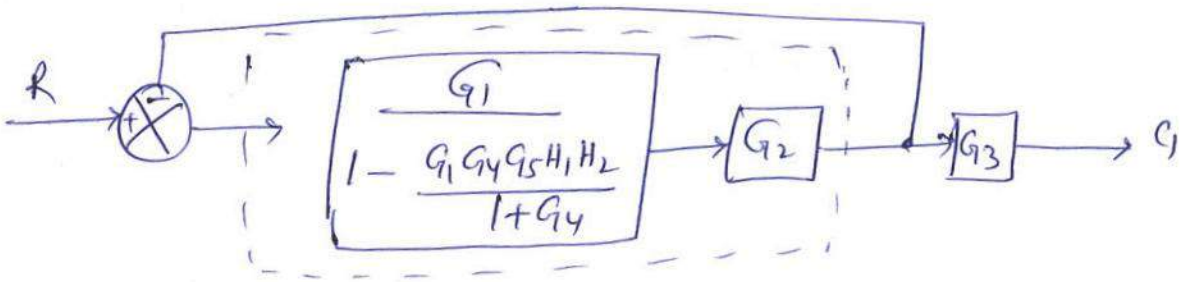
Step2:- Combining the blocks in Cascade and Splitting the Summing point.



Step3:- Eliminating the feedback path

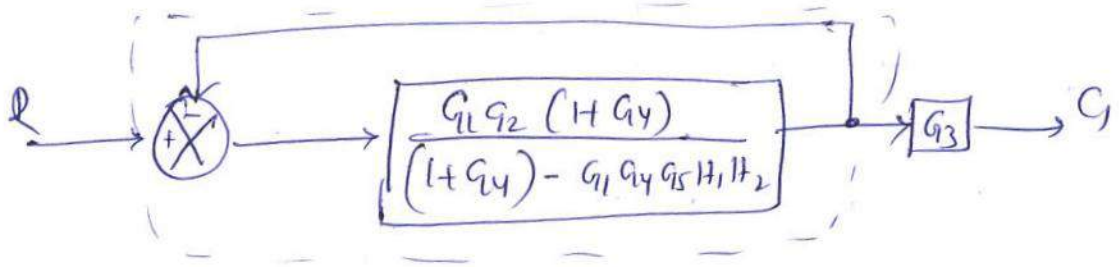


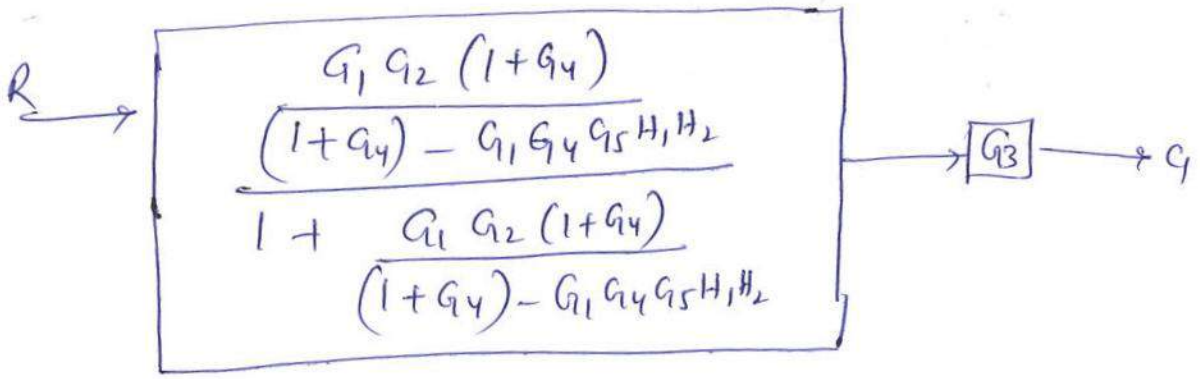
Step4: Combining the blocks in Cascade.



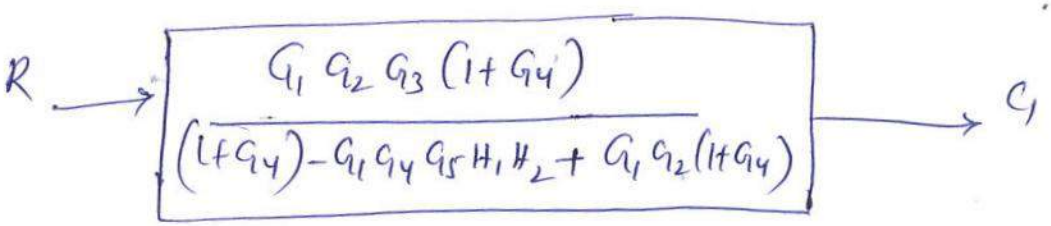
Step5:

Eliminating the feedback path.





Step6: Combining the blocks in Cascade.

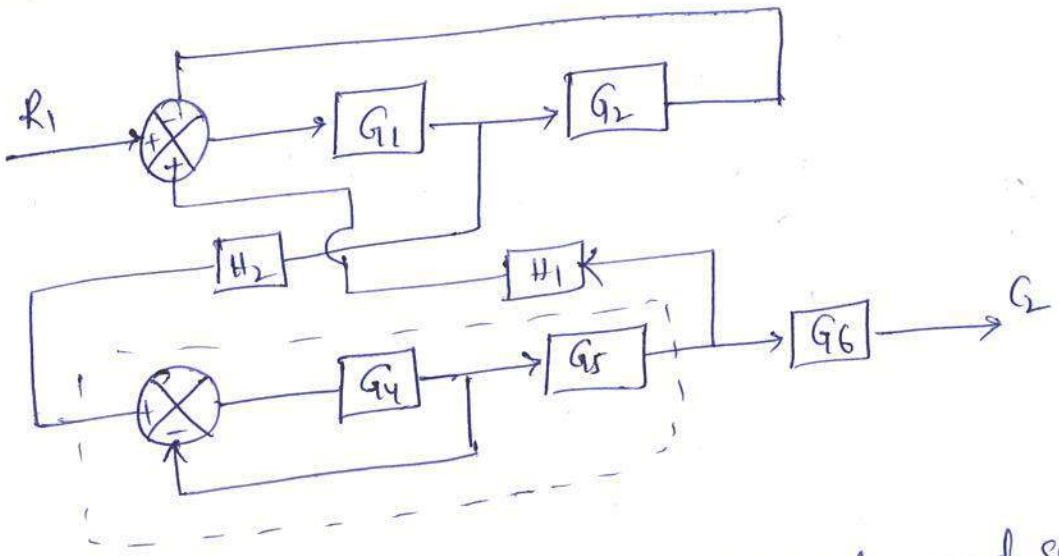


$$\therefore \frac{C_1}{R} = \frac{G_1 G_2 G_3 (1+G_4)}{(1+G_4) - G_1 G_4 G_5 H_1 H_2 + G_1 G_2 (1+G_4)}$$

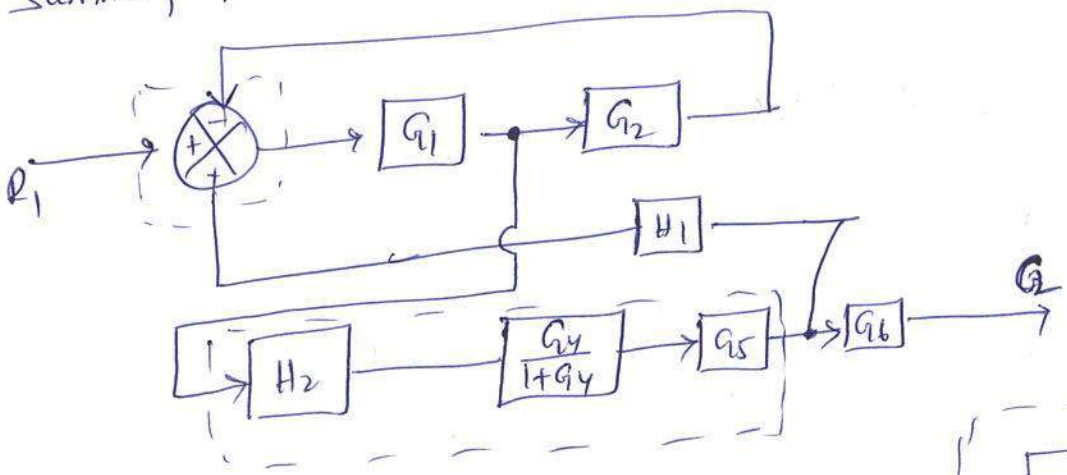
Case2 :- To find C_2/R_1

In this Case set $R_2 = 0$ and Consider only one o/p C_2. Hence we Can remove the Summing point which adds R_2 and need not Consider G_3, since G_3 is on the open path. The resulting block diagram shown in the fig.

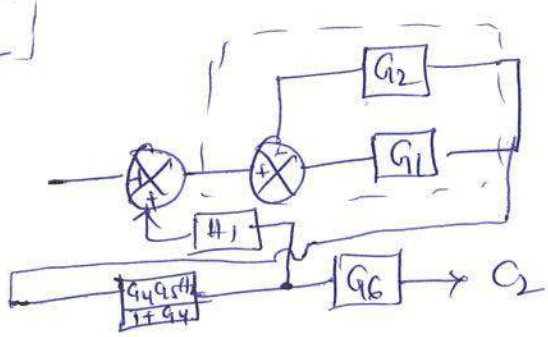
Step1 :- Eliminate the feedback path



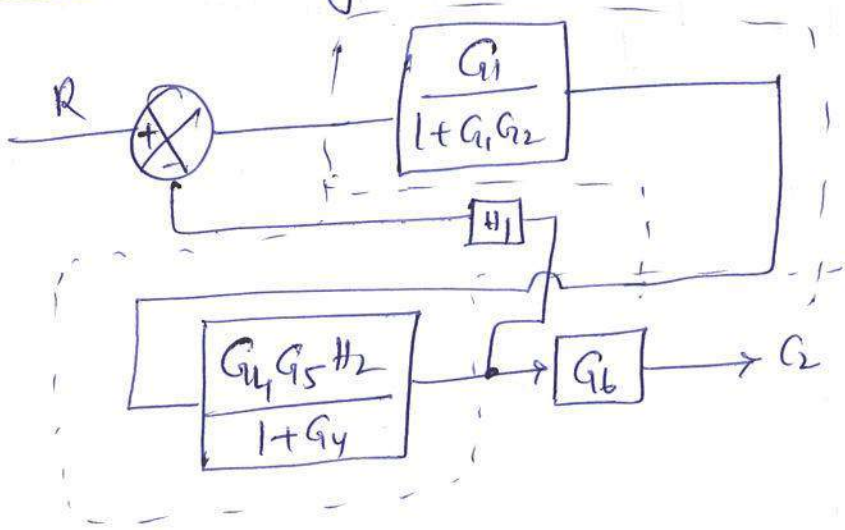
Step2 :- Combining blocks in Cascade and splitting the Summing point.



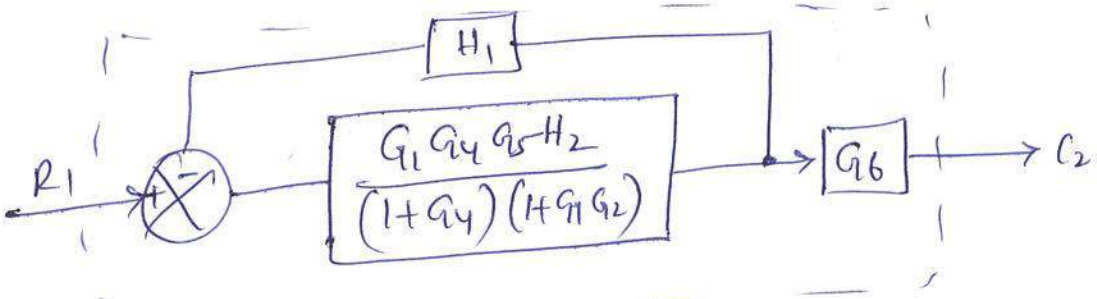
Step3 :- Eliminating the feedback path

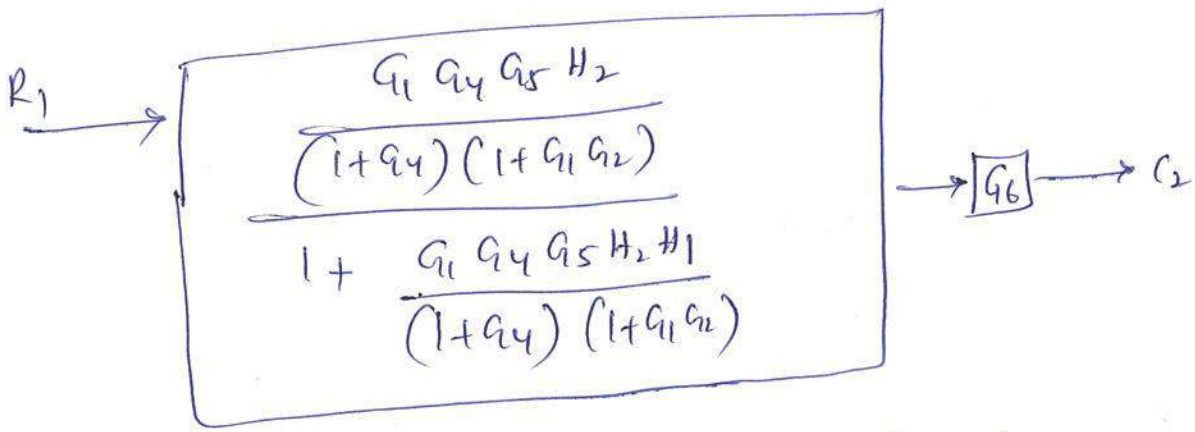


Step4: Combing the blocks in Cascade.

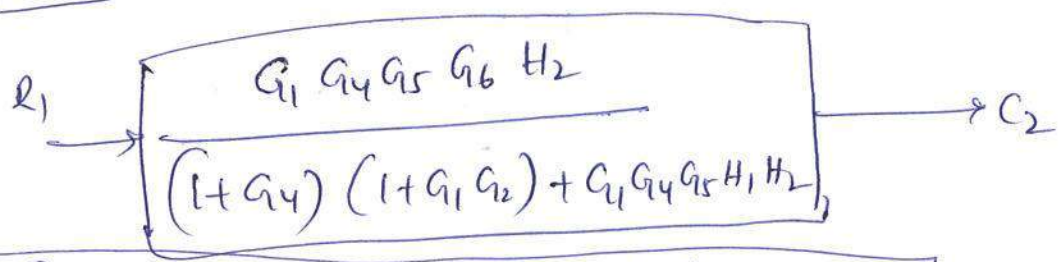


Step5: Eliminating the feedback path





Step6: Combing the blocks in Cascade.



$$\frac{C_2}{R_1} = \frac{G_1G_4G_5G_6H_2}{(1+G_4)(1+G_1G_2)+G_1G_4G_5H_1H_2}$$

Problem 5:

Construct the block diagram of armature Controlled dc motor.

Sol: The differential equations governing the armature Controlled dc motor are

$$V_a = i_a R_a + L_a \frac{di_a}{dt} + e_b \quad \text{---} (1)$$

$$T = k_t i_a \quad \text{---} (2)$$

$$T = J \frac{d\omega}{dt} + B\omega \quad \text{---} (3)$$

$$e_b = k_b \omega \quad \text{---} (4)$$

$$\omega = \frac{d\theta}{dt}. \quad \text{---} (5)$$

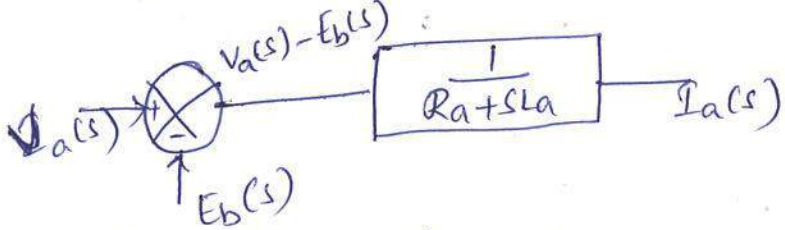
- On taking the Laplace Transform of eq (1).

$$V_a(s) = I_a(s) R_a + L_a s I_a(s) + E_b(s) \quad \text{---} (6)$$

where $V_a(s)$, $E_b(s)$ are the inputs and $I_a(s)$ is the output.

eq (6) $\Rightarrow$ $V_a(s) - E_b(s) = I_a(s)[R_a + L_a s]$

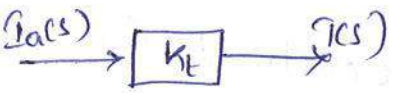
$$\therefore I_a(s) = \frac{1}{R_a + sL_a}[V_a(s) - E_b(s)]$$



- On taking the Laplace transform of eq (2)

$$T(s) = k_t I_a(s) \quad \text{---} (7)$$

from eq (7), $I_a(s)$ is the input and $T(s)$ is the o/p.

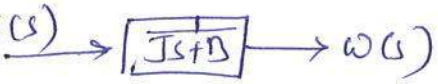


- On taking Laplace Transform of eq (3)

$$T(s) = Js\,\omega(s) + B\omega(s) \quad \text{---(8)}$$

& $\omega(s)$ is the output and $T(s)$ is the i/p.

$$\omega(s)(Js+B) = T(s)$$



$$\omega(s) = \frac{1}{Js+B}\,T(s) \quad \text{---(9)}$$

- On taking Laplace Transform of eq (4)

$$E_b(s) = k_b\,\omega(s) \quad \text{---(10)}$$

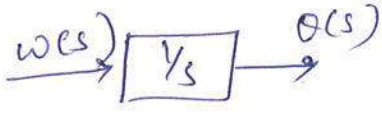
$\omega(s)$ is the i/p $E_b(s)$ is the o/p

- On taking Laplace Transform of eq (5)

$$\omega(s) = s\theta(s) \qquad \left[\because \omega_t = \frac{d\theta}{dt}\right]$$

i/p is $\omega(s)$, $\theta(s)$ is o/p.

$$\theta(s) = \frac{1}{s}\,\omega(s) \quad \text{---(11)}$$



By using above block diagram, the overall block diagram can be drawn as.

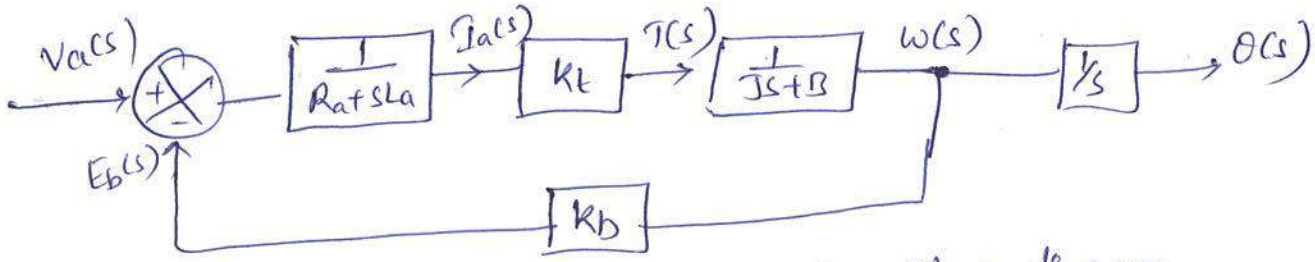


fig: Armature Controlled dc motor Block diagram.

Problem 6:

Construct the block diagram of field Controlled dc motor.

Sol: The differential equations governing the field Controlled dc motor are

$$V_f = I_f R_f + L_f \frac{di_f}{dt} \quad \text{——— (1)}$$

$$T = K_{tf} I_f \quad \text{——— (2)}$$

$$T = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} \quad \text{——— (3)}$$

- On taking Laplace Transform of eq (1).

$$V_f(s) = R_f I_f(s) + L_f s I_f(s)$$

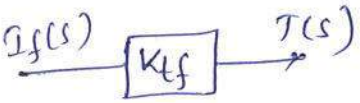
$V_f(s)$ is the input and $I_f(s)$ is the o/p.

$$I_f(s)(R_f + sL_f) = V_f(s)$$

$$I_f(s) = \frac{1}{R_f + L_f s} V_f(s)$$

- On taking Laplace Transform of eq (2).

$$T(s) = K_{tf} I_f(s)$$

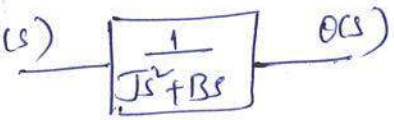


$T(s)$ is the output $I_f(s)$ is the i/p.

- On taking Laplace Transform of eq (3).

$$T(s) = J s^2\theta(s) + B\theta(s)s$$

$$= \theta(s)[Js^2 + Bs]$$



$T(s)$ is the i/p $\theta(s)$ is the o/p

$$\theta(s) = \frac{1}{Js^2 + Bs} T(s)$$

The overall block diagram can be drawn as

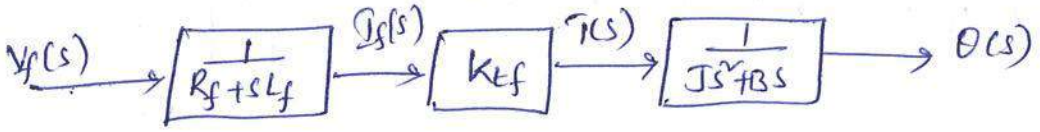
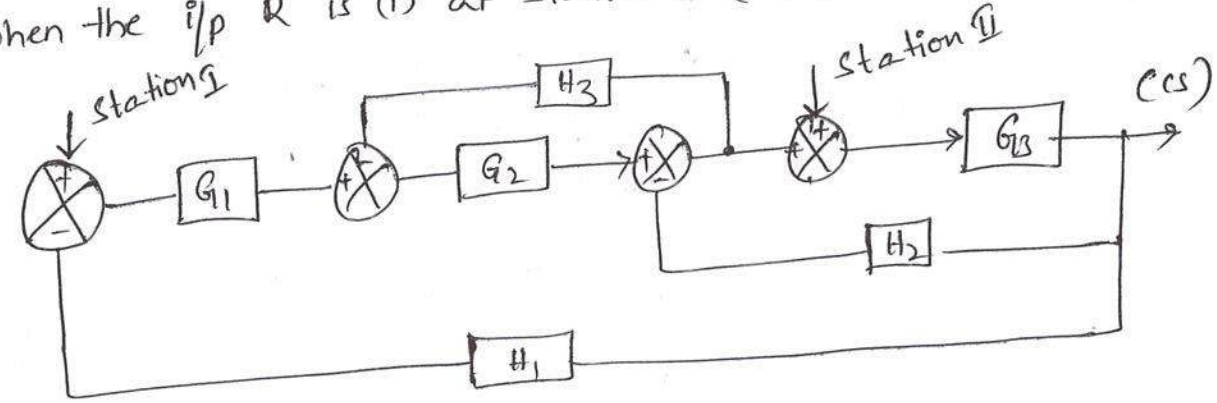


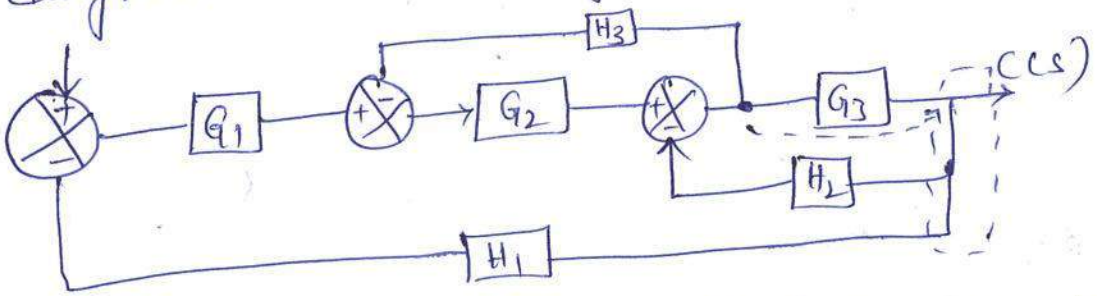
fig: Block diagram of field controlled dc motor.

Problem : 7

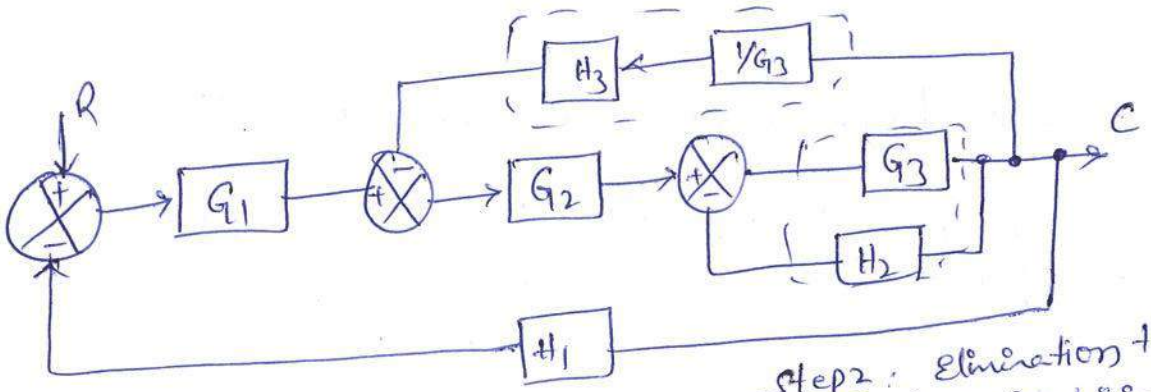
For the system represented by the block diagram shown in fig. Evaluate the closed loop transfer function when the i/p R is (i) at station 1 (ii) at station II.



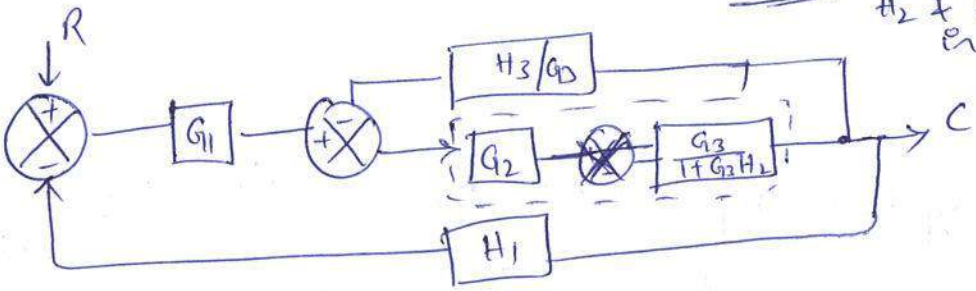
Sol: Consider the input R is at station 1 and so the input at station II is made zero. Let the o/p be C_1. Since there is no input at station II that summing point can be removed and resulting block diagram is shown in fig.



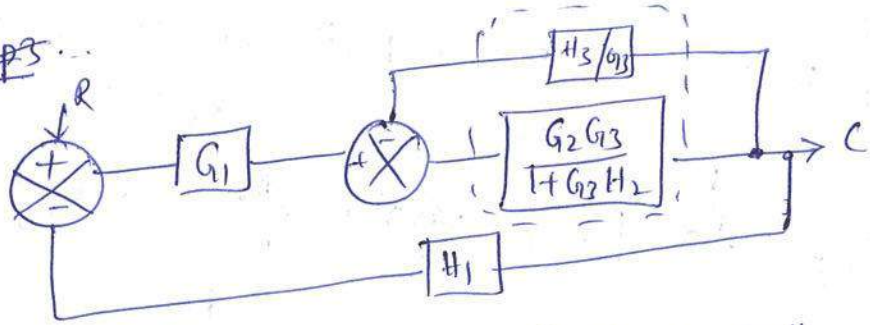
Step1 Shift the take off point of feedback H_3 beyond G_3 and rearrange the branch points



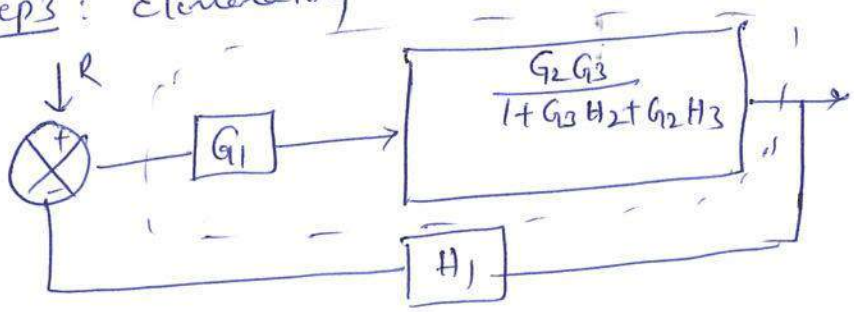
Step2: Elimination the feedback H_2 + Combining blocks in Cascade.



Step3...

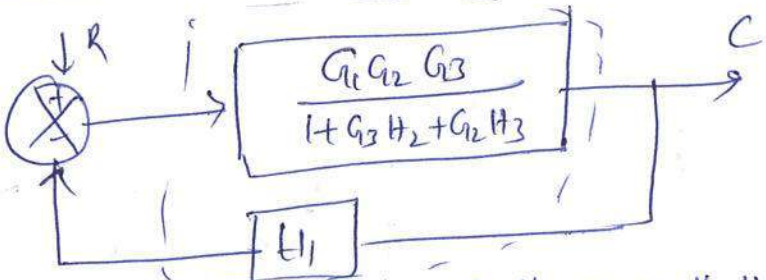


Step3: Eliminating the feedback path



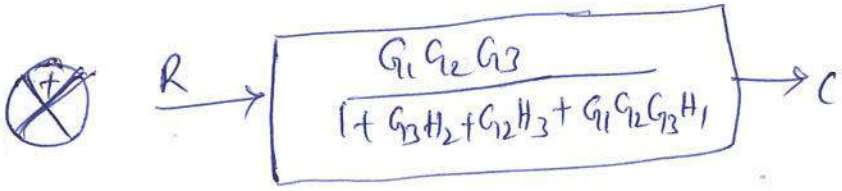
$$\frac{\frac{G_2G_3}{1+G_3H_2}}{1+\frac{G_2G_3}{1+G_3H_2}\cdot\frac{H_3}{G_3}} = \frac{G_2G_3}{1+G_3H_2+G_2H_3}$$

Step4:- Combining the blocks in Cascade.



$$\frac{\frac{G_1G_2G_3}{1+G_3H_2+G_2H_3}}{1+\frac{G_1G_2G_3H_1}{1+G_3H_2+G_2H_3}} \Rightarrow \frac{G_1G_2G_3}{1+G_3H_2+G_2H_3+G_1G_2G_3H_1}$$

Step5:- Eliminating feedback path H_1

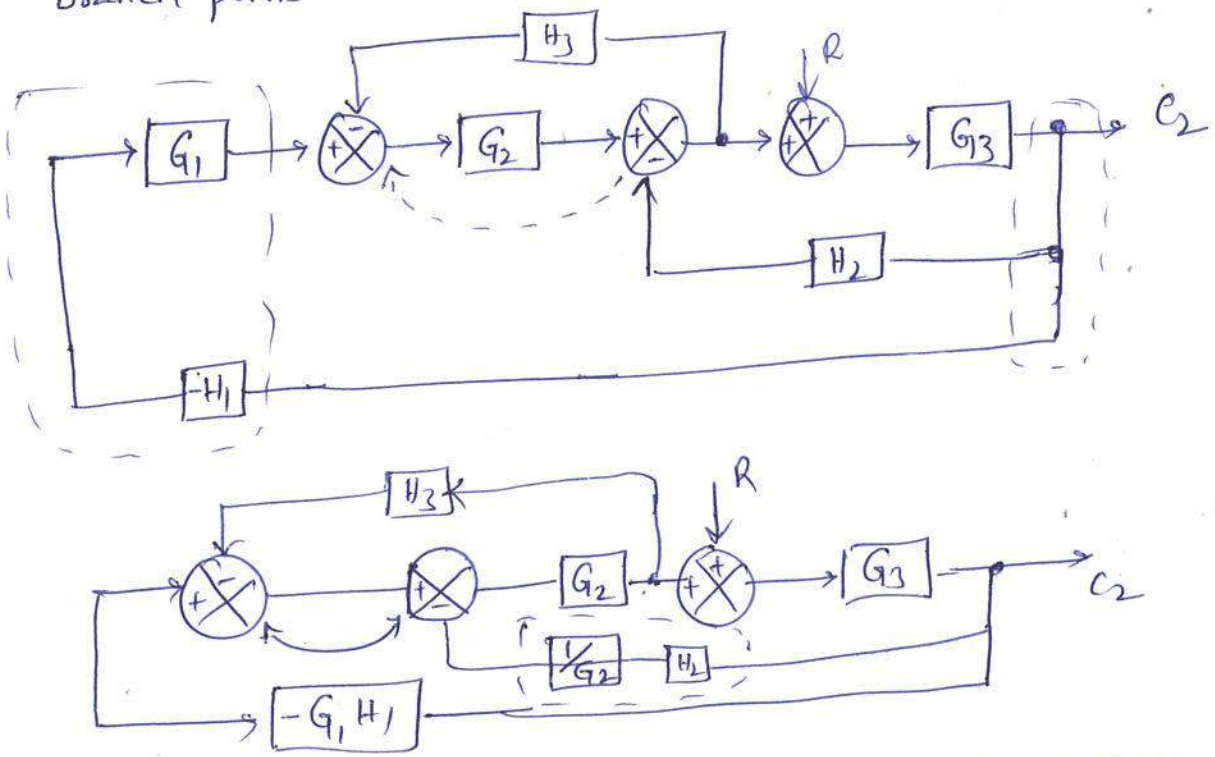


$$\Rightarrow \frac{C(s)}{R(s)}$$

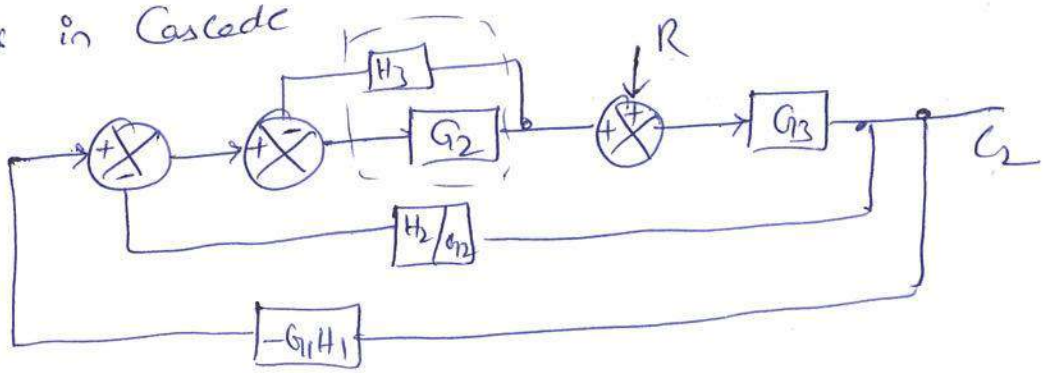
(ii) Consider the input R at station II, the input at station I is made zero. Let output be C_2, since there is no input in station I that Corresponding Summing point Can be removed and a negative sign Can be attached to the feedback path gain H_1.

The resulting block diagram is shown in fig.

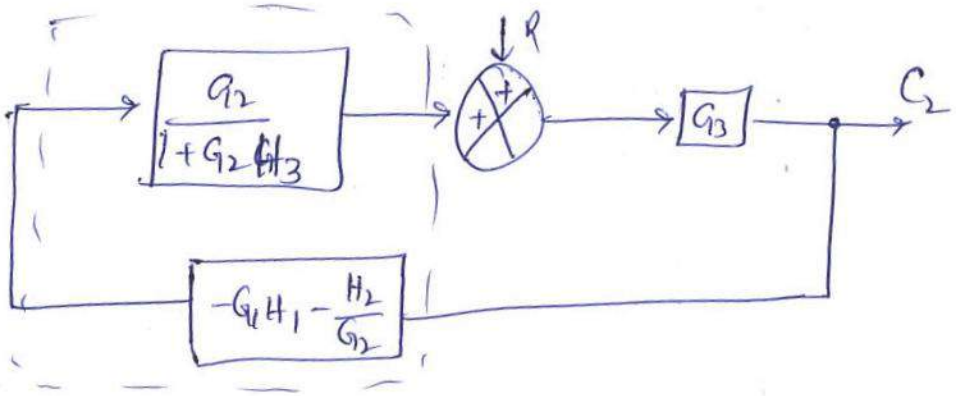
Step1: Combing the blocks in Cascade, shifting the Summing point of H_2 before G_2 and rearranging the branch points.

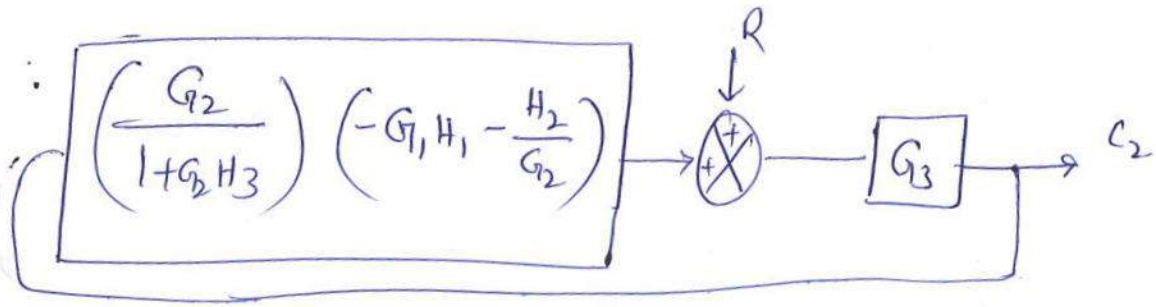


Step2 Interchanging Summing points and Combining the blocks in Cascade

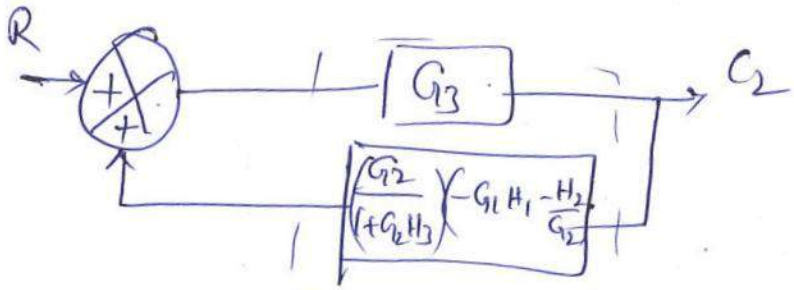


Step3: Combing the parallel blocks and eliminating feedback path

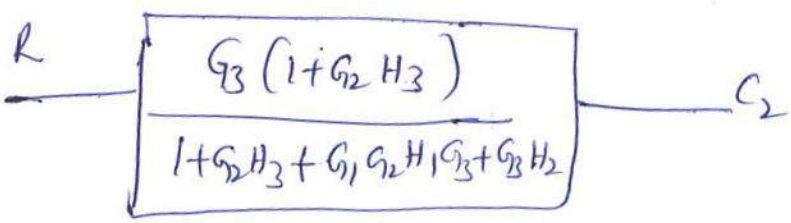




for the above fig. we can draw as.



Step4: eliminating the feedback path. $\because \frac{G}{1-GH}$



$$\frac{G_3}{1-G_3\left[\left(\frac{G_2}{1+G_2H_3}\right)\left(\frac{-G_1G_2H_1-H_2}{G_2}\right)\right]}$$

$$\Rightarrow \frac{G_3}{1-\frac{G_3((-G_1G_2H_1)-H_2)}{1+G_2H_3}}$$

$$\Rightarrow \frac{G_3(1+G_2H_3)}{1+G_2H_3+G_1G_2H_1G_3+G_3H_2}$$

$$T.F = \frac{C_2}{R_2} = \frac{G_3(1+G_2H_3)}{1+G_2H_3+G_1G_2H_1G_3+G_3H_2}$$

Signal flow Graph:

✓ The Signal flow graph is used to represent the Control System graphically and it was developed by S.J. Mason.

✓ The signal flow graph depicts the flow of signals from one point of a system to another and gives the relationship among the signals.

✓ A signal flow graph consists of a network in which nodes are connected by the directed path or branches.

✓ Each node represents a system variable and each branch connected between two nodes act as a signal multiplier.

✓ Each branch connected between two nodes acts as signal multiplier. Each branch has a gain or transmittance. When the signal pass through a branch, it gets multiplied by the gain of the Branch.

✓ The signal flows in one direction only and the direction of the flow can be indicated as an arrow.

Terms used in Signal flow graph:

Node: A Node is a point representing a variable or a signal.

Branch: A branch is directed line segment joining two nodes. The arrow on the branch indicates the direction of signal flow and the gain of a branch is the transmittance.

Transmittance: The gain acquired by the signal when it travels from one node to another is called transmittance. The transmittance can be real or Complex.

Input node (source): It is node that has only outgoing branches.

Output node (sink): It is a node that has only incomming branches.

Mixed node: It has a node that has both outgoing and incoming branches.

path: A path is a traversal of Connected branches in the direction of the branch arrows. The path should not Cross a node more than once.

Open path:

A open path starts at a node and ends at another node.

Closed path:

Closed path starts and ends at same node

Forward path:

It is a path from an input node to an output node that does not cross any nodes more than once.

Forward path gain:

It is the product of the branch transmittances (gains) of a forward path.

Individual loop:

It is a closed path starting from a node and after passing through a certain part of a graph arrives at the same node without crossing any node more than once.

Loop gain:

It is the product of the branch transmittances (gains) of a loop.

Non-touching loops:

If the loops does not have a common node then they are said to be non-touching loops.

Properties of Signal flow graph:

The basic properties of signal flow graph are the following:

1. The Algebraic equations which are used to Construct Signal flow graph must be in the form of cause and effect relationship.
2. Signal flow graph is applicable to linear systems only.
3. A node in the Signal flow graph represents the Variable or Signal.
4. A node adds the signal of all incoming branches and transmits the sum to all outgoing branches.
5. A mixed node which has both incomming and outgoing signals can be treated as output node by adding an outgoing branch of unity transmittance.
6. A branch indicates functional dependence of on Signal on the other.
7. The Signal travel along branches only in the marked direction and when it travel it gets multiplied by the gain or transmittance of the branch.
8. The signal flow graph of a system is not unique. Be arranging the system equations different types of signal flow graphs can be drawn for a given system.

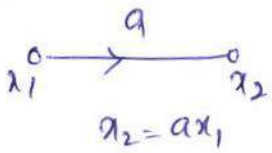
Signal flow graph Algebra.

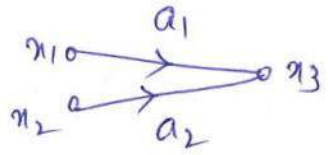
Signal flow graph for a system can be reduced to obtain the transfer function of the system using the following rules.

Rules:

Rule 1: Incoming signal to a node through a branch is given by the product of a signal at previous node and the gain of the branch

Ex:





$x_3 = a_1x_1 + a_2x_2$

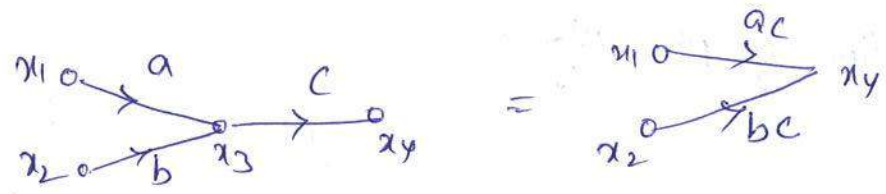
Rule 2: Cascaded branches can be ~~obtai~~ combined to give a single branch whose transmittance is equal to the product of individual branch transmittance.

Ex:

Rule 3: parallel branches may be represented by single branch whose transmittance is the sum of individual branch transmittances.

Ex:

Rule 4: A mixed node can be eliminated by multiplying the transmittance of outgoing branch (from the mixed node) to the transmittance of all incoming branches to the mixed node.

Ex: 

Rule 5:

A loop gain may be eliminated by writing equations at the i/p and o/p node and rearranging the equations to find the ratio of o/p to i/p. This ratio gives the gain of resultant branch.

Ex:

Proof

$$x_2 = ax_1 + cx_3$$

$$x_3 = bx_2$$

put $x_2 = ax_1 + cx_3$ in the equation x_3.

$$\therefore x_3 = b(ax_1 + cx_3)$$

$$x_3 = abx_1 + bcx_3$$

$$x_3 - bcx_3 = abx_1$$

$$x_3(1-bc) = abx_1$$

$$\frac{x_3}{x_1} = \frac{ab}{1-bc}.$$

Signal flow graph Reduction.

The Signal flow graph of a System can be reduced either by using the rules of a signal flow graph algebra (i.e.,) by writing equations at every

node and then rearranging these equations to get the ratio of o/p and i/p (Transfer function).

S.J. Mason developed a formula called "Mason's Gain formula" which can be used directly to find the transfer function of the system.

Mason's Gain formula:

The Mason's gain formula is used to determine the transfer function of the system from the signal flow graph of the system.

Let $R(s)$ = Input to the system.

$C(s)$ = output of the system.

Transfer function of the system. $T(s) = \frac{C(s)}{R(s)}$

Mason's Gain formula states the overall gain of the system (transfer function) as follows

$$\text{Overall gain } T = \frac{1}{\Delta} \sum_k P_k \Delta_k$$

Where $T = T(s)$ = Transfer function of the system.

P_k = forward path gain of kth forward path.

$\Delta = 1 - (\text{Sum of individual loop gains})$

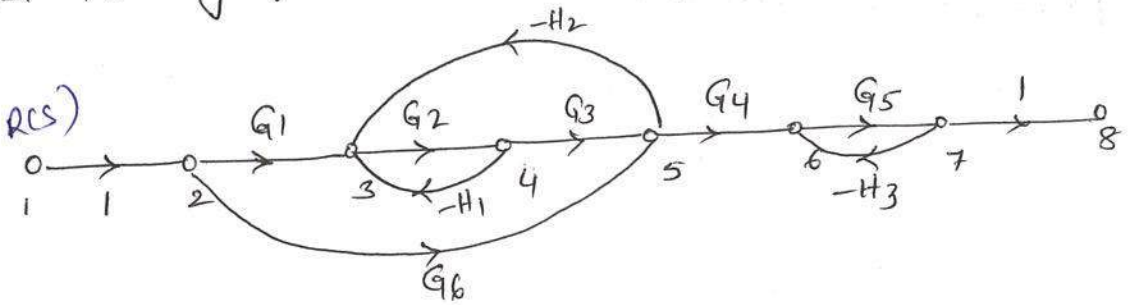
$+ (\text{sum of gain products of all possible combinations of two non-touching loops})$

$- (\text{Sum of gain products of all possible Combinations of three - non touching loops}) + \cdots$

Δ_K = Δ for that part of the graph which is not touching k^{th} forward path.

Problems 1:

Find the overall transfer function of the system whose signal flow graph is shown in fig.



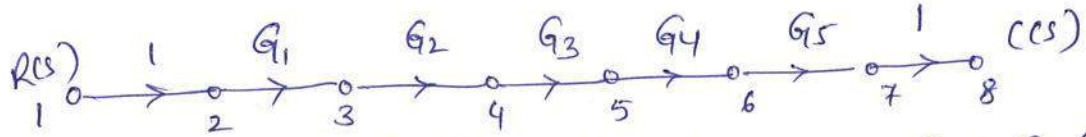
Sol: Mason's Gain formula:

Transfer function $T = \frac{1}{\Delta} \sum_K P_K \Delta_K$

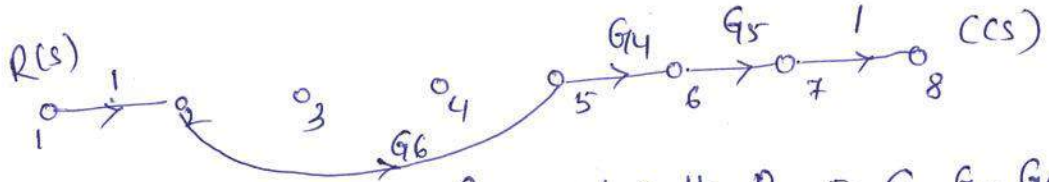
where K is 2 only two forward paths

I Forward path Gains, K = 2

Let forward path Gains be P_1 and P_2



Gain of forward path $P_1 = G_1 G_2 G_3 G_4 G_5$



Gain of forward path $P_2 = G_4 G_5 G_6$

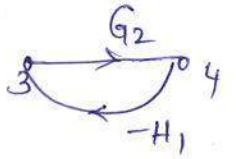
II Individual Loop gain

There are three individual loops

Let individual loop gains be P_{11}, P_{21} and P_{31}

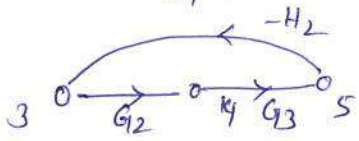
Loop gain of individual loop-1

$P_{11} = -G_2 H_1$



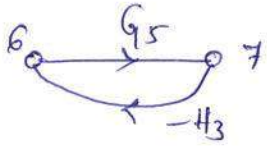
Loop gain of individual loop 2

$P_{21} = -(G_2 G_3 H_2)$



Loop gain of individual loop 3

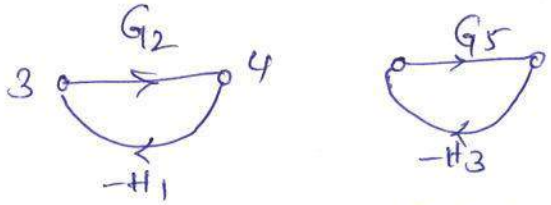
$P_{31} = -G_5 H_3$.



III Gain products of Two non-Touching Loops.

There are two combinations of non-touching loops.

Let the gain products of two non-touching loops be P_{12} and P_{22}.



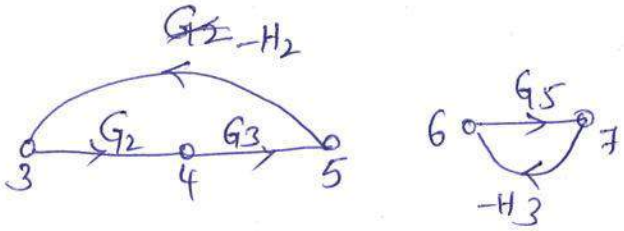
first Combination of two non-touching loops

Gain product of first Combination of two non-touching loops

$P_{12} = P_{11} P_{31}$

$= (-G_2 H_1)(-G_5 H_3)$

$= +G_2 G_5 H_1 H_3$.



Second Combination of two non-touching loops

Gain product of second Combination of two non-touching loops

$P_{22} = P_{21} P_{31}$

$= (-G_2 G_3 H_2)(-G_5 H_3)$

$= G_2 G_3 G_5 H_2 H_3$.

(iv) Calculation of Δ and Δ_k

$$\Delta = 1 - (P_{11} + P_{21} + P_{31}) + (P_{12} + P_{32})$$

$$= 1 - (-G_2H_1 - G_2G_3H_2 - G_5H_3) + (G_2G_5H_1H_3 + G_2G_3G_5H_2H_3)$$

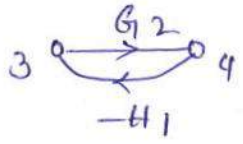
$$= 1 + G_2H_1 + G_2G_3H_2 + G_5H_3 + G_2G_5H_1H_3 + G_2G_3G_5H_2H_3$$

Δ_1 = 1, Since there is no part of graph which is not touching with first forward path

$$\Delta_2 = 1 - P_{11}$$

$$= 1 - (-G_2H_1)$$

$$= 1 + G_2H_1$$



(the part of the graph which is non-touching with second forward path).

(v) Transfer function T:

By Mason's Gain formula, the transfer function

$$T = \frac{1}{\Delta}\sum_k P_k\Delta_k$$

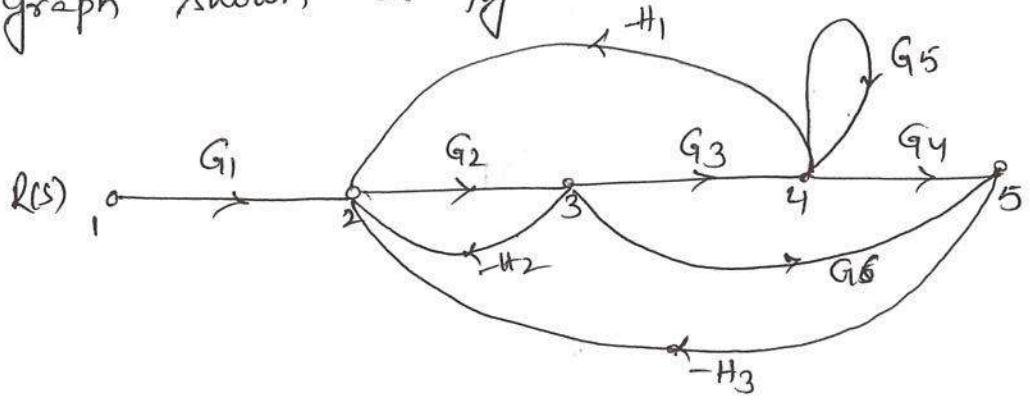
$$= \frac{1}{\Delta}(P_1\Delta_1 + P_2\Delta_2) \quad [\because K = 2 \text{ two feed paths}].$$

$$= \frac{G_1G_2G_3G_4G_5 + G_4G_5G_6(1 + G_2H_1)}{1 + G_2H_1 + G_2G_3H_2 + G_5H_3 + G_2G_5H_1H_3 + G_2G_3G_5H_2H_3}$$

$$T = \frac{G_1G_2G_3G_4G_5 + G_4G_5G_6 + G_2G_4G_5G_6H_1}{1 + G_2H_1 + G_2G_3H_2 + G_5H_3 + G_2G_5H_1H_3 + G_2G_3G_5H_2H_3}$$

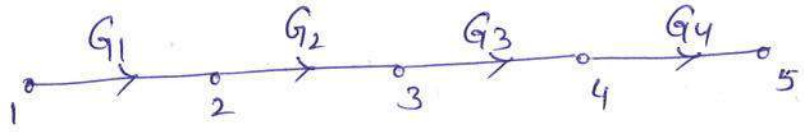
Problem 2:

Find the overall gain C(s)/R(s) for the signal flow graph shown in fig.

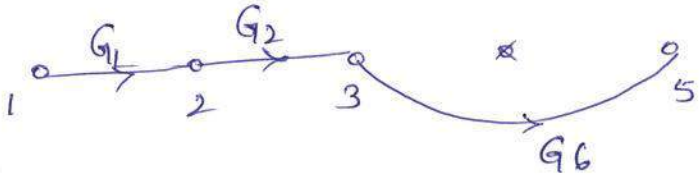


Sol: I. Forward path Gains

There are two forward paths, k = 2



$P_1 = G_1 G_2 G_3 G_4$

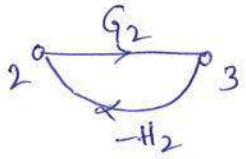


$P_2 = G_1 G_2 G_6$

II Individual Loop Gains:

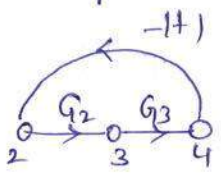
There are five individual loops.

(1)



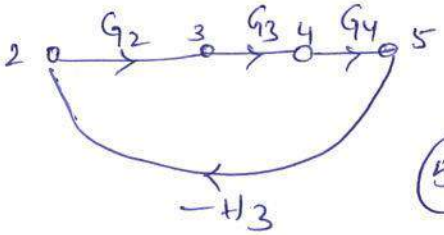
$P_{11} = -G_2 H_2$

(2)



$P_{12} = -G_2 G_3 H_1$

(3)

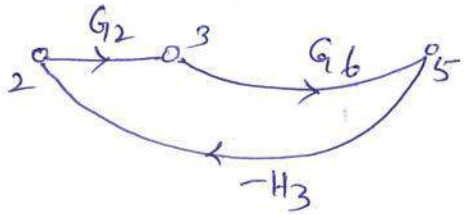


$P_{13} = -G_2 G_3 G_4 H_3$

(4)

$P_{14} = G_5$

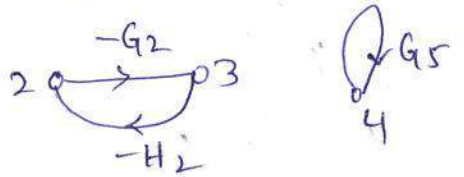
(5)



$P_{15} = -G_2 G_6 H_3$

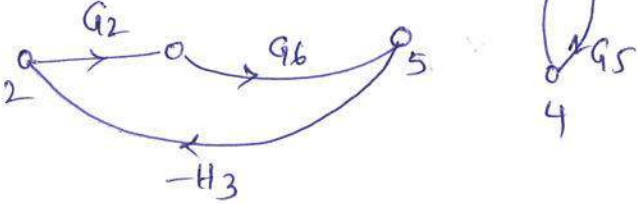
(III) Gain products of two non-touching loops

There are two combinations of two non-touching loops.



First combination of two non-touching loops

$P_{21} = -G_2 H_2 G_5$



second combination of two non touching loops

$P_{22} = -G_2 G_6 H_3 G_5$

(IV) Calculation of Δ and Δ_k

$$\Delta = 1 - (P_{11} + P_{12} + P_{13} + P_{14} + P_{15}) + (P_{21} + P_{22})$$

$$= 1 + G_2 G_3 H_1 + G_2 H_2 + G_2 G_6 H_3 + G_2 G_3 G_4 H_3 - G_5 - G_2 G_5 H_2 - G_2 G_6 G_5 H_3$$

$\Delta_k = \Delta_1 = 1$ (since there is no part of the graph which is not touching forward path 1)

$\Delta_2 = 1 - G_5$ (the part of the Graph which is not touching forward path-2)

(V) Transfer function, T

By Mason's Gain formula $T = \frac{1}{\Delta} \sum_k P_k \Delta_k$

$$= \frac{1}{\Delta} [P_1 \Delta_1 + P_2 \Delta_2]$$

$$= \frac{1}{\Delta} [G_1 G_2 G_3 G_4 + G_1 G_2 G_6 (1 - G_5)]$$

$$T = \frac{G_1 G_2 G_3 G_4 + G_1 G_2 G_6 - G_1 G_2 G_5 G_6}{1 + G_2 G_3 H_1 + G_2 H_2 + G_2 G_6 H_3 + G_2 G_3 G_4 H_3 - G_5 - G_2 G_5 H_2 - G_2 G_6 G_5 H_3}$$

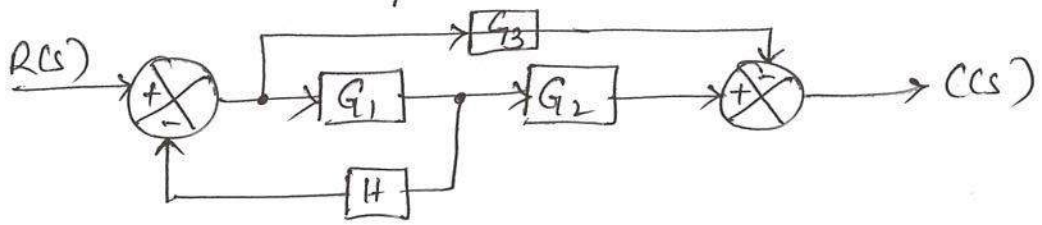
Conversion of Block diagram to Signal flow graph.

Procedure:

1. Assume nodes at input, output, at every summing point, at every branch point and in between Cascade blocks.
2. Draw the nodes seperately as Small Circles and number the in the order 1, 2, 3 ... etc.
3. From the block diagram find the gain between each node in the main forward path and Connect all the Corresponding Circles by straight line and mark the gain between the nodes.
4. Draw the feed forward paths between various nodes and mark the gain of the forward path along with sign.
5. Draw the feedback path between various nodes and mark the gain of feedback path along with sign.

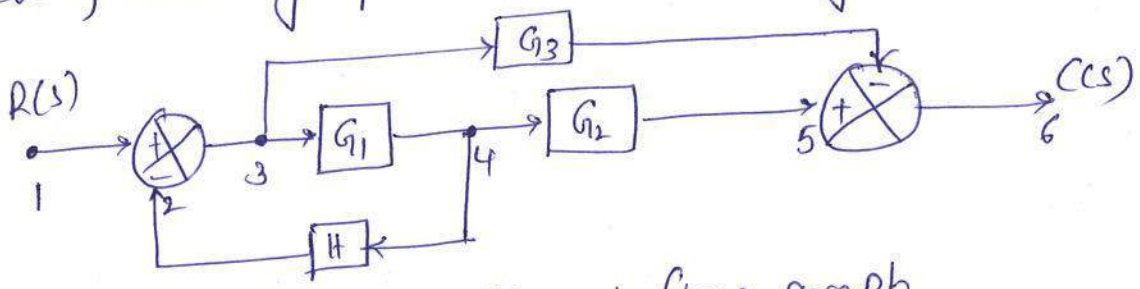
Problem 3

Convert the given block diagram to signal flow graph and determine C(s)/R(s)

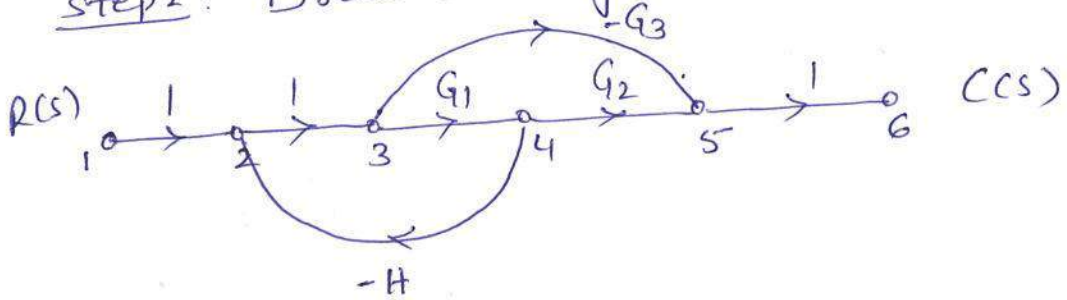


Sol: Step1:

The nodes are assigned at input, output, at every summing point and branching point..

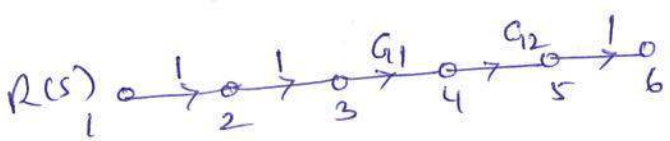


step2: Draw the signal flow graph

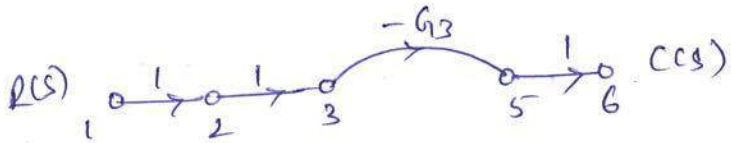


I. Forward path Gain

There are two forward paths



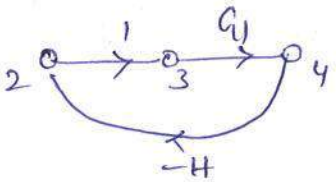
$P_1 = G_1 G_2$



$P_2 = -G_3$.

II. Individual Loop gain

There is only one individual loop



$P_{11} = -G_1 H$

III Gain product of two non touching loops

There are no combinations of non-touching loops

(iv) Calculation of Δ and Δ_k

$$\Delta = 1 - (\text{Sum of individual loops}) \cdots$$

$$= 1 - (P_{11})$$

$$= 1 + G_1 H$$

$$\Delta_k = \Delta_1 = 1$$

$$\Delta_2 = 1$$

(Since there are no part of the graph which is non touching with forward path 1 and 2.)

(v) Transfer function, T

By Mason's gain formula.

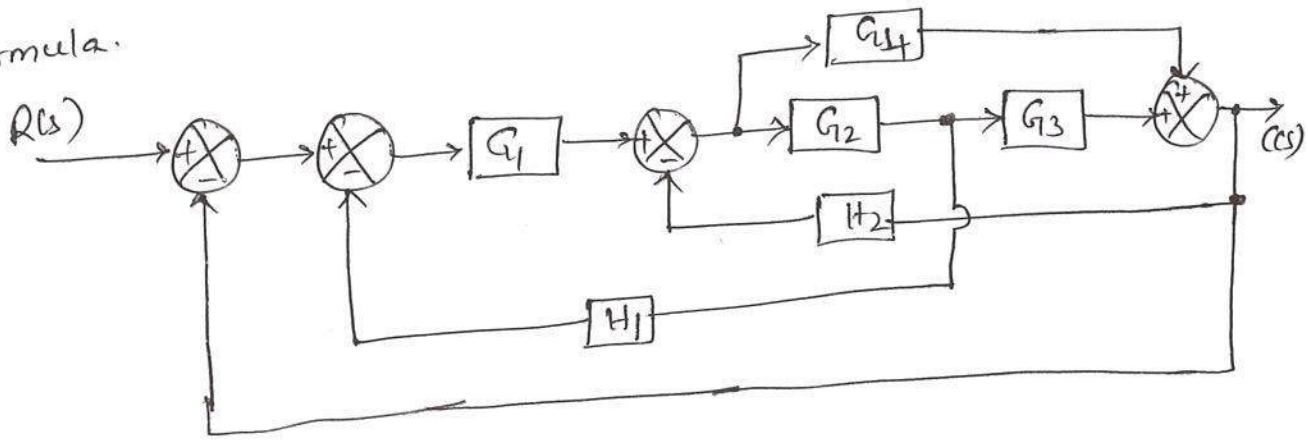
$$T(s) = \frac{1}{\Delta} \sum_k P_k \Delta_k$$

$$= \frac{1}{\Delta} [P_1 \Delta_1 + P_2 \Delta_2]$$

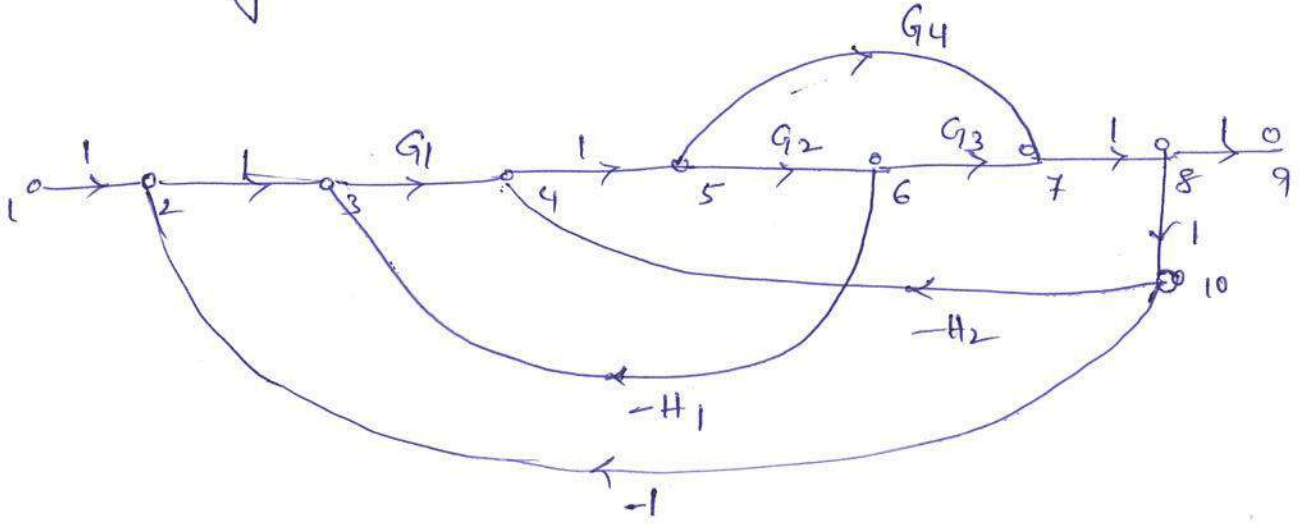
$$\boxed{T = \frac{G_1 G_2 - G_3}{1 + G_1 H}}$$

Problem 4

Convert the block diagram to signal flow graph and determine the transfer function using mason's Gain formula.

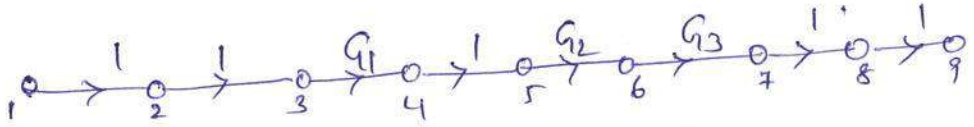


Sol: The nodes are assigned at i/p, o/p, at every summing point and branch point & Cascaded gains.

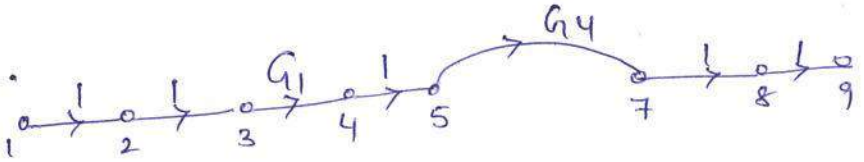


I Forward path Gain:

There are two forward paths (k = 2)



$P_1 = G_1 G_2 G_3$

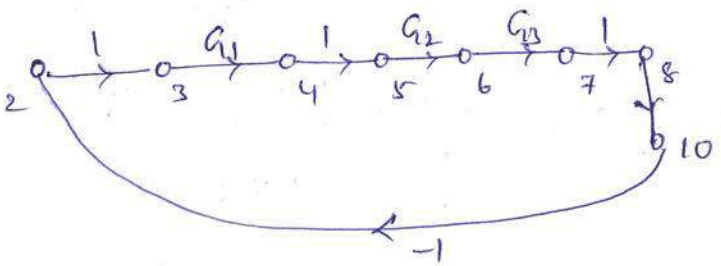


$P_2 = G_1 G_4$.

II. Individual loop Gains:

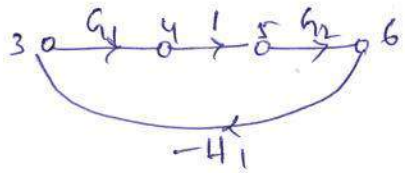
There are five individual loop gains.

(1)



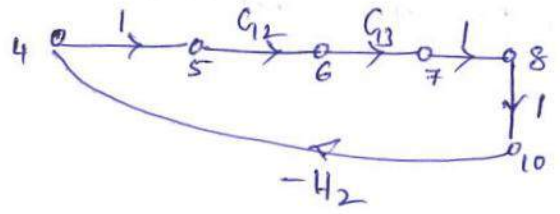
$P_{11} = -G_1 G_2 G_3$

(2)



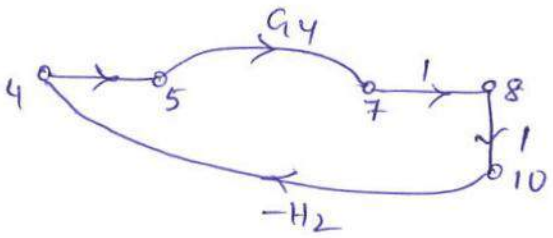
$P_{12} = -G_1 G_2 H_1$

(3)



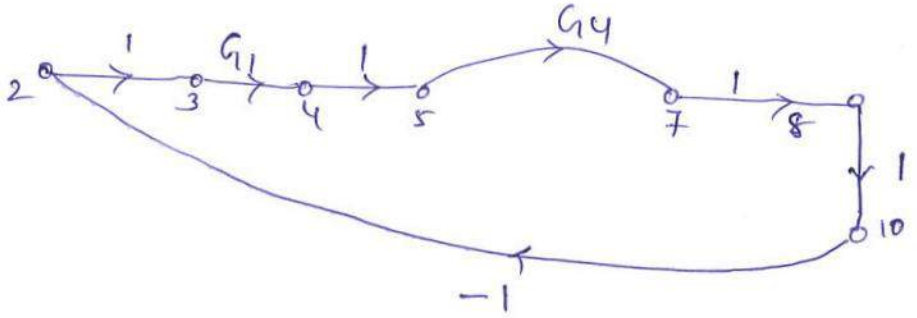
$P_{13} = -G_2 G_3 H_2$

(4)



$P_{14} = -G_4 H_2$

(5)



$P_{15} = -G_1 G_4$

(iii) Gain products of two non-touching loops

There are no possible Combinations of two non-touching loops,

(iv) Calculation of Δ and Δ_k

$\Delta = 1 -$ Sum of ~~non-touching~~ Individual loops

$= 1 - (P_{11} + P_{12} + P_{13} + P_{14} + P_{15})$

$= 1 + G_1 G_2 G_3 + G_1 G_2 H_1 + G_2 G_3 H_2 + G_1 G_4 + G_4 H_2$

$\Delta_k = \Delta_1 = 1, \quad \Delta_2 = 1$

(v) Transfer function T

$$T = \frac{1}{\Delta} \sum_k P_k \Delta_k \Rightarrow \frac{1}{\Delta} [P_1 \Delta_1 + P_2 \Delta_2]$$

$$T = \frac{G_1 G_2 G_3 + G_1 G_4}{1 + G_1 G_2 G_3 + G_1 G_2 H_1 + G_2 G_3 H_2 + G_1 G_4 + G_4 H_2}$$

Problem 5

Construct a signal flow graph for armature Controlled dc motor.

Sol The differential equations governing the armature Controlled dc motor are

$$V_a = i_a R_a + L_a \frac{di_a}{dt} + e_b$$

$$T = k_t i_a$$

$$T = J \frac{d\omega}{dt} + B\omega$$

$$e_b = k_b \omega$$

$$\omega = \frac{d\theta}{dt}$$

Laplace Transform of the above equations are

$$V_a(s) = I_a(s) R_a + L_a s I_a(s) + E_b(s) \quad \text{---(1)}$$

$$T(s) = k_t I_a(s) \quad \text{---(2)}$$

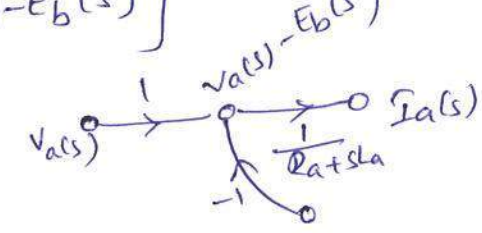
$$T(s) = J s \omega(s) + B\omega(s) \quad \text{---(3)}$$

$$E_b(s) = k_b \omega(s) \quad \text{---(4)}$$

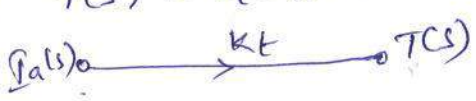
$$\omega(s) = s\theta(s) \quad \text{---(5)}$$

from Eq(1) $V_a(s) - E_b(s) = I_a(s)[R_a + sL_a]$

$$\therefore I_a(s) = \frac{1}{R_a + sL_a}[V_a(s) - E_b(s)]$$



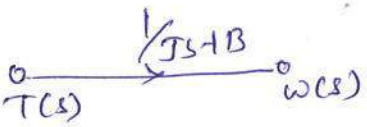
from eq(2) $T(s) = k_t I_a(s)$



from (3)

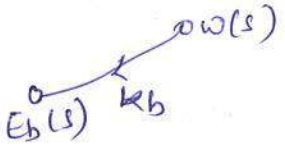
$$T(s) = \omega(s)\,[Js+B]$$

$$\therefore \quad \omega(s) = \frac{1}{Js+B}\,T(s)$$



from (4)

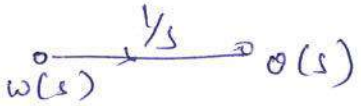
$$E_b(s) = k_b\,\omega(s)$$



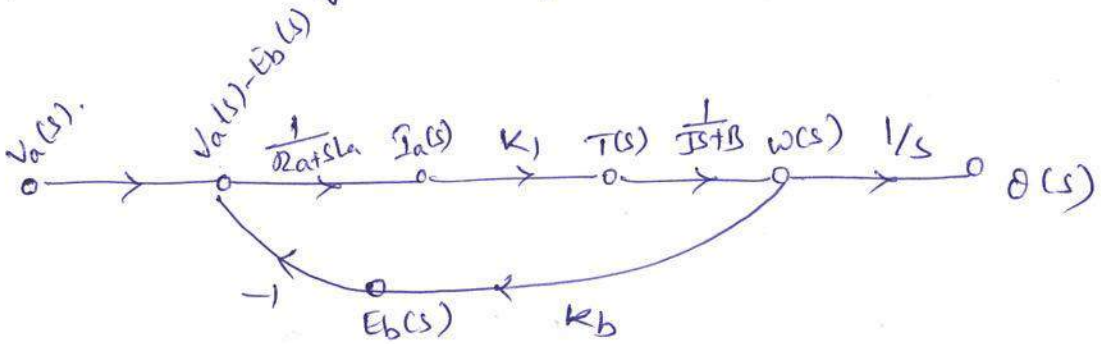
from (5)

$$\omega(s) = s\,\theta(s)$$

$$\theta(s) = \frac{1}{s}\,\omega(s)$$



The overall signal flow graph.



Problem 6

Construct signal flow graph for field Controlled dc motor.

Sol The differential equation governing the system are

$$V_f = i_f R_f + L_f \frac{di_f}{dt}$$

$$T = k_t\, i_f$$

$$T = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt}.$$

Laplace Transform of above equations are.

$$V_f(s) = I_f(s)\,R_f + L_f\,s\,I_f(s) \quad \text{——(1)}$$

$$T(s) = k_t\,I_f(s) \quad \text{——(2)}$$

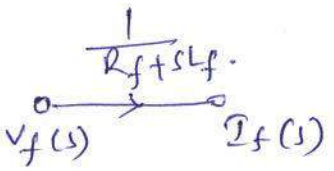
$$T(s) = J\,s^2\theta(s) + B\,\theta(s) \quad \text{——(3)}$$

from eq (1)

$$V_f(s) = I_f(s) R_f + L_f s I_f(s)$$

$$= I_f(s) (R_f + sL_f)$$

$$I_f(s) = \frac{1}{R_f + sL_f} [V_f(s)]$$



from eq (2)

$$T(s) = K_t I_f(s)$$

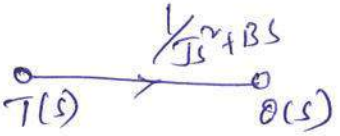
$$\frac{T}{I_f} = K_t$$

from eq (3)

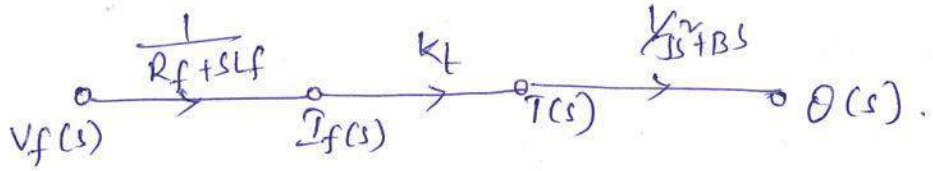
$$T(s) = Js^2\theta(s) + Bs\,\theta(s)$$

$$= \theta(s) [Js^2 + Bs]$$

$$\theta(s) = \frac{1}{Js^2 + Bs} T(s)$$



∴ The overall signal flow graph is.



Control Systems

Unit-I Notes

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Asst. prof
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1.16 SHORT QUESTION AND ANSWER

Q1.1 ***What is system?***

When a number of elements or components are connected in a sequence to perform a specific function, the group thus formed is called a system.

Q1.2 ***What is control system?***

A system consists of a number of components connected together to perform a specific function. In a system when the output quantity is controlled by varying the input quantity, then the system is called control system.

The output quantity is called controlled variable or response and input quantity is called command signal or excitation.

Q1.3 ***What are the two major type of control systems?***

The two major type of control systems are open loop and closed loop systems.

Q1.4 *Define open loop system.*

The control system in which the output quantity has no effect upon the input quantity are called open loop control system. This means that the output is not fedback to the input for correction.

Q1.5 *Define closed loop system.*

The control systems in which the output has an effect upon the input quantity so as to maintain the desired output value are called closed loop control systems.

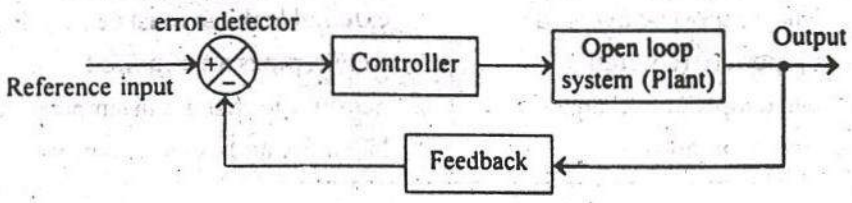
Q1.6 ***What is feedback? What type of feedback is employed in control system?***

The feedback is a control action in which the output is sampled and a proportional signal is given to input for automatic correction of any changes in desired output.

Negative feedback is employed in control system.

Q1.7 ***What are the components of feedback control system?***

The components of feedback control system are plant, feedback path elements, error detector and controller.



Q1.8 Why is negative feedback invariably preferred in a closed loop system?

The negative feedback results in better stability in steady state and rejects any disturbance signals. It also has low sensitivity to parameter variations. Hence negative feedback is preferred in closed loop systems.

Q1.9 What are the characteristics of negative feedback?

The characteristics of negative feedback are as follows :

(i) accuracy in tracking steady state value
(ii) rejection of disturbance signals
(iii) low sensitivity to parameter variations
(iv) reduction in gain at the expense of better stability.

Q1.10. What is the effect of positive feedback on stability?

The positive feedback increases the error signal and drives the output to instability. But sometimes the positive feedback is used in minor loops in control systems to amplify certain internal signals or parameters.

Q1.11. Distinguish between open loop and closed loop system.

Open loop	Closed loop
(i) Inaccurate & unreliable	(i) Accurate & reliable
(ii) Simple and economical	(ii) Complex and costlier
(iii) The changes in output due to external disturbances are not corrected automatically.	(iii) The changes in output due to external disturbances are corrected automatically.
(iv) They are generally stable.	(iv) Great efforts are needed to design a stable system.

12 What is servomechanism?

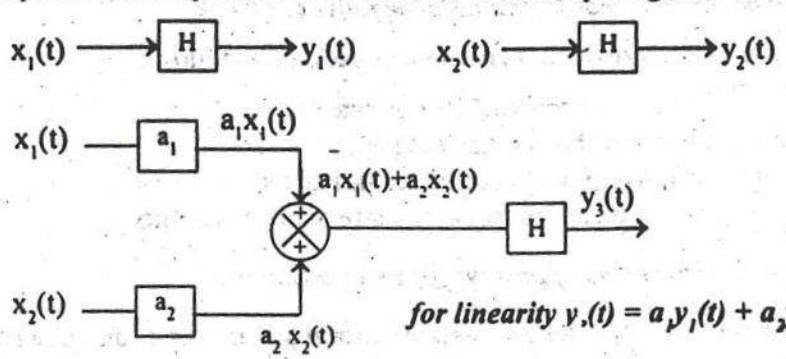
The servomechanism is a feedback control system in which the output is mechanical position (or time derivatives of position e.g. velocity and acceleration).

State the principle of homogenity (or) State the principle of superposition.

The principle of superposition and homogenity states that if the system has responses $y_1(t)$ and $y_2(t)$ for the inputs $x_1(t)$ and $x_2(t)$ respectively then the system response to the linear combination of these input $a_1x_1(t) + a_2x_2(t)$ is given by linear combination of the individual outputs $a_1y_1(t) + a_2y_2(t)$, where a_1 and a_2 are constants.

Q1.14 *Define linear system.*

A system is said to be linear if it obeys the principle of superposition and homogenity. The principle of superposition states that the response of a system to a weighed sum of signals is equal to the corresponding weighed sum of the responses of the system to each of the individual input signals.



for linearity $y_3(t) = a_1 y_1(t) + a_2 y_2(t)$.

Q1.15 *What is time invariant system?*

A system is said to be time invariant if its input output characteristics do not change with time. A linear time invariant system can be represented by constant coefficient differential equations. (In linear time varying systems the coefficients of the differential equation governing the system are function of time).

Q1.16 *Define transfer function.*

The transfer function of a system is defined as the ratio of Laplace transform of output to Laplace transform of input with zero initial conditions. It is also defined as the Laplace transform of the impulse response of system with zero initial conditions.

Q1.17 *State whether transfer function technique is applicable to non-linear system and whether the transfer function is independent of the input of a system.*

(i) The transfer function technique is not applicable to non-linear system.

(ii) The transfer function of a system is independent of input and depends only on system parameters but the output of a system depends on input.

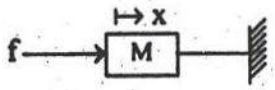
Q1.18 *What are the basic elements used for modelling mechanical translational system?*

The model of mechanical translational system can be obtained by using three basic elements mass, spring and dashpot.

Q1.19 ***Write the force balance equation of ideal mass element.***

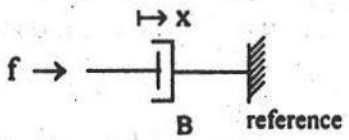
Let a force f be applied to an ideal mass M. The mass will offer an opposing force, f_m which is proportional to acceleration.

Therefore, $f = f_m = M\dfrac{d^2x}{dt^2}$

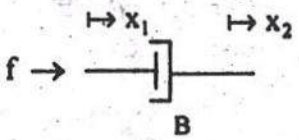


Q1.20 ***Write the force balance equation of ideal dashpot.***

Let a force f be applied to an ideal dashpot, with viscous frictional coefficient B. The dashpot will offer an opposing force, f_b which is proportional to velocity.



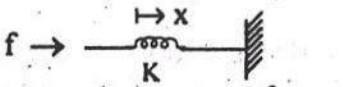
$f = f_b = B\dfrac{dx}{dt}$



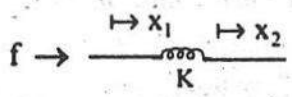
$f = f_b = B\dfrac{d}{dt}(x_1 - x_2)$

Q1.21 ***Write the force balance equation of ideal spring?***

Let a force f be applied to an ideal spring with spring constant K. The spring will offer an opposing force f_k which is proportional to displacement.



$f = f_k = Kx$



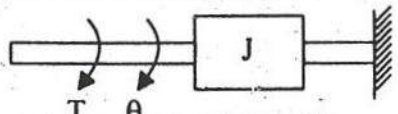
$f = f_k = K(x_1 - x_2)$

Q1.22 ***What are the basic elements used for modelling mechanical rotational system?***

The model of mechanical rotational system can be obtained by using three basic elements mass with Moment of Inertia, J, dash-pot with rotational frictional coefficient, B and torsional spring with stiffness, K.

Q1.23 ***Write the torque balance equation of an ideal rotational mass element.***

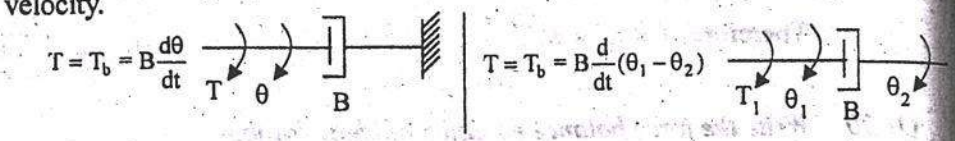
Let a torque T be applied to an ideal mass with moment of inertia, J. The mass will offer an opposing torque T_j which is proportional to angular acceleration.



$T = T_j = J\dfrac{d^2\theta}{dt^2}$

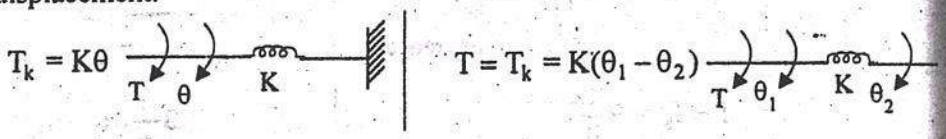
Q1.24 *Write the torque balance equation of an ideal rotational dash-pot.*

Let a torque T be applied to a rotational dash-pot with frictional coefficient B. The dashpot will offer an opposing torque which is proportional to angular velocity.



Q1.25 *Write the torque balance equation of ideal rotational spring.*

Let a torque T be applied to an ideal rotational spring with spring constant K. The spring will offer an opposing torque T_k which is proportional to angular displacement.



Q1.26 *Name the two types of electrical analogous for mechanical system.*

The two types of analogies for the mechanical system are force-voltage and force-current analogy.

Q1.27 *Write the analogous electrical elements in force-voltage analogy for the elements of mechanical translational system.*

Force, f	→	Voltage, e	Mass, M	→	Inductance, L
Velocity, v	→	Current, i	Stiffness, K	→	Inverse of capacitance, 1/C
Displacement, x	→	Charge, q	Newton's second law, $\Sigma f = 0$	→	Kirchoff's voltage law, $\Sigma v = 0$
Frictional coefficient, B	→	Resistance, R			

Q1.28 *Write the analogous electrical elements in force-current analogy for the elements of mechanical translational system.*

Force, f	→	Current, i	Mass, M	→	Capacitance, C
Velocity, v	→	Voltage, v	Stiffness, K	→	Inverse of Inductance, 1/L
Displacement, x	→	Flux, ϕ	Newton's second law, $\Sigma f = 0$	→	Kirchoff's current law, $\Sigma i = 0$
Frictional coefficient, B	→	Conductance, G = 1/ R			

Q1.29 ***Write the analogous electrical elements in torque-voltage analogy for the elements of mechanical rotational system.***

Torque, T	Voltage, e
Angular velocity, ω	Current, i
Angular displacement, θ	Charge, q.
Frictional coefficient, B	Resistance, R
Moment of inertia, J	Inductance, L
Stiffness of spring, K	Inverse of capacitance, 1/C
Newton's second law, $\Sigma T = 0$	Kirchoff 's voltage law, $\Sigma v = 0$.

Q1.30 ***Write the analogous electrical elements in torque-current analogy for the elements of mechanical rotational system.***

Torque, T	Current, i
Angular velocity, ω	Voltage, v
Angular displacement, θ	Flux, ϕ
Frictional coefficient , B	Conductance, G=1/R
Moment of inertia, J	Capacitance, C
Stiffness of spring, K	Inverse of inductance, 1/L
Newton's second law, $\Sigma T = 0$	Kirchoff 's current law, $\Sigma i = 0$

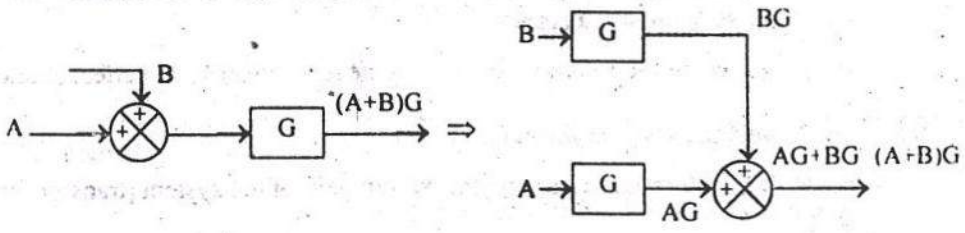
Q1.31 ***What is block diagram? What are the basic components of block diagram?***

A block diagram of a system is a pictorial representation of the functions performed by each component of the system and shows the flow of signals. The basic elements of block diagram are block, branch point and summing point.

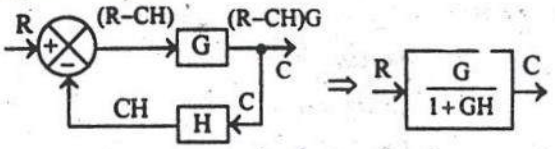
Q1.32 ***What is the basis for framing the rules of block diagram reduction technique?***

The rules for block diagram reduction technique are framed such that any modification made on the diagram does not alter the input output relation.

Q.133 ***Write the rule for moving the summing point ahead of a block.***



Q1.34 *Write the rule for eliminating negative feedback loop.*



Proof

$$C = (R - CH)\,G$$
$$C = RG - CHG$$
$$C + CHG = RG$$
$$C(1+HG) = RG$$
$$\frac{C}{R} = \frac{G}{1+GH}$$

Q1.35 *What is a signal flow graph?*

A signal flow graph is a diagram that represents a set of simultaneous linear algebraic equations. By taking Laplace transform, the time domain differential equations governing a control system can be transferred to a set of algebraic equations in s-domain. The signal flow graph of the system can be constructed using these equations.

Q1.36 *What is transmittance?*

The transmittance is the gain acquired by the signal when it travels from one node to another node in signal flow graph.

Q1.37 *What is sink and source?*

Source is the input node in the signal flow graph and it has only outgoing branches. Sink is a output node in the signal flow graph and it has only incoming branches.

Q1.38 *Define non-touching loop.*

The loops are said to be non-touching if they do not have common nodes.

Q1.39 *What are the basic properties of signal flow graph?*

The basic properties of signal flow graph are

(i) Signal flow graph is applicable to linear systems.

(ii) It consists of nodes and branches. A node is a point representing a variable or signal. A branch indicates functional dependence of one signal on the other.

(iii) A node adds the signals of all incoming branches and transmits this sum to all outgoing branches.

(iv) Signals travel along branches only in the marked direction and is multiplied by the gain of the branch.

(v) The algebraic equations must be in the form of cause and effect relationship.

Q1.40 *Write the Mason's gain formula.*

Mason's gain formula states that the overall gain of the system [transfer function] as follows,

$$\text{Overall gain, } T = \frac{1}{\Delta}\sum_{K} P_K \Delta_K$$

$T = T(s)$ = Transfer function of the system.

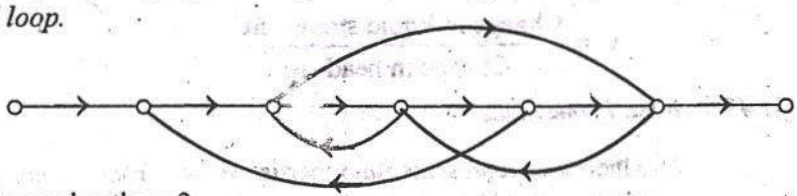
K = Number of forward paths in the signal flow graph.

P_K = Forward path gain of Kth forward path.

$$\Delta = 1 - \begin{bmatrix}\text{sum of individual}\\ \text{loop gains}\end{bmatrix} + \begin{bmatrix}\text{sum of gain products of all possible}\\ \text{combinations of two non - touching loops}\end{bmatrix} - \begin{bmatrix}\text{sum of gain products of all possible}\\ \text{combinations of three non - touching loops}\end{bmatrix} + \ldots\ldots$$

$$\Delta_K = \begin{bmatrix}\Delta \text{ for that part of the graph which}\\ \text{is not touching } K^{th} \text{ forward path}\end{bmatrix}$$

Q1.41 For the given signal flow graph, identify the number of forward path and number of individual loop.



Number of forward paths = 2

Number of individual loops = 4

Q1.42 What are the basic elements of thermal system.

The basic elements of thermal system are thermal resistance and thermal capacitance.

Q1.43 Define thermal resistance.

The thermal resistance for heat transfer between two substances is defined as the ratio of change in temperature and change in heat flow rate.

$$\text{Thermal resistance, } R = \frac{\text{Change in temperature, } °C}{\text{Change in heat flow rate, Kcal / sec}}$$

Q1.44 Define thermal capacitance.

Thermal capacitance is defined as the ratio of change in heat stored and change in temperature.

$$\text{Thermal capacitance, } C = \frac{\text{Change in heat stored, Kcal}}{\text{Change in temperature, } °C}$$

Q1.45 ***Mention the electrical analogous of simple thermal system.***

The electrical analogous of simple I[st] order thermal system is R-C parallel circuit.

Q1.46 ***What are the basic elements of hydraulic system.***

The basic elements of hydraulic system are resistance, capacitance and inductance.

Q1.47 ***Define hydraulic resistance.***

The resistance for liquid flow is defined as the change in the level difference necessary to cause the unit change in flow rate.

$$R = \frac{\text{Change in level difference, m}}{\text{Change in flow rate, } m^3/\text{sec}}$$

Q1.48 ***Define hydraulic capacitance.***

The capacitance C of tank is defined to be the change in quantity of stored liquid necessary to cause a unit change in head (height).

$$C = \frac{\text{Change in liquid stored, } m^3}{\text{Change in head, m}}$$

Q1.49 ***What is inertance?***

The inertance represents fluid inertia derived from the inertial forces required to accelerate a fluid in a pipe. It is a energy storing element. The energy storage due to inertance is negligible compared to capacitance element.

Q1.50 ***What are the basic elements of pneumatic systems?***

The basic elements of pneumatic systems are pneumatic resistance and pneumatic capacitance.

Q1.51 ***Define pneumatic resistance.***

The gas flow resistance, R may be defined as follows

$$R = \frac{\text{Change in gas pressure difference, } N/m^2}{\text{Change in gas flow rate, } m^3/\text{sec}}$$

Q1.52 ***Define pneumatic capacitance.***

The capacitance of the pressure vessel may be defined by

$$C = \frac{\text{Change in gas stored, } m^3}{\text{Change in gas pressure, } N/m^2}$$

1.17 EXERCISES

E1.1 For the mechanical system shown in fig E1.1 derive the transfer function. Also draw the force-voltage and force-current analogous circuits.

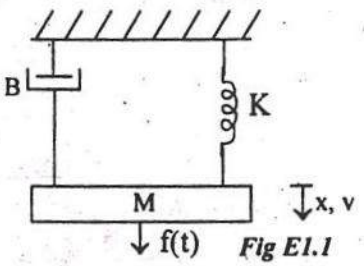


Fig E1.1

E1.2 For the mechanical system shown in fig E1.2 draw the force-voltage and force-current analogous circuits.

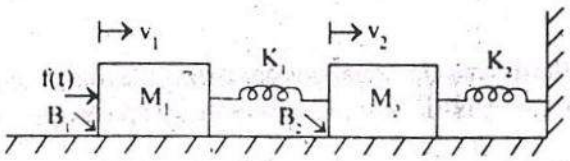


Fig E1.2

E1.3 Write the differential equations governing the mechanical system shown in fig E1.3. Also draw the force-voltage and force-current analogous circuit.

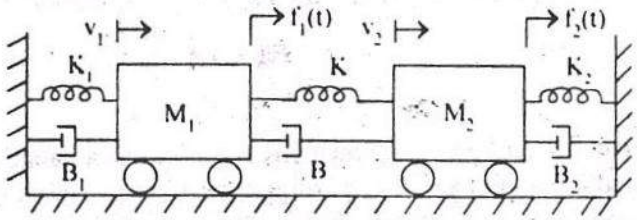


Fig E1.3

E1.4 Write the differential equations governing the mechanical system shown fig E1.4. Draw the force-voltage and force-current analogous circuits.

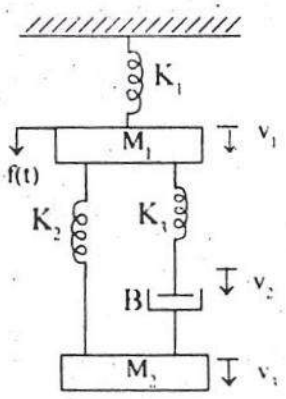


Fig E1.4

E1.5 Consider the mechanical translational system shown in fig E1.5, Draw(a) force-voltage and (b) force-current analogous circuits.

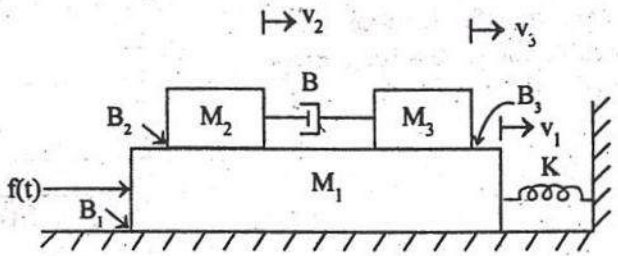


Fig E1.5

E1.6 Write the differential equations governing the rotational mechanical system shown in fig E1.6. Also draw the torque-voltage and torque-current analogous circuits.

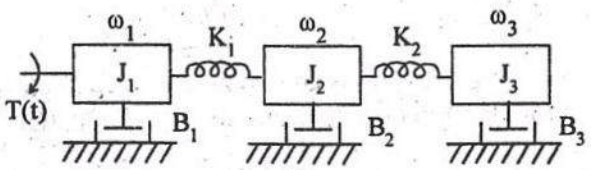


Fig E 1.6

E1.7 In an electrical circuit the elements resistance, capacitance and inductance are connected in parallel across the voltage source E as shown in fig E1.7, Draw(a) Translation mechnical analogous system (b) Rotational mechanical analogous system.

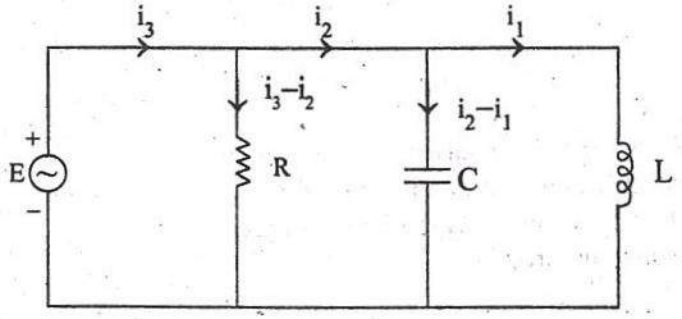


Fig E1.7.

E1.8 Consider the block diagram shown in fig E1.8. Using the block diagram reduction technique, find C/R

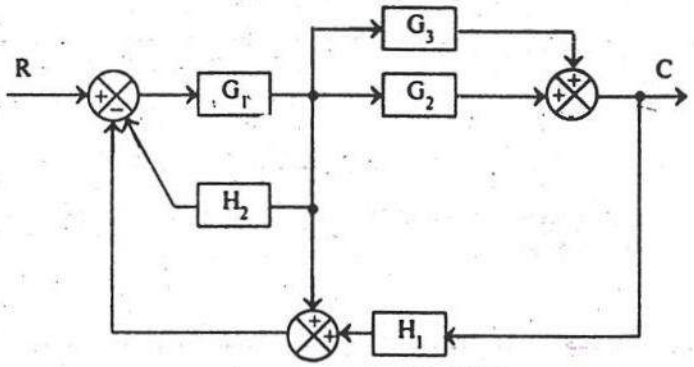


Fig E1.8

E1.9 Consider the block diagram shown in fig E1.9. Using the block diagram reduction technique, obtain C/R.

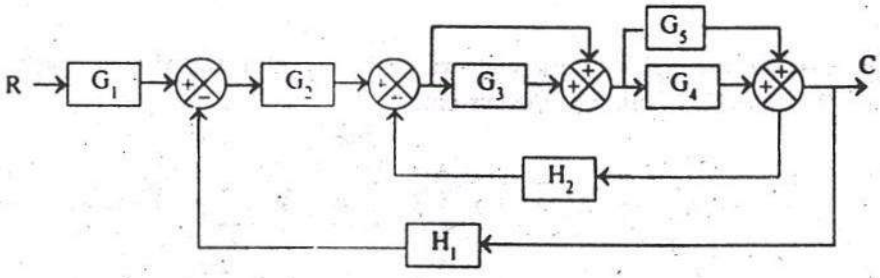


Fig E1.9

E.1.10 For the block diagram shown in fig E1.10. obtain C/R.

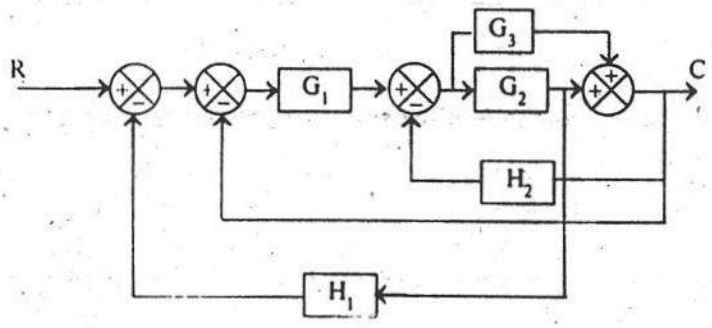


Fig E1.10

E1.11 For the block diagram shown in fig E1.11 obtain C/R

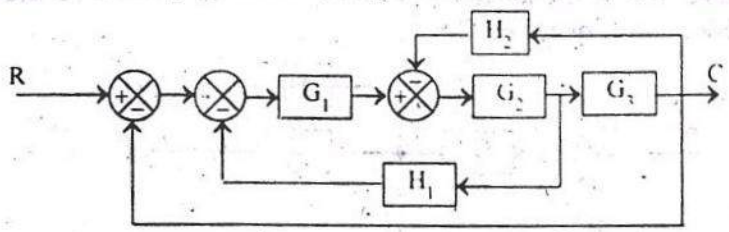


Fig E1.11

E1.12 Consider the system shown in fig E1.12. obtain the transfer function by use of Mason's gain formula.

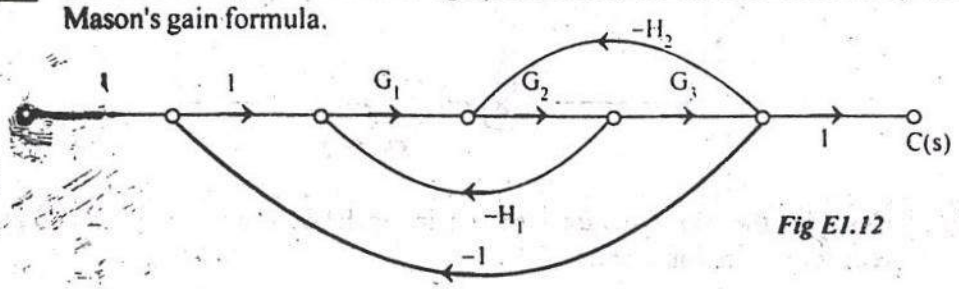


Fig E1.12

E1.13 Convert the block diagram shown in fig E1.13 to signal flow graph and find the transfer function of the system.

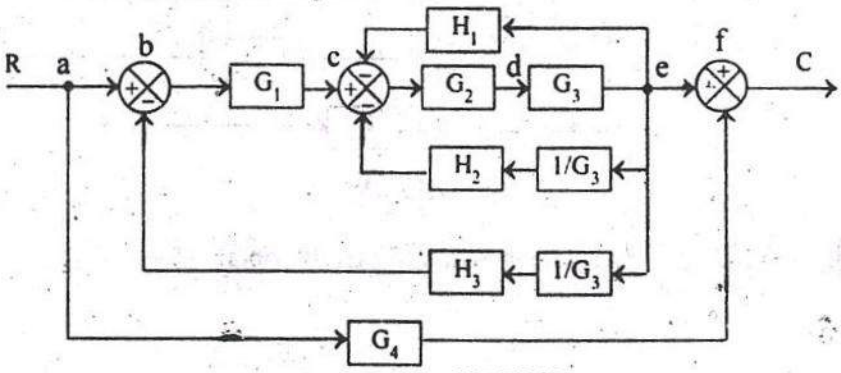


Fig E1.13

E1.14 Find the transfer function of the system whose signal flow graph is shown in fig E1.14.

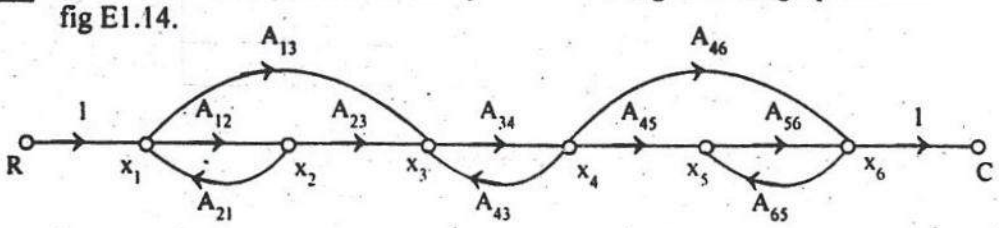


Fig E1.14

E1.15 Find the transfer function of the system having signal flow graph as shown in fig E1.15. **15**

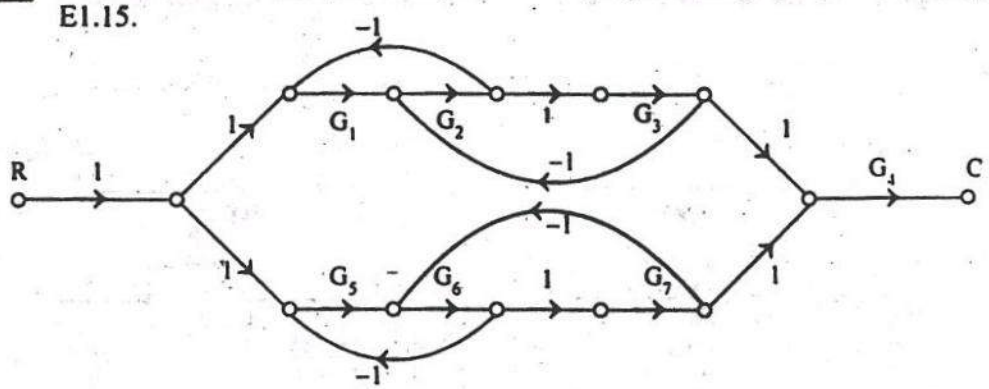


Fig E1.15

E1.16 Consider the signal flow graph shown fig E1.16. Find $\frac{X_8}{X_1}$

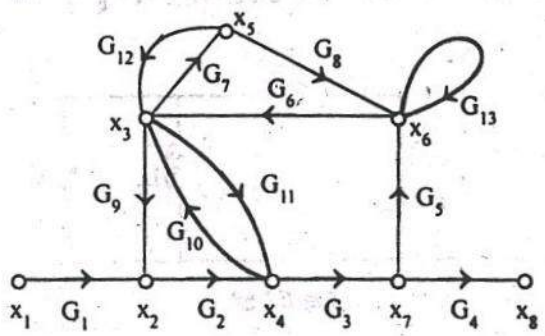


Fig E1.16

E1.17 Consider the signal flow graph shown in fig E1.17 obtain $\frac{x_8}{x_1}$ and $\frac{x_8}{x_2}$

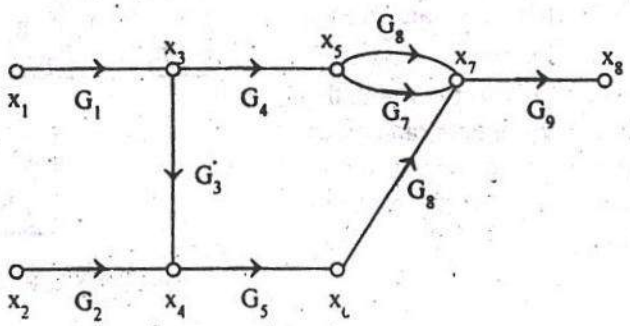


Fig E1.17

160 **E1.18** Find the transfer functions of the networks shown in fig E1.18.

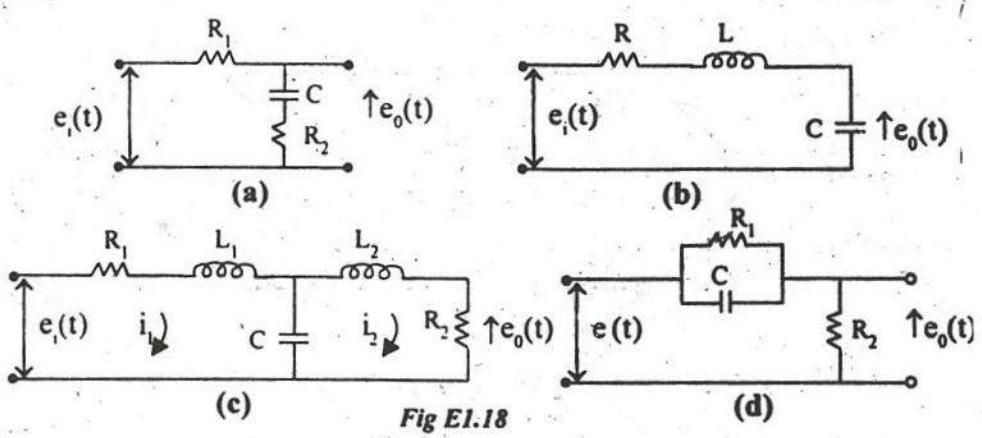


Fig E1.18

E1.19 Find the transfer function of the circuit shown in fig E1.19

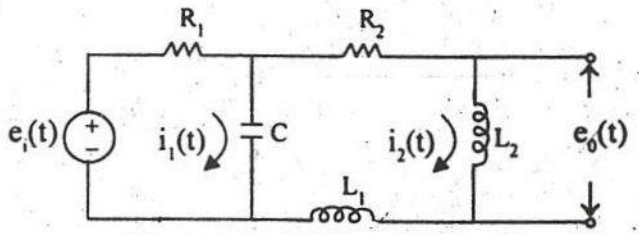


Fig E1.19

E1.20 A process plant consists of two tanks of capacitances C_1 and C_2 respectively. If the flow rate into the top tank is Q_1, find the transfer function relating this flow with the liquid level in the bottom tank. Each tank has a resistance R in its outlet pipe. Assume tanks to be non-interacting.

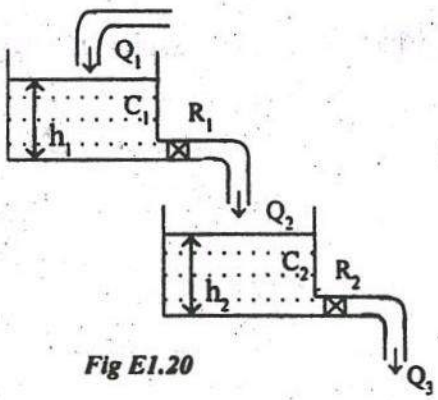
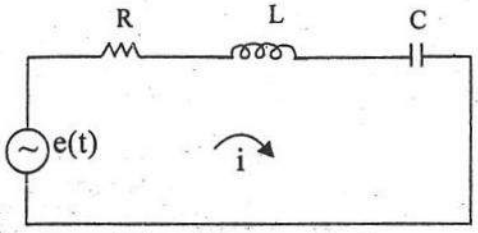


Fig E1.20

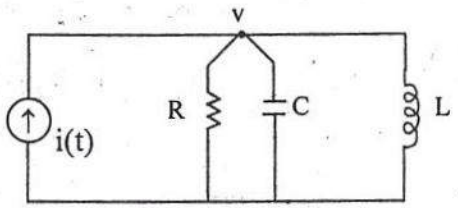
ANSWER FOR EXERCISE PROBLEMS

E1.1 The transfer function is $\frac{X(s)}{F(s)} = \frac{1}{(Ms^2 + Bs + K)}$



$f(t) \rightarrow e(t)$ $\quad M \rightarrow L$
$v \rightarrow i$ $\quad B \rightarrow R$
$K \rightarrow 1/C$

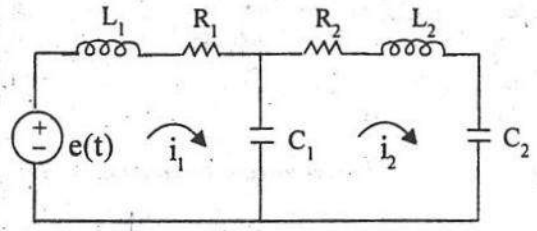
Force-voltage analogous circuit



$f(t) \rightarrow i(t)$ $\quad M \rightarrow C$
$v \rightarrow v$ $\quad B \rightarrow 1/R$
$K \rightarrow 1/L$

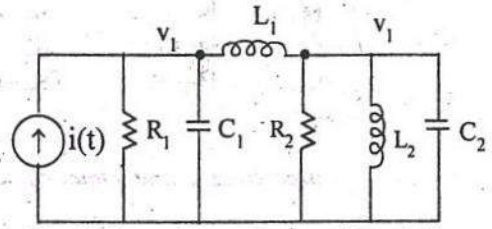
Force-current analogous circuit

E1.2



$f(t) \rightarrow e(t)$ $\quad M_1 \rightarrow L_1$ $\quad B_2 \rightarrow R_2$
$v_1 \rightarrow i_1$ $\quad M_2 \rightarrow L_2$ $\quad K_1 \rightarrow 1/C_1$
$v_2 \rightarrow i_2$ $\quad B_1 \rightarrow R_1$ $\quad K_2 \rightarrow 1/C_2$

Force-voltage analogous circuit



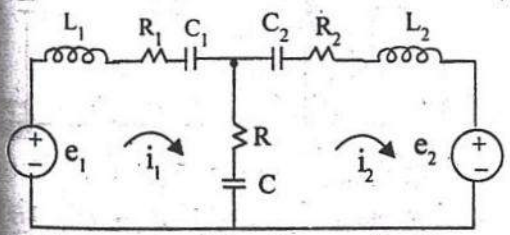
$f(t) \rightarrow i(t)$ $\quad M_1 \rightarrow C_1$ $\quad B_2 \rightarrow 1/R_2$
$v_1 \rightarrow v_1$ $\quad M_2 \rightarrow C_2$ $\quad K_1 \rightarrow 1/L_1$
$v_2 \rightarrow v_2$ $\quad B_1 \rightarrow 1/R_1$ $\quad K_2 \rightarrow 1/L_2$

Force-current analogous circuit

E1.3

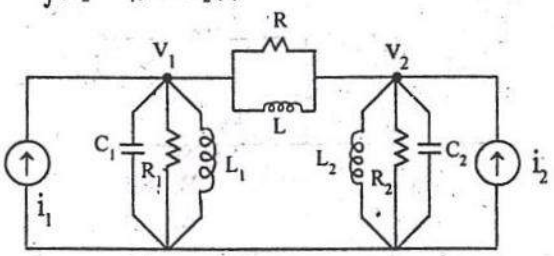
$$M_1\frac{dv_1}{dt} + B_1v_1 + B(v_1 - v_2) + K_1\int v_1dt + K\int(v_1 - v_2)dt = f_1(t)$$

$$M_2\frac{dv_2}{dt} + B_2v_2 + B(v_2 - v_1) + K_2\int v_2dt + K\int(v_2 - v_1)dt = f_2(t)$$



$f_1 \rightarrow e_1$ $\quad M_1 \rightarrow L_1$ $\quad B \rightarrow R$
$f_2 \rightarrow e_2$ $\quad M_2 \rightarrow L_2$ $\quad K_1 \rightarrow 1/C_1$
$v_1 \rightarrow i_1$ $\quad B_1 \rightarrow R_1$ $\quad K_2 \rightarrow 1/C_2$
$v_2 \rightarrow i_2$ $\quad B_2 \rightarrow R_2$ $\quad K \rightarrow 1/C$

Force-voltage analogous circuit



$f_1 \rightarrow i_1$ $\quad M_1 \rightarrow C_1$ $\quad B \rightarrow 1/R$
$f_2 \rightarrow i_2$ $\quad M_2 \rightarrow C_2$ $\quad K_1 \rightarrow 1/L_1$
$v_1 \rightarrow v_1$ $\quad B_1 \rightarrow 1/R_1$ $\quad K_2 \rightarrow 1/L_2$
$v_2 \rightarrow v_2$ $\quad B_2 \rightarrow 1/R_2$ $\quad K \rightarrow 1/L$

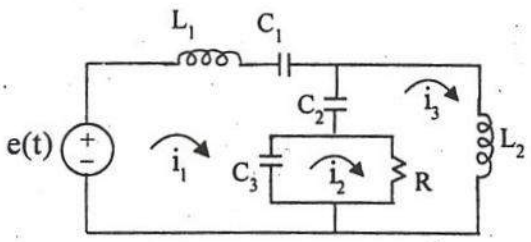
Force-current analogous circuit

E1.4

$$M_1\frac{dv_1}{dt} + K_1\int v_1 dt + K_2\int (v_1 - v_3)dt + K_3\int (v_1 - v_2)dt = f(t)$$

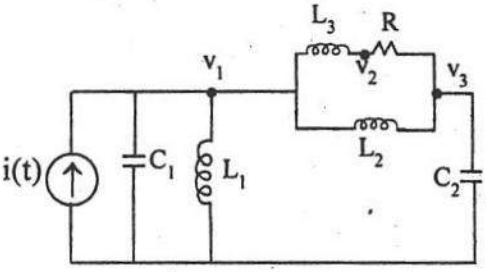
$$K_3\int (v_2 - v_1)dt + B(v_2 - v_3) = 0$$

$$M_2\frac{dv_3}{dt} + B(v_3 - v_2) + K_2\int (v_3 - v_1)dt = 0$$



$f(t)\rightarrow e(t)$ $M_1\rightarrow L_1$ $K_1\rightarrow 1/C_1$
$v_1\rightarrow i_1$ $M_2\rightarrow L_2$ $K_2\rightarrow 1/C_2$
$v_2\rightarrow i_2$ $B\rightarrow R$ $K_3\rightarrow 1/C_3$
$v_3\rightarrow i_3$

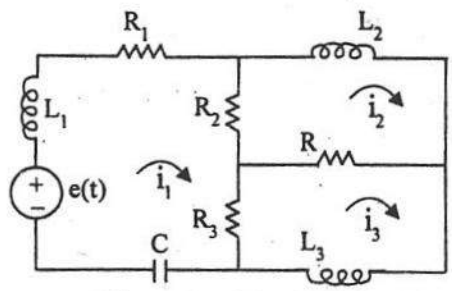
Force-voltage analogous circuit



$f(t)\rightarrow i(t)$ $M_1\rightarrow C_1$ $K_1\rightarrow 1/L_1$
$v_1\rightarrow v_1$ $M_2\rightarrow C_2$ $K_2\rightarrow 1/L_2$
$v_2\rightarrow v_2$ $B\rightarrow 1/R$ $K_3\rightarrow 1/L_3$
$v_3\rightarrow v_3$

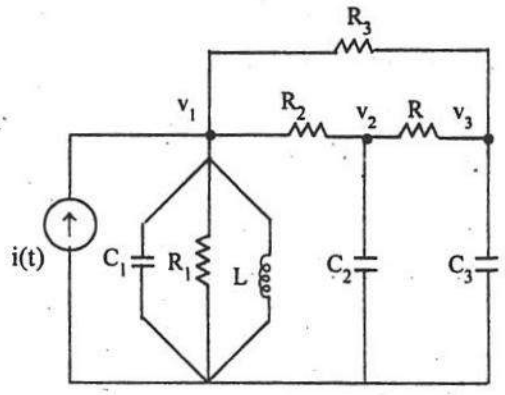
Force-current analogous circuit

E1.5



$f(t)\rightarrow e(t)$ $M_1\rightarrow L_1$ $B_1\rightarrow R_1$
$v_1\rightarrow i_1$ $M_2\rightarrow L_2$ $B_2\rightarrow R_2$
$v_2\rightarrow i_2$ $M_3\rightarrow L_3$ $B_3\rightarrow R_3$
$v_3\rightarrow i_3$ $B\rightarrow R$ $K\rightarrow 1/C$

Force voltage-analogous circuit



$f(t)\rightarrow i(t)$ $M_1\rightarrow C_1$ $B_1\rightarrow 1/R_1$
$v_1\rightarrow v_1$ $M_2\rightarrow C_2$ $B_2\rightarrow 1/R_2$
$v_2\rightarrow v_2$ $M_3\rightarrow C_3$ $B_3\rightarrow 1/R_3$
$v_3\rightarrow v_3$ $B\rightarrow 1/R$ $K\rightarrow 1/L$

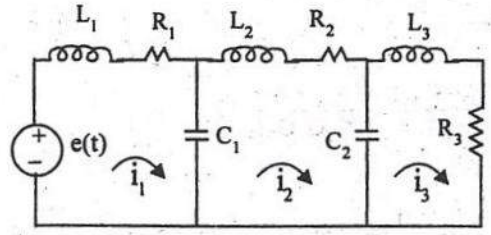
Force-current analogous circuit

E1.6

$$J_1\frac{d\omega_1}{dt}+B_1\omega_1+K_1\int(\omega_1-\omega_2)dt=T(t)$$

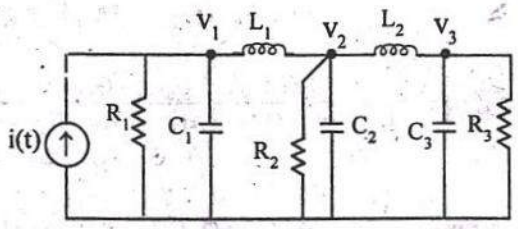
$$J_2\frac{d\omega_2}{dt}+B_2\omega_2+K_1\int(\omega_2-\omega_1)dt+K_2\int(\omega_2-\omega_3)dt=0$$

$$J_3\frac{d\omega_3}{dt}+B_3\omega_3+K_2\int(\omega_3-\omega_2)dt=0$$



$T(t)\rightarrow e(t)$ $J_1\rightarrow L_1$ $B_1\rightarrow R_1$ $K_1\rightarrow 1/C_1$
$\omega_1\rightarrow i_1$ $J_2\rightarrow L_2$ $B_2\rightarrow R_2$ $K_2\rightarrow 1/C_2$
$\omega_2\rightarrow i_2$ $J_3\rightarrow L_3$ $B_3\rightarrow R_3$
$\omega_3\rightarrow i_3$

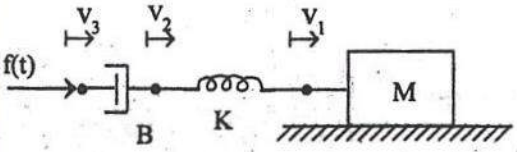
Torque-voltage analogous circuit



$T(t)\rightarrow i(t)$ $J_1\rightarrow C_1$ $B_1\rightarrow 1/R_1$ $K_1\rightarrow 1/L_1$
$\omega_1\rightarrow v_1$ $J_2\rightarrow C_2$ $B_2\rightarrow 1/R_2$ $K_2\rightarrow 1/L_2$
$\omega_2\rightarrow v_2$ $J_3\rightarrow C_3$ $B_3\rightarrow 1/R_3$
$\omega_3\rightarrow v_3$

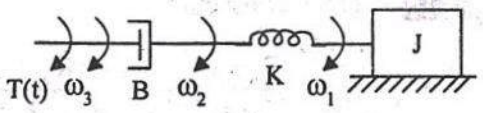
Torque-current analogous circuit

E1.7



$e(t)\rightarrow f(t)$ $R\rightarrow B$
$i_1\rightarrow v_1$ $1/C\rightarrow K$
$i_2\rightarrow v_2$ $L\rightarrow M$
$i_3\rightarrow v_3$

Analogous mechanical translational system



$e(t)\rightarrow T(t)$ $R\rightarrow B$
$i_1\rightarrow \omega_1$ $1/C\rightarrow K$
$i_2\rightarrow \omega_2$ $L\rightarrow J$
$i_3\rightarrow \omega_3$

Analogous mechanical rotational system

E1.8

$$\frac{C}{R}=\frac{G_1G_2+G_1G_3}{1+G_1H_2+G_1+G_1G_2H_1+G_1G_3H_1}$$

E1.9

$$\frac{C}{R}=\frac{G_1G_2(1+G_3)(G_4+G_5)}{[1+(1+G_3)(G_4+G_5)H_2+(1+G_3)(G_4+G_5)G_2H_1]}$$

E1.10

$$\frac{C}{R}=\frac{G_1G_2+G_1G_3}{[1+G_1G_2H_1+G_1G_2+G_1G_3+G_2H_2+G_3H_2]}$$

E1.11 $\dfrac{C}{R}=\dfrac{G_1G_2G_3}{1+G_2G_3H_2+G_1G_2H_1+G_1G_2G_3}$ **E1.12** $\dfrac{C(s)}{R(s)}=\dfrac{G_1G_2G_3}{1+G_1G_2H_1+G_2G_3H_2+G_1G_2G_3}$

E1.13 $\dfrac{G_1G_2G_3+G_4+G_2G_3G_4H_1+G_2G_4H_2+G_1G_2G_4H_3}{1+G_2G_3H_1+G_2H_2+G_1G_2H_3}$

E1.14 $\dfrac{C}{R}=\dfrac{A_{12}A_{23}A_{34}A_{45}A_{56}+A_{13}A_{34}A_{45}A_{56}+A_{12}A_{23}A_{34}A_{46}+A_{13}A_{34}A_{46}}{1-(A_{12}A_{21}+A_{34}A_{43}+A_{56}A_{65})+(A_{12}A_{21}A_{34}A_{43}+A_{12}A_{21}A_{56}A_{65}+A_{34}A_{43}A_{56}A_{65})-(A_{12}A_{21}A_{34}A_{43}A_{56}A_{65})}$

E1.15 $\dfrac{C}{R}=\dfrac{G_1G_2G_3G_4\,(1+G_5G_6+G_6G_7)+G_4G_5G_6G_7(1+G_1G_2+G_2G_3)}{1+G_1G_2+G_2G_3+G_5G_6+G_6G_7+G_1G_2G_5G_6+G_5G_6G_2G_3+G_1G_2G_6G_7+G_2G_3G_6G_7}$

E1.16 $\dfrac{x_8}{x_1}=\dfrac{[G_1G_2G_3G_4]\,[1-(G_7G_{12}+G_6G_7G_8+G_{13})+G_7G_{12}G_{13}]}{1-\{\,G_2G_9G_{10}+G_{10}G_{11}+G_2G_3G_5G_6G_9+G_3G_5G_6G_{11}+G_7G_{12}+G_6G_7G_8+G_{13}\,\}+G_2G_9G_{10}G_{13}+G_{10}G_{11}G_{13}+G_7G_{12}G_{13}}$

E1.17 $\dfrac{x_8}{x_1}=G_1G_4G_6G_9+G_1G_4G_7G_9+G_1G_3G_5G_8G_9$

$\dfrac{x_8}{x_2}=G_2G_5G_8G_9$

E1.18 Transfer function.

a) $\dfrac{E_o(s)}{E_i(s)}=\dfrac{1+s\,R_2\,C}{1+s(R_1+R_2)\,C}$ (b) $\dfrac{E_o(s)}{E_i(s)}=\dfrac{1}{s^2LC+s\,R\,C+1}$

(c) $\dfrac{E_o(s)}{E_i(s)}=\dfrac{s\,R_2\,C}{(s^2L_1C+s\,R_1\,C+1)\,(s^2L_2C+s\,R_2\,C+1)-1}$

(d) $\dfrac{E_o(s)}{E_i(s)}=\dfrac{sR_1R_2C+R_2}{sR_1R_2C+(R_1+R_2)}$

E1.19 $\dfrac{C(s)}{E(s)}=\dfrac{s^2\,L_2\,C}{[s\,R_1\,C+1]\left[s^2(L_1+L_2)\,C+s\,R_2\,C+1\right]-1}$

E1.20 $\dfrac{H_2(s)}{Q_1(s)}=\dfrac{R_2}{(1+T_1s)(1+T_2s)}$ Where $T_1=R_1C_1$ and $T_2=R_2C_2$

2. Time Response Analysis of Standard Test Signals

Time Response:

The time response of the system is the output of the Closed loop system as a function of time, denoted by C(t). The output in s-domain, C(s) is given by the product of the transfer function and input R(s)

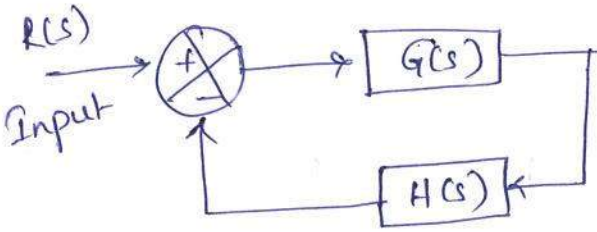


fig: Closed loop System.

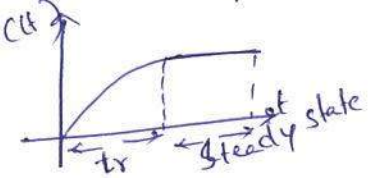
The closed loop transfer function $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$

Response in s-domain $C(s) = R(s)\frac{G(s)}{1+G(s)H(s)}$

Response in time domain $c(t) = L^{-1}[C(s)]$

$= L^{-1}\left[\frac{G(s)}{1+G(s)H(s)}\right]$

The time Response of a Control System Consists of two parts: the transient and steady state Response. The "transient Response" shows the response of the System when the input changes from one state to another. The "steady state response" shows the response to as time, t approaches infinity.



Test Signals:

Test Signals are used to predict the performance of the system. The commonly used test signals are: impulse, step, ramp, acceleration and sinusoidal.

The standard Test signals are:

1a. Step signal b. Unit Step signal.

2 a. Ramp signal b. Unit Ramp signal.

3 a. Parabolic signal b. Unit parabolic signal.

4. Impulse signal

5. Sinusoidal signal.

Step Signal:

The step signal is a signal whose value changes from zero to A at $t=0$ and remains constant at A for $t>0$. The step signal resembles an actual steady input to a system. A special case of a step signal is unit step in which A is unity.

The Mathematical representation of the step signal is:

$$r(t) = A\,u(t)$$

where $u(t) = 1 \; ; \; t \geq 0$

$u(t) = 0 \; ; \; t < 0$

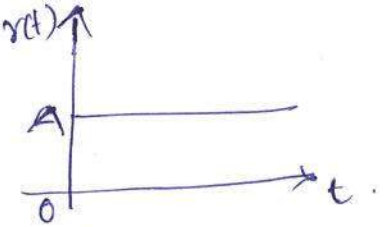


fig: Step signal

Ramp Signal:

The Ramp Signal is a signal whose value increases linearly with time from an initial value of zero at t=0. The Ramp Signal resembles a constant velocity input to the system. A special case of ramp signal is unit ramp signal in which the value of A is unity. The mathematical representation of ramp signal is.

$$r(t) = At \; ; \; t \geq 0$$
$$= 0 \; ; \; t < 0$$

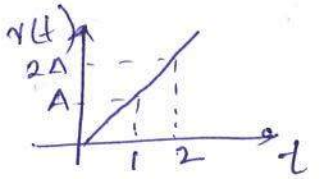


fig: Ramp signal

Parabolic Signal:

In parabolic signal, the instantaneous value varies as square of the time from an initial value of zero at t=0. The sketch of the signal with respect to time resembles a parabola. The parabolic signal resembles a constant acceleration input to the system. A special case of parabolic signal is unit parabolic signal in which A is unity.

The mathematical representation of the parabolic signal is

$$r(t) = \frac{At^2}{2} \; ; \; t \geq 0$$
$$= 0 \; ; \; t < 0$$

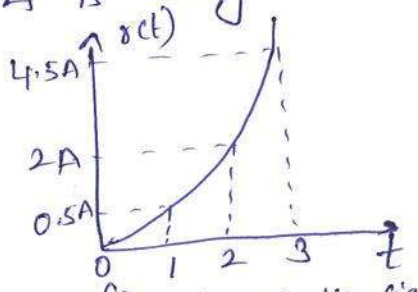


fig: parabolic signal.

Impulse signal:

A signal which is available for very short duration is called impulse signal. An ideal impulse signal is a unit impulse signal which is defined as a signal having zero values at all times except at $t=0$. At $t=0$, the magnitude become infinite.

It is denoted by $\delta(t)$ and Mathematically expressed as

$$\delta(t) = 0 \quad \text{for } t \neq 0$$

$$\text{and } \lim_{t_1 \to 0} \int_{-t_1}^{+t_1} \delta(t)\,dt = 1$$

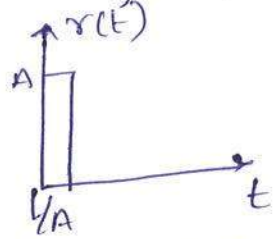


fig: Unit impulse.

Order of a System:

The order of the system is given by the order of the differential equations governing the system. If the system is governed by n^{th} order differential equation, then the system is called n^{th} order system. The order can be determined from the transfer function of the system.

The transfer function of the system can be obtained by taking Laplace transform of the differential equation governing the system and rearranging them as ratio of two polynomials in s.

Transfer function $T(s) = K\frac{P(s)}{Q(s)}$

Where K = Constant

$P(s)$ = Numerator polynomial

$Q(s)$ = Denominator "

The order of the system is given by the maximum power of s in the denominator polynomial, $Q(s)$

If $Q(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n$

then n is the order of the system.

If $n=0$, then the system is zero order system

$n=1$, " " " " first " "

$n=2$ " " " " second " "

The order can be specified for both open loop system and closed loop system. Also the value of n gives the number of poles in the transfer function. Hence the order is given by the number of poles of the transfer function.

Response of first order system for Unit step Input.

The closed loop transfer function of first order system $\frac{C(s)}{R(s)} = \frac{1}{1+Ts}$

If the input is unit step then $r(t)=1 \Rightarrow R(s) = \frac{1}{s}$

∴ The Response in s-domain $C(s) = R(s) \cdot \frac{1}{1+Ts}$

$$C(s) = \left(\frac{1}{1+Ts}\right) R(s)$$

$$= \frac{1}{s} \cdot \frac{1}{1+Ts}$$

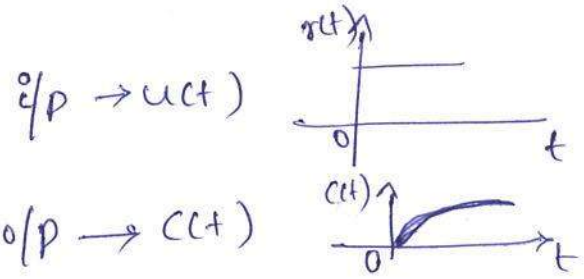
$$= \frac{1}{s} \cdot \frac{1}{T\left(\frac{1}{T}+s\right)}$$

$$= \frac{1}{s} \cdot \frac{1/T}{\left(\frac{1}{T}+s\right)}$$

$$C(s) = \frac{1}{s} - \left(\frac{1}{s+\frac{1}{T}}\right)$$

$$C(t) = L^{-1}\left[\frac{1}{s} - \frac{1}{s+1/T}\right]$$

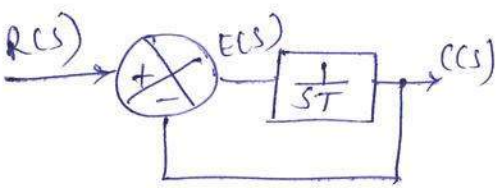
$$= 1 - e^{-1/T \cdot t}, \quad t \geq 0$$



Error $e(t) = r(t) - C(t)$

$= 1 - 1 + e^{-1/T \cdot t}$

$= + e^{-1/T \cdot t}$. → Error in case of the first order system when i/p is unit step signal.



$$T(s) = \frac{1}{1+sT}$$

fig:- first order system.

[∵ R(t) is unit step input
$u(t) = 1 \; ; t \geq 0$]

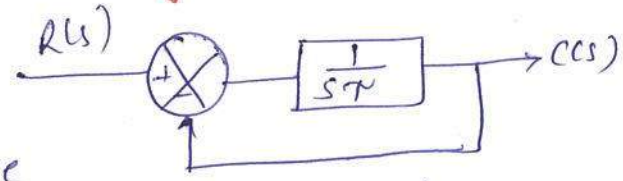
[∵ $R(s) = L.T[r(t)]$
$= L.T[1] = 1/s$].

$\left[\because L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}\right]$

$\left[\because L^{-1}\left[\frac{1}{s}\right] = 1\right]$

Time Response of first order system for unit impulse:

If the input $r(t) = \delta(t)$ ↳ unit Impulse



$$T(s) = \frac{1}{1+sT}$$

$R(s) = L[\delta(t)]$

$= 1$ [∵ Laplace Transform of Impulse is '1'].

[∵ $\frac{G}{1+GH} = \frac{\frac{1}{sT}}{1+\frac{1}{sT} \cdot 1}$

$= \frac{1}{sT+1}$].

$$\frac{C(s)}{R(s)} = \frac{1}{1+sT}$$

$$C(s) = R(s)\left[\frac{1}{1+sT}\right] \qquad [\because R(s) = 1]$$

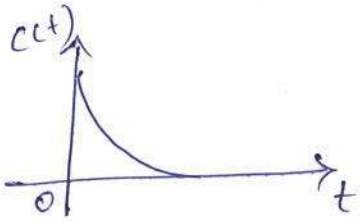
$$= 1 \cdot \left[\frac{1}{1+sT}\right]$$

$$= \frac{1}{1+sT}$$

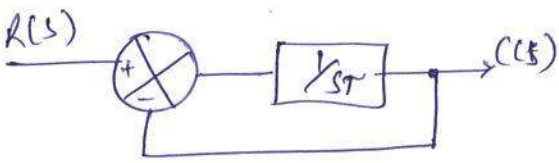
$$c(t) = L^{-1}\left[\frac{1}{1+sT}\right] \Rightarrow L^{-1}\left[\frac{1}{T(1/T+s)}\right] \Rightarrow L^{-1}\left[\frac{\frac{1}{T}}{s+1/T}\right]$$

$$= L^{-1}\left[\frac{1/T}{1/T+s}\right] \Rightarrow L^{-1}\left[\frac{a}{s+a}\right]$$

$$= \frac{1}{T}\left[e^{-\frac{1}{T}t}\right] \quad t \geq 0$$



Time Response of first order system to Unit Ramp Signal :



Input $R(t) = r(t) = t$ → Unit Ramp

$$T(s) = \frac{1}{sT+1}$$

$$R(s) = \frac{1}{s^2} \qquad [\because r(t) = t,\ t \geq 0]$$

$$[\because L[r(t)] = 1/s^2 = R(s)]$$

Transfer function $\frac{C(s)}{R(s)} = \frac{1}{1+sT}$

output $C(s) = R(s)\frac{1}{1+sT}$

$$= \frac{1}{s^2} \cdot \frac{1}{1+sT} \rightarrow \text{Convert easy tranformably equation.}$$

$$= \frac{1-sT}{s^2} + \frac{T^2}{1+sT}$$

$$= \frac{(1-sT)}{s^2} + \frac{T^2}{(1+sT)} \Rightarrow \frac{1-sT}{s^2} + \frac{T^2}{T(s+1/T)}$$

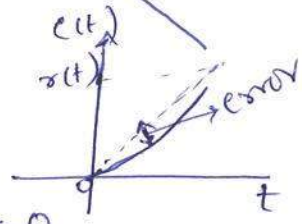
$$= \frac{1-sT}{s^2} + \frac{T^2}{T^2(1/T+s)}$$

$$= \frac{1}{s^2} \cdot \frac{sT+1}{s} \cdot \frac{1}{s+\frac{1}{T}}$$

$$c(t) = L^{-1}\left[\frac{1}{s^2} - \frac{T}{s} + \frac{T}{(1/T+s)}\right] \qquad c(t) = L^{-1}\left[\frac{1}{s^2} - \frac{1}{s+1/T}\right]$$

$$c(t) = t - T + Te^{-\frac{1}{T}t} \qquad = t - e^{-1/T}, \quad t \geq 0$$

$$e(t) = r(t) - c(t) \Rightarrow t - t + T + Te^{-1/T}t$$



Error $e(t) = r - r e^{-k_p t}$

Steady state error $e_{ss} = \lim_{t \to \infty} e(t)$

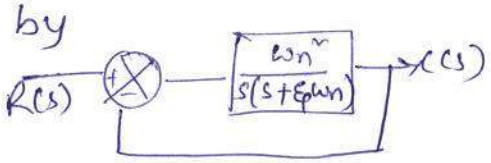
$$= \lim_{t \to \infty} \left[r - r e^{-k_p t} \right]$$

$$= r - 0$$

$$\boxed{e_{ss} = r}$$

Second order System:

The standard form of Closed loop transfer function of second order system is given by



$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Where ω_n = undamped natural frequency rad/sec

ξ = Damping ratio.

The Damping ratio is defined as the ratio of the actual damping to the Critical Damping.

$$\xi = \frac{\text{Actual Damping}}{\text{Critical Damping}}$$

The response $c(t)$ of second order system depends on the value of damping ratio.

Case: $\xi = 0$ — undamped System

$0 < \xi < 1$ — Under damped system

$\xi = 1$ — Critically damped system

$\xi > 1$ — Overdamped system.

The Characteristic equation of the 2nd Order system is $s^2 + 2\xi\omega_n s + \omega_n^2 = 0$

roots of given equation are:

$$s_1 = \frac{-2\xi\omega_n + \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2}$$

$$s_2 = \frac{-2\xi\omega_n \pm \sqrt{4\omega_n^2(\xi^2-1)}}{2}$$

$$s_1, s_2 \Rightarrow -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$$

When $\xi = 0$; $s_1, s_2 = \pm j\omega_n$ → roots are purely Immaginary & the system is undamped

When $\xi = 1$; $s_1, s_2 = -\omega_n$ → roots are real and Equal and the system is Critically damped

When $\xi > 1$; $s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$ → roots are real and unequal and the system is overdamped.

When $0 < \xi < 1$; $s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$

$$= -\xi\omega_n \pm \omega_n\sqrt{(-1)(1-\xi^2)}$$

$$= -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$

$$= -\xi\omega_n \pm j\omega_d$$ → roots are complex Conjugate the system is under damped.

Where $\omega_d = \omega_n\sqrt{1-\xi^2}$

ω_d is called damped frequency rad/sec.

Response of undamped second order system for unit step input:

The standard form of closed loop transfer function of second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

for undamped system, $\xi = 0$

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

When the input is unit step $r(t) = 1$, and $R(s) = 1/s$

$\therefore$ The Response in s-domain $C(s) = R(s) \frac{\omega_n^2}{s^2 + \omega_n^2}$

$$= \frac{1}{s} \cdot \frac{\omega_n^2}{(s^2 + \omega_n^2)}$$

By partial fraction expansion

$$C(s) = \frac{\omega_n^2}{s(s^2 + \omega_n^2)} = \frac{A}{s} + \frac{B}{s^2 + \omega_n^2}$$

A is obtained by multiplying $C(s)$ by s and

Letting $s = 0$.

$$A = C(s)s = \frac{\omega_n^2}{s(s^2 + \omega_n^2)} \times s = \frac{\omega_n^2}{s^2 + \omega_n^2}\Big|_{s=0}$$

$$= \frac{\omega_n^2}{\omega_n^2} = 1.$$

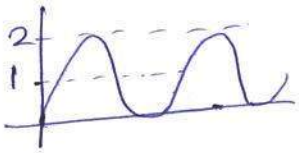
By ~~ab~~ multiplying $C(s)$ by $(s^2 + \omega_n^2)$ and let $s^2 = -\omega_n^2$ (or) $s = j\omega_n$

$$B = C(s)(s^2 + \omega_n^2) = \frac{\omega_n^2}{s(s^2 + \omega_n^2)} \times (s^2 + \omega_n^2)$$

$$= \frac{\omega_n^2}{s}\Big|_{s=j\omega_n}.$$

$= \frac{\omega_n^2}{j\omega_n} = -j\omega_n = -s$

$\therefore C(s) = \frac{A}{s} + \frac{B}{s^2+\omega_n^2} = \frac{1}{s} - \frac{s}{s^2+\omega_n^2}$



The time domain response

$c(t) = L^{-1}[C(s)] = L^{-1}\left[\frac{1}{s} - \frac{s}{s^2+\omega_n^2}\right]$

$= 1 - \cos\omega_n t$

Response of second order system for underdamped Case and when the input is unit step

The standard form of closed loop transfer function of second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

for underdamp system $0 < \xi < 1$ and roots of the characteristic equation are complex.

The roots of denominator are $s = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$

since $\xi < 1$, $\xi^2 < 1$

$\therefore s = -\xi\omega_n \pm \omega_n\sqrt{(-1)(1-\xi^2)}$

$= -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$

The damped frequency oscillation $\omega_d = \omega_n\sqrt{1-\xi^2}$

$\therefore s = -\xi\omega_n \pm j\omega_d$.

The Response in s-domain $C(s) = R(s)\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

for unit step input $r(t) = 1$, $R(s) = 1/s$

$\therefore C(s) = \omega_n^2 / s(s^2+2\xi\omega_n s+\omega_n^2)$.

By partial fraction Expansion $C(s) = \frac{A}{s} + \frac{Bs+C}{s^2+2\xi\omega_n s+\omega_n^2}$

$$A = s\,(C(s))\Big|_{s=0} = \frac{\omega_n^2}{\omega_n^2} = 1$$

for B & C

$$\frac{\omega_n^2}{s(s^2+2\xi\omega_n s+\omega_n^2)} = \frac{1}{s} + \frac{Bs+C}{s^2+2\xi\omega_n s+\omega_n^2}$$

On Cross multiplication and equating the coefficients of like power of s we get.

$$\omega_n^2 = s^2 + 2\xi\omega_n s + \omega_n^2 + (Bs+C)\,s$$

$$\omega_n^2 = s^2 + 2\xi\omega_n s + \omega_n^2 + Bs^2 + Cs$$

equating Coefficients of s^2 we get $0 = 1+B \quad \therefore B = -1$

" " s " $0 = 2\xi\omega_n + C \quad \therefore C = -2\xi\omega_n$

$$C(s) = \frac{1}{s} - \frac{(s+2\xi\omega_n)}{s^2+2\xi\omega_n s+\omega_n^2}$$

Adding and subtracting $\xi^2\omega_n^2$ to the denominator of second term in the above $C(s)$.

$$= \frac{1}{s} - \frac{s+2\xi\omega_n}{s^2+2\xi\omega_n s+\omega_n^2+\xi^2\omega_n^2-\xi^2\omega_n^2}$$

$$= \frac{1}{s} - \frac{s+2\xi\omega_n}{(s^2+2\xi\omega_n s+\xi^2\omega_n^2)+(\omega_n^2-\xi^2\omega_n^2)}$$

$$= \frac{1}{s} - \frac{s+2\xi\omega_n}{(s+\xi\omega_n)^2+\omega_n^2(1-\xi^2)}$$

$$= \frac{1}{s} - \frac{s+2\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} \qquad [\because \omega_d = \sqrt{1-\xi^2}]$$

$$= \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}$$

Multiplying and dividing by ω_d in the third term of the above eq.

$$= \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n}{\omega_d}\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}$$

On taking Inverse Laplace Transform

$$c(t) = 1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi\omega_n}{\omega_d} e^{-\xi\omega_n t}\sin\omega_d t.$$

$$= 1 - e^{-\xi\omega_n t}\left[\cos\omega_d t + \frac{\xi\omega_n}{\omega_n\sqrt{1-\xi^2}}\sin\omega_d t\right]$$

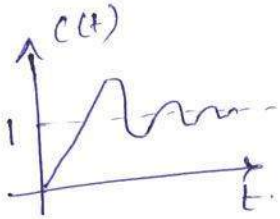
$$= 1 - e^{-\xi\omega_n t}\left[\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right]$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left(\sqrt{1-\xi^2}\cos\omega_d t + \xi\sin\omega_d t\right]$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left(\sin\omega_d t\cdot\xi + \cos\omega_d t\cdot\sqrt{1-\xi^2}\right).$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left(\sin\omega_d t\cos\theta + \cos\omega_d t\sin\theta\right)$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t+\theta)$$



where, $\left(\theta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right)$.

Response of second Order System for Critically damped Case and when input is Unit step:

The standard form of closed loop transfer function of second order system is $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

for Critical damping $\xi = 1$

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\omega_n s+\omega_n^2} = \frac{\omega_n^2}{(s+\omega_n)^2}$$

When input is unit step, $r(t)=1$ and $R(s)=1/s$

$\therefore$ The Response in s-domain

$$C(s) = R(s)\frac{\omega_n^2}{(s+\omega_n)^2}$$

$$= \frac{1}{s}\cdot\frac{\omega_n^2}{(s+\omega_n)^2}$$

By partial fraction expansion, we can write

$$C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{(s+\omega_n)^2} + \frac{C}{s+\omega_n}$$

$$A = sC(s)\Big|_{s=0} = \frac{\omega_n^2}{(s+\omega_n)^2}\Big|_{s=0} = \frac{\omega_n^2}{\omega_n^2} = 1$$

$$B = (s+\omega_n)^2 C(s)\Big|_{s=-\omega_n} = \frac{\omega_n^2}{s}\Big|_{s=-\omega_n} = -\omega_n$$

$$C = \frac{d}{ds}\left[(s+\omega_n)^2 C(s)\right]\Big|_{s=-\omega_n}$$

$$= \frac{d}{ds}\left[\frac{\omega_n^2}{s}\right)\Big|_{s=-\omega_n}$$

$$= \frac{-\omega_n^2}{s^2}\Big|_{s=-\omega_n} = \frac{-\omega_n^2}{(-\omega_n)^2} = -1$$

$$\therefore C(s) = \frac{A}{s} + \frac{B}{(s+\omega_n)^2} + \frac{C}{s+\omega_n}$$

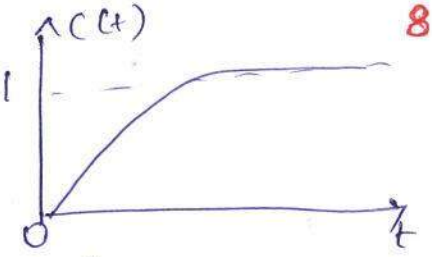
$$= \frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{s+\omega_n}$$

The Response in time domain

$$C(t) = L^{-1}[C(s)] = L^{-1}\left[\frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{s+\omega_n}\right]$$

$$C(t) = 1 - \omega_n t \cdot e^{-\omega_n t} - e^{-\omega_n t}$$

$$C(t) = 1 - e^{-\omega_n t}(1 + \omega_n t)$$



The Response of critically damped second order system have no oscillations.

Response of second order system for overdamped case and when input is unit step:

The standard form of closed loop transfer function of second order system is $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$

for overdamped system $\xi > 1$, the roots of the denominator of transfer function is real and distinct

Let the roots of the denominator be s_a, s_b.

$$s_a, s_b = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1} = -\left[\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}\right]$$

$$\therefore s_1 = \xi\omega_n - \omega_n\sqrt{\xi^2 - 1}$$

$$= \xi\omega_n + \omega_n\sqrt{\xi^2 - 1}$$

Let $s_1 = -s_a$

$s_2 = -s_b$

The closed loop transfer function can be written in terms of s_1 and s_2.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+s_1)(s+s_2)}$$

for unit step input $r(t) = 1$ and $R(s) = 1/s$

$$\therefore C(s) = R(s)\frac{\omega_n^2}{(s+s_1)(s+s_2)}$$

$$= \frac{\omega_n^2}{s(s+s_1)(s+s_2)}$$

By partial fraction expansion we can write

$$C(s) = \frac{\omega_n^2}{s(s+s_1)(s+s_2)} = \frac{A}{s} + \frac{B}{s+s_1} + \frac{C}{s+s_2}$$

$$A = s\,C(s)\Big|_{s=0} = s\,\frac{\omega_n^2}{s(s+s_1)(s+s_2)}\Big|_{s=0}$$

$$= \frac{\omega_n^2}{s_1 s_2}$$

$$= \frac{\omega_n^2}{\left[\xi\omega_n - \omega_n\sqrt{\xi^2-1}\right]\left[\xi\omega_n + \omega_n\sqrt{\xi^2-1}\right]}$$

$$= \frac{\omega_n^2}{\xi^2\omega_n^2 - \omega_n^2(\xi^2-1)} = \frac{\omega_n^2}{\omega_n^2} = 1.$$

$$B = (s+s_1)\,C(s)\Big|_{s=-s_1} = \frac{\omega_n^2}{s(s+s_2)}\Big|_{s=-s_1}$$

$$= \frac{\omega_n^2}{-s_1(-s_1+s_2)}$$

$$= \frac{-\omega_n^2}{s_1\left[-\xi\omega_n + \omega_n\sqrt{\xi^2-1} + \xi\omega_n + \omega_n\sqrt{\xi^2-1}\right]}$$

$$= \frac{-\omega_n^2}{\left[2\omega_n\sqrt{\xi^2-1}\right]s_1} = \frac{-\omega_n}{2\sqrt{\xi^2-1}} \cdot \frac{1}{s_1}$$

$$C = (s)(s+s_2)\Big|_{s=-s_2} = \frac{\omega_n^2}{s(s+s_1)}\Big|_{s=-s_2}$$

$$= \frac{\omega_n^2}{-s_2(-s_2+s_1)}$$

$$= \frac{\omega_n^2}{-s_2\left[-\xi\omega_n - \omega_n\sqrt{\xi^2-1} + \xi\omega_n - \omega_n\sqrt{\xi^2-1}\right]}$$

$$= \frac{\omega_n^2}{\left[2\omega_n\sqrt{\xi^2-1}\right]s_2}$$

$$= \frac{\omega_n}{2\sqrt{\xi^2-1}} \cdot \frac{1}{s_2}$$

$$\therefore C(s) = \frac{1}{s} - \frac{\omega_n}{2\sqrt{\xi^2-1}} \cdot \frac{1}{s_1}\frac{1}{(s+s_1)} + \frac{\omega_n}{2\sqrt{\xi^2-1}} \cdot \frac{1}{s_2}\frac{1}{(s+s_2)}$$

on taking inverse Laplace transform we get.

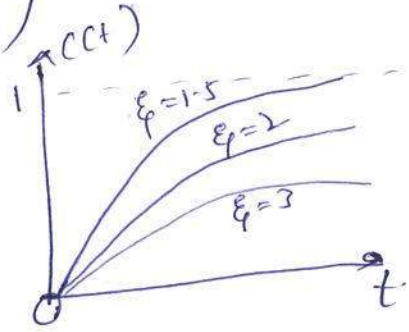
The Response in time domain.

$$C(t) = 1 - \frac{\omega_n}{2\sqrt{\xi^2-1}} \cdot \frac{1}{s_1} e^{-s_1 t} + \frac{\omega_n}{2\sqrt{\xi^2-1}} \frac{1}{s_2} e^{-s_2 t}$$

$$C(t) = 1 - \frac{\omega_n}{2\sqrt{\xi^2-1}} \left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_1 t}}{s_2} \right)$$

Where $s_1 = \xi\omega_n - \omega_n\sqrt{\xi^2-1}$

and $s_2 = \xi\omega_n + \omega_n\sqrt{\xi^2-1}$



The Response of overdamped Second order system has no oscillations but it takes longer time for the response to reach the final steady value.

Initial value Theorem :

If the transform $F(s)$ of an unknown function $f(t)$ is known then it is possible to determine the initial value of $f(t)$. that is the value of $f(t)$ at $t=0^+$

If $f(t)$ & it's first derivative are Laplace transformable Then initial value of $f(t)$ is

$$f(0^+) = \lim_{s\to\infty} sF(s)$$

$$F(s) = \frac{1}{s+1}$$

$$f(0^+) = \lim_{s\to\infty} s\,\frac{1}{(s+1)}$$

$$f(0^+) = 1$$

Final value Theorem :

If the transform $F(s)$ of an unknown function $f(t)$ is known then it is possible to determine the final value of $f(t)$, i.e., the value of $f(t)$ at $t=\infty$

If $f(t)$ & it's 1st derivative are Laplace transformable then the final value of $f(t)$

$$f(\infty) = \lim_{s\to 0} sF(s)$$

$$F(s) = \frac{3s}{2s^2+s}$$

$$f(\infty) = \lim_{s\to 0} s\,\frac{3s}{2s^2+s} \Rightarrow \lim_{s\to 0} \frac{s^2\,3}{s^2(2+1/s)}$$

$$= \frac{3}{\infty} = 0$$

$$\boxed{f(\infty) = 0}$$

$$F(s) = \frac{2s+1}{3s^2+s}$$

$$f(\infty) = \lim_{s\to 0} s\,\frac{(2s+1)}{3s^2+s}$$

$$= \lim_{s\to 0} \frac{s(2s+1)}{s(3s+1)}$$

$$\boxed{f(\infty) = 1}$$

Applications of the initial and final Value theorem

for $X(s) = \frac{N(s)}{D(s)}$

n - order of polynomial N(s)

d - order of polynomial D(s)

Initial value:

$$x(0^+) = \lim_{s \to \infty} sX(s) = \begin{cases} 0 & d > n+1 \\ \text{finite} \neq 0 & d = n+1 \\ \infty & d < n+1 \end{cases}$$

Ex:- $X(s) = \frac{1}{s+1}$ $\quad x(0^+) = ?$

final value:

$$\text{If } x(\infty) = \lim_{s \to 0} sX(s) = 0$$

$$= \lim_{s \to 0} X(s) < \infty \quad \text{, No poles at } s = 0.$$

Time domain specifications:

The transient response characteristics of a Control System to a unit step input is specified interms of the following time domain specifications.

1. Delay time, t_d
2. Rise time, t_r
3. peak time, t_p
4. Maximum overshoot, M_p
5. Settling time, t_s.

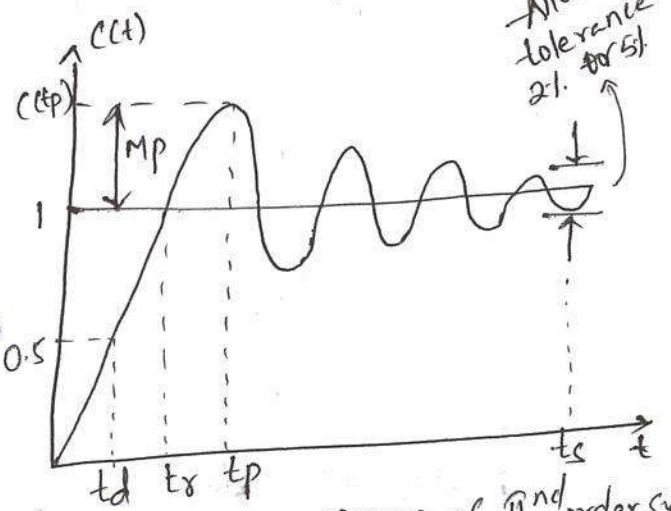


fig: Damped oscillatory response of 2nd order system

1. Delay time:

It is the time taken for response to reach 50% of the final value, for the very first time.

Rise time (t_r):

It is the time taken for response to raise from 0 to 100% for the very first time.

✓ for under damped system, the rise time is calculated from 0 to 100%.

✓ for overdamped system it is the time taken by the response to raise from 10% to 90%.

✓ for critically damped system, it is the time taken for response to raise from 5% to 95%

Peak time (t_p):

It is the time taken for the response to reach the peak value for the very first time (or) It is the time taken for the response to reach the peak overshoot Mp.

Peak overshoot (Mp)

It is defined as the ratio of the maximum peak value measured from final value to the final value.

Let final value $= C(\infty)$

Maximum value $= C(t_p)$

Peak overshoot $Mp = \frac{C(t_p) - C(\infty)}{C(\infty)}$

Peak overshoot, % $Mp = \frac{C(t_p) - C(\infty)}{C(\infty)} \times 100$.

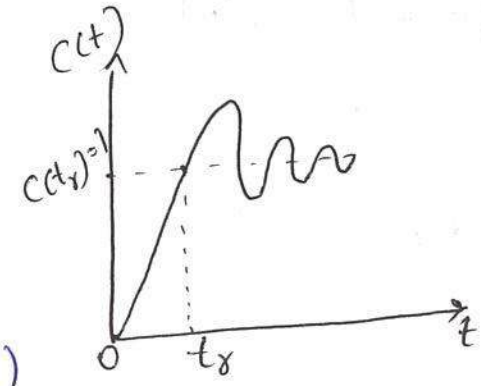
Settling time (t_s):

It is defined as the time taken by the response to reach and stay within a specified error. (usual tolerance is 2% or 5%).

Design specifications for second order system based on the time response:

Rise time:

Response of second order system for underdamped case is given by.



$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

At $t = t_r$, $c(t_r) = 1$

$$\therefore c(t_r) = 1 - \frac{e^{-\xi\omega_n t_r}}{\sqrt{1-\xi^2}} \sin(\omega_d t_r + \theta) = 1$$

$$\therefore -\frac{e^{-\xi\omega_n t_r}}{\sqrt{1-\xi^2}} \sin(\omega_d t_r + \theta) = 0$$

Since $-e^{-\xi\omega_n t} \neq 0$

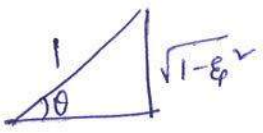
the term, $\sin(\omega_d t_r + \theta) = 0$

$\sin\phi = 0$, when $\phi = 0, \pi, 2\pi, 3\pi \cdots$

$\therefore \omega_d t_r + \theta = \pi$

$\omega_d t_r = \pi - \theta$

$\therefore$ Rise time, $t_r = \frac{\pi - \theta}{\omega_d}$



Right angle triangle

Here $\theta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$

damped frequency of oscillation $\omega_d = \omega_n\sqrt{1-\xi^2}$

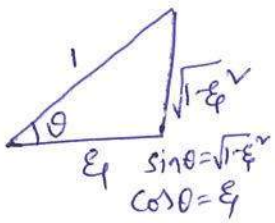
$$\therefore \text{Rise time } t_r = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}}{\omega_n\sqrt{1-\xi^2}}$$

Peak time (t_p)

To find the expression for peak time, t_p, differentiate $C(t)$ with respect to t and equate to 0.

i.e., $\left.\frac{d}{dt}C(t)\right|_{t=t_p} = 0.$

The unit step response of second order system is given by $C(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta)$



Differentiating $C(t)$ with respect to t.

$$\frac{d}{dt}C(t) = \frac{-e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}(-\xi\omega_n)\sin(\omega_d t+\theta) + \left(\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\right)\cos(\omega_d t+\theta)\cdot\omega_d$$

put $\omega_d = \omega_n\sqrt{1-\xi^2}$

$$\frac{d}{dt}C(t) = \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}(\xi\omega_n)\sin(\omega_d t+\theta) - \frac{\omega_n\sqrt{1-\xi^2}}{\sqrt{1-\xi^2}}e^{-\xi\omega_n t}\cos(\omega_d t+\theta)$$

$$= \frac{\omega_n e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\xi\sin(\omega_d t+\theta) - \sqrt{1-\xi^2}\cos(\omega_d t+\theta)\right]$$

$$= \frac{\omega_n}{\sqrt{1-\xi^2}}e^{-\xi\omega_n t}\left[\cos\theta\sin(\omega_d t+\theta) - \sin\theta\cos(\omega_d t+\theta)\right]$$

(from the fig).

$$= \frac{\omega_n}{\sqrt{1-\xi^2}}e^{-\xi\omega_n t}\left[\sin(\omega_d t+\theta)\cos\theta - \cos(\omega_d t+\theta)\sin\theta\right]$$

$$= \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t}\left[\sin\left((\omega_d t+\theta)-\theta\right)\right]$$

$$= \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t}\left[\sin\left((\omega_d t+\theta)-\theta\right)\right]$$

$$= \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin(\omega_d t)$$

at $t=t_p$, $\frac{d}{dt}C(t)=0$.

$$\therefore \frac{d}{dt}C(t)\Big|_{t=t_p} = \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t_p} \sin(\omega_d t_p) = 0.$$

$$\therefore \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t_p} \sin(\omega_d t_p) = 0$$

Since, $e^{-\xi\omega_n t_p} \neq 0$ the term $\sin(\omega_d t_p)=0$.

$\sin\phi = 0$, when $\phi = 0, \pi, 2\pi, 3\pi$

$\therefore \omega_d t_p = \pi$

$\therefore$ Peak time $t_p = \frac{\pi}{\omega_d}$

The damped frequency of oscillation, $\omega_d = \omega_n\sqrt{1-\xi^2}$

$\therefore$ peak time $t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$

Peak Overshoot:

$$\%\text{ Peak overshoot } \%M_p = \frac{C(t_p)-C(\alpha)}{C(\alpha)} \times 100$$

where $C(t_p)$ = peak response at $t=t_p$

$C(\alpha)$ = final steady state value

The unit step response of second order system is given by $C(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$

At $t = \alpha$, $C(t) = C(\alpha) = 1 - \frac{e^{-\alpha}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$

$= 1 - 0 = 1$

At $t = t_p$ $C(t) = C(t_p) = 1 - \frac{e^{-\xi\omega_n t_p}}{\sqrt{1-\xi^2}} \sin(\omega_d t_p + \theta)$

Put $t_p = \frac{\pi}{\omega_d}$

$$C(t_p) = 1 - \frac{e^{-\xi\omega_n \frac{\pi}{\omega_d}}}{\sqrt{1-\xi^2}} \sin\left(\omega_d \frac{\pi}{\omega_d} + \theta\right)$$

The dampded frequency of oscillation $\omega_d = \omega_n\sqrt{1-\xi^2}$

$$C(t_p) = 1 - \frac{e^{-\xi\omega_n \frac{\pi}{\omega_n\sqrt{1-\xi^2}}} \sin(\pi + \theta)}{\sqrt{1-\xi^2}}$$

$= 1 + \frac{e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} \sin\theta$ $(\because \sin(\pi+\theta) = -\sin\theta)$.

$$C(t_p) = 1 + \frac{e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} \sqrt{1-\xi^2}$$

$$C(t_p) = 1 + e^{-\xi\pi/\sqrt{1-\xi^2}}$$

$$\therefore \%M_p = \frac{c(t_p) - C(\alpha)}{C(\alpha)} \times 100$$

$$\% M_p = \frac{1 + e^{-\xi\pi/\sqrt{1-\xi^2}} - 1}{1} \times 100.$$

$$\therefore \% M_p = 1 + e^{-\xi\pi/\sqrt{1-\xi^2}} \times 100$$

Settling time (t_s)

The Response of the second order system has two Components, they are:

1. Decaying exponential Component, $\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}$
2. Sinusoidal Component, $\sin(\omega_d t + \theta)$

In this the decaying exponential term dampens (or) reduces the oscillations produced by sinusoidal Component. Hence the settling time is decided by the exponential Component.

from the exponential Component can found the settling time.

for 2% tolerence error band at $t = t_s$, $\frac{e^{-\xi\omega_n t_s}}{\sqrt{1-\xi^2}} = 0.02$

for least values of ξ, $e^{\xi\omega_n t} = 0.02$

On taking natural Logerthemic we get.

$-\xi\omega_n t_s = \ln(0.02)$

$-\xi\omega_n t_s = -4$

$\therefore$ Settling time $t_s = \frac{4}{\xi\omega_n} = 4T$; for 2% error.

where $T = \frac{1}{\xi\omega_n}$ = Time Constant of the system.

for 5% error, $e^{\xi\omega_n t} = 0.05$

On taking natural logarithm we get

$-\xi\omega_n t_s = \ln(0.05)$

$-\xi\omega_n t_s = -3$

$\therefore$ Settling time $t_s = \frac{3}{\xi\omega_n} = 3T$; for 5% error.

percentage error Settling time $t_s = \ln(\%\text{error})/\xi\omega_n$.

Problem 1:

Measurement Conducted on a Servo mechanism show the System response to be $C(t) = 1 + 0.2\,e^{-60t} - 1.2\,e^{-10t}$ When subject to a unit step input. Obtain an expression for Closed loop transfer function. Determine the undamped natural frequency and damping ratio.

Sol: Given that $C(t) = 1 + 0.2\,e^{-60t} - 1.2\,e^{-10t}$

on taking Laplace transform of $C(t)$ we get.

$$C(s) = \frac{1}{s} + 0.2\,\frac{1}{s+60} - 1.2\,\frac{1}{s+10}$$

$$= \frac{(s+60)(s+10) + 0.2(s)(s+10) - 1.2(s)(s+60)}{s(s+60)(s+10)}$$

$$= \frac{s^2 + 70s + 600 + 0.2s^2 + 2s - 1.2s^2 - 72s}{s(s+60)(s+10)}$$

$$= \frac{600}{s(s+60)(s+10)}$$

Since input is unit step $R(s) = 1/s$

$$\therefore C(s) = R(s)\,\frac{600}{s(s+60)(s+10)} \quad [\because \text{since } R(s) = 1/s]$$

$$T.F = \frac{C(s)}{R(s)} = \frac{600}{s(s+60)(s+10)}$$

The damping ratio and natural frequency of oscillation can be estimated by comparing the system transfer function with standard form of second order transfer function

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$= \frac{600}{s^2 + 70s + 600}$$

On comparing we get

$\omega_n^2 = 600$

$\therefore \omega_n = \sqrt{600} = 24.49$ rad/sec

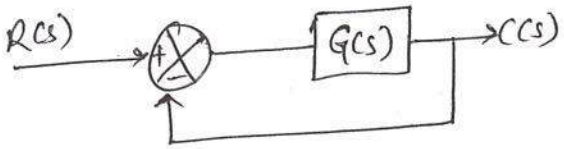
$2\xi\omega_n = 70$

$\xi = \frac{70}{2\omega_n}$

$= 1.43$

Problem 2

Obtain the response of unity feedback system whose open loop transfer function is $G(s) = \frac{4}{s(s+5)}$ and when the input is unit step.



Sol: The closed loop Transfer function $\frac{C(s)}{R(s)} = \frac{G}{1+GH}$

$$= \frac{\frac{4}{s(s+5)}}{1 + \frac{4}{s(s+5)}} \Rightarrow \frac{\frac{4}{s(s+5)}}{\frac{s(s+5)+4}{s(s+5)}}$$

$$= \frac{4}{s(s+5)+4}$$

$$= \frac{4}{s^2+5s+4} \Rightarrow \frac{4}{(s+4)(s+1)}$$

The Response in s domain $C(s) = R(s)\frac{4}{(s+4)(s+1)}$

$$= \frac{1}{s} \cdot \frac{4}{(s+4)(s+1)}$$

Since the input is unit step $R(s) = 1/s$.

By partial fraction expansion.

$$C(s) = \frac{4}{s(s+1)(s+4)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+4}$$

$$\Rightarrow \quad 4 = A(s+1)(s+4) + B(s)(s+4) + C(s)(s+1)$$

$s \Rightarrow 0 \qquad 4 = 4A + B$

$$A = 1$$

$s \Rightarrow -4 \qquad 4 = +12B$

$$C = +1/3$$

$s \Rightarrow -1 \qquad 4 = -3B$

$$B = -4/3.$$

The time domain response $c(t)$

$$C(s) = \frac{1}{s} - \frac{4/3}{s+1} + \frac{1/3}{s+4}$$

$$c(t) = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{4/3}{s+1} + \frac{1/3}{s+4}\right]$$

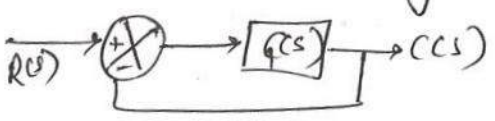
$$c(t) = 1 - \frac{4}{3}e^{-t} + \frac{1}{3}e^{-4t}$$

=

P.Q

Problem 3

The unity feedback system is Characterized by an open loop transfer function $G(s) = \frac{k}{s(s+10)}$. Determine the gain k, so that the system will have a damping ratio of 0.5 for this value of k.



Determine settling time, peak overshoot and time to peak overshoot for a unit step input.

Sol: The unity feedback system is shown in fig 15

The closedloop transfer function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$G(s) = \frac{k}{s(s+10)}$$

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{k}{s(s+10)}}{1+\frac{k}{s(s+10)}\cdot 1} \Rightarrow \frac{k}{s(s+10)+k}$$

$$= \frac{k}{s^2+10s+k}$$

Compare withe second order Transferfunction.

$$\frac{C(s)}{R(s)} = \frac{\omega n^2}{s^2+2\xi\omega n s+\omega n^2} = \frac{k}{s^2+10s+k}$$

$\omega n^2 = k$

$\omega n = \sqrt{k}$

$2\xi\omega n = 10$

$\xi = 0.5$ (Given)

$\omega n = \sqrt{k}$

$2 \times 0.5\sqrt{k} = 10$

$\sqrt{k} = 10$

$\therefore k = 100$

$\omega n = 10$ rad/sec.

$\therefore$ The Value of Gain $k = 100$.

Percentage peak overshoot $\%Mp = e^{-\xi\pi/\sqrt{1-\xi^2}} \times 100$

$= e^{-0.5\pi/\sqrt{1-(0.5)^2}} \times 100$

$= 0.163 \times 100$

$= 16.3\%$

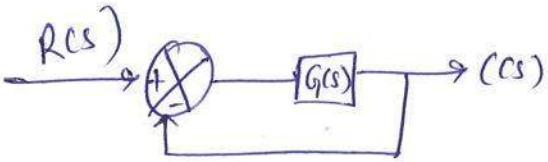
Peak time $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$

$= \frac{\pi}{10\sqrt{1-(0.5)^2}}$

$= 0.363$ sec.

Problem 4:

A unity feedback Control system has an open loop transfer function $G(s) = \frac{10}{s(s+2)}$. Find rise time, percentage overshoot, peak time and settling time for a step input of 12 units.



The closed loop transfer function $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$

$= \frac{\frac{10}{s(s+2)}}{1+\frac{10}{s(s+2)}} \Rightarrow \frac{10}{s(s+2)+10}$

$= \frac{10}{s^2+2s+10}$

$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} = \frac{10}{s^2+2s+10}$

$\omega_n^2 = 10$

$\omega_n = \sqrt{10} = 3.162$ rad/sec

$2\xi\omega_n = 2$

$\xi\omega_n = 1$

$\xi = \frac{1}{3.162} = 0.316.$

$\theta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} = 1.249$

$\omega_d = \omega_n\sqrt{1-\xi^2} = 3$ rad/sec.

Rise time $t_r = \frac{\pi - \theta}{\omega_d}$

$= \frac{\pi - 1.249}{3} = 0.63$ sec

Peak overshoot % $M_p = e^{-\xi\pi/\sqrt{1-\xi^2}} \times 100$

$= 35.12\%$.

Peak overshoot $= \frac{35.12}{100} \times 12 = 4.214$ units.

peak time $t_p = \frac{\pi}{\omega_d}$

$= \frac{\pi}{3} = 1.047$ sec

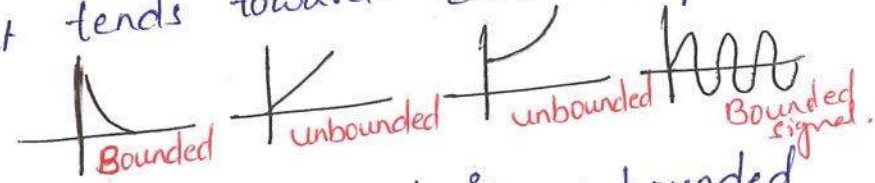
Time Constant $T = \frac{1}{\xi\omega_n} = \frac{1}{0.316 \times 3.162} = 1$ sec.

Settling time for 5% error $t_s = 3T = 3$ sec

for 2% error $t_s = 4T = 4$ sec.

Concept of Stability:

1. A system is stable if it's output is bounded for any bounded input.
2. A system is asimptotically stable if in the absence of input, the output tends towards zero irrespective of initial conditions.



3. A system is stable if it's output is unbounded for any bounded input.
4. Stability of the system depends on the poles.

5. If all the poles are located in left half of S-plane then the system is called stable system.

6. As poles approaches origin the stability decreases.

7. When poles (non-repeated) are located on immaginary axis then the system is called "marginally stable".

8. If poles located on immaginary axis are repeated the system is "unstable system".

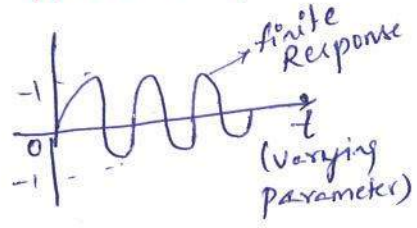
9. If the poles located in right half of S-plane the the system is "unstable system".

10. The poles which are close to origin are called "dominent poles."

Classification of System based on stability:

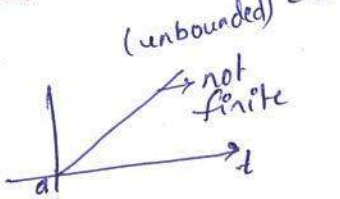
1. Absolutely stable system
2. unstable system
3. Conditionally stable system

This is the system which gets a stable output for variation of it's parameters. (All variations of it's parameters is going to get the finite output)



For All variation of Parameters, if the o/p is not finite (or) not bounded then the system is called "unstable system.

Ex:



Conditionally stable System will provide finite response in the limited period of parameter variations. (After specific Condition the system may unstable).

Techniques used to Calculate Stability:

1. Routh-Hurwitz Criterion
2. Root Locus
3. Bode plot
4. Nyquist plot
5. Nicholas chart.

Routh Hurwitz Criterion:

✓ The closed loop transfer function of a system is a ratio of two polynomials in s. The denominator polynomial of closed loop transfer function is called the characteristic equation of the system.

✓ The roots of the characteristic equation of a stable system should lies on the left half of the s-plane. Hence the roots should have negative real parts.

✓ The first step in analysing the stability of a system is to examine it's characteristic equation. The necessary condition for stability is that all the coefficients of the polynomial be positive. If some of the coefficients are zero or negative. it can be concluded that the system is not stable.

✓ When all the coefficients are positive, the system

necessarily stable. Some of the roots may lie on the right of the s-plane or the imaginary axis.

✓ In order for all the roots to have negative real parts, it is necessary but not sufficient that all of the coefficients of the characteristic equation be positive. If all the coefficients of the characteristic equation are positive, then the system may be stable and on should proceed further to examine the sufficient conditions of stability.

The Routh stability Criterion is based on ordering the coefficients of the characteristic equation:

$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n = 0$. where $a_0 > 0$

s^n :	a_0	a_2	a_4	a_6	a_8 ...
s^{n-1} :	a_1	a_3	a_5	a_7	a_9 ..
s^{n-2} :	b_0	b_1	b_2	b_3	b_4 --
s^{n-3} :	c_0	c_1	c_2	c_3	c_4 ---
⋮					
s^1 :	g_0				
s^0 :	h_0.				

** Statement: " The necessary and sufficient condition for stability is that all of the elements in the first column of the Routh array be positive. If the condition

is not met, the system is unstable and the number of sign changes in the elements of the first column of the routh array corresponds to the number of roots of the characteristic equation in the right half of the s-plane".

Note: If the order of sign of the first column element is +, +, −, + and +, then + to − is considered as onesign, and − to + as another sign change.

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_3}{a_1}$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

$$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

In the process of Constructing Routh array the missing terms are Considered as zeros.

In the Construction of Routh array, one may come across the following Cases.

Case 1: Normal Routh array (Non-zero elements in the first Column of Routh array).

Case 2: A row of all zeros

Case 3: first element of a row is zero but some or other elements are not zero.

Case I: Normal Routh array:

- The sign changes are noted to find the number of roots lying on the right half of the s-plane and the stability of the system can be estimated.
- If there is no sign change in the first column of Routh array then all the roots are lying on left half of s-plane and the system is stable.
- If there is sign change in the first column of routh array, then the system is unstable and the number of roots lying on the right half of s-plane is equal to number of sign changes. The remaining roots are lying on the left half of s-plane.

Case II: A Row of all zeros:

An all zero row indicates the existence of an even polynomial as a factor of the given characteristic equation. In an even polynomial the exponents of 's' are even integers or zero only. This even polynomial factor is called the auxillary polynomial. The coefficients of the auxillary polynomial will always be the elements of the row directly above the row of zeros in the array.

The roots of an even polynomial occur in pairs that are equal in magnitude and opposite sign. Hence these roots can be purely immaginary, purely real or Complex. The purely imaginary & purely real roots occur in pairs. The Complex roots occur in groups of four and Complex roots have quadrantal symmetry.

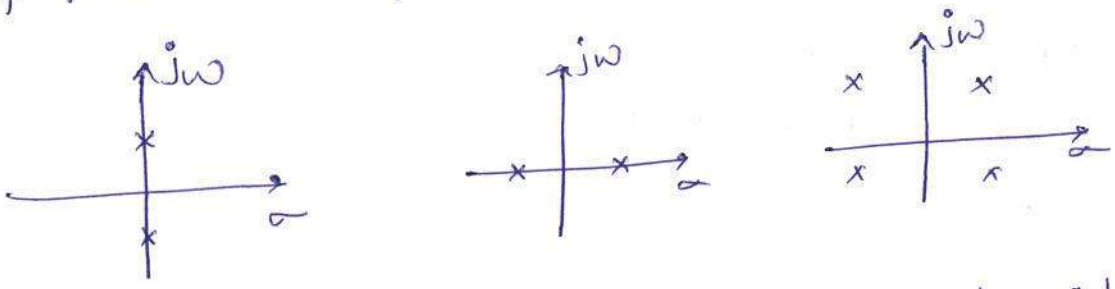


fig: The roots of an even polynomial.

Method I:

1. Determine the Auxillary polynomial, A(s)
2. Differentiate the " " with respect to s to get $\frac{d}{ds}A(s)$
3. The row of zeros is replaced with coefficients of $\frac{dA(s)}{ds}$
4. Continue the Construction of the array in the usual manner. and the array is interpreted as follows.

a. If there are sign changes in the first Column of routh array then the system is unstable. The number of roots lying on right half of the s-plane is equal to the number sign changes. The number of roots on imaginary axis can be estimated from

the roots of auxillary polynomial. The remaining roots are lying on the left half of s-plane.

b. If there are no sign changes in the first column of routh array then the all zeros row indicate the existence of purely immaginary roots and so the system is limitedly, (or) Marginally stable. The roots of auxillary equation lies on imaginary axis and the remaining roots lies on left half of s-plane.

Method: 2

1. Determine the Auxillary polynomial, A(s)
2. Divide the Characteristic equation by auxillary polynomial
3. Construct Routh array using the Coefficients of quotient polynomial.
4. The array is interpretted as follows.

a. If there are sign changes in the first column of routh array of quotient polynomial then the system is unstable. The number of roots of quotient polynomial lying on right half of s-plane is given by number of sign changes in first column of routh array.

The Roots of auxillary polynomial are directly Calculate to find wheather they are purely imaginary or purely real or Complex.

The total number of roots on right half of s-plane is given by the sum of number of sign changes and the number of roots on imaginary axis can be estimated from the roots of auxillary polynomial. The remaining roots are lying on the left half of s-plane

b. If there is no sign changes in the first column of routh array of quotient polynomial then the system is limitedly or marginally stable. since there is no sign change all roots of quotient polynomial are lying on the left half of s-plane.

The roots of auxillary polynomial are directly calculated to find wheather they are purely imaginary or purely real or complex. The number of roots lying on imaginary axis and on the right half of s-plane can be estimated from the roots of auxillary polynomial. The remaining roots are lying on the left half of s-plane.

Case (ii): First element of a row is zero:

While constructing routh array, if a zero is encountered as first element of a row then all the elements of the next row will be infinite. To overcome this problem let $0 \to \epsilon$ and complete the construction of array in the usual way (as that of Case 1)

finally let $\epsilon \to 0$ and determine the values of the

elements of the array which are functions of ϵ. The resultant array is interpreted as follows.

Note: If all the elements of row are zeros then the solution is attempted by Considering the polynomial as Case II. Even if there is a single element zero on s' row, it is Considered as a row of all zeros.

a. If there is no sign change in first Column of routh array and if there is no row will all zeros then all the roots are lying on left half of s-plane and the system is stable.

b. If there are sign changes in first column of routh array and there is no row with all zeros, then some of the roots are lying on the right half of s-plane and the system is unstable. The number of roots lying on the right half of s-plane is equal to number of sign changes and the remaining roots are lying on the left half of s-plane.

c. If there is a row of all zeros after letting $\epsilon \to 0$, then there is a possibility of roots on imaginary axis. Determine the auxillary polynomial to eliminate the imaginary roots. The Routh array is constructed using the coefficients of quotient polynomial and the characteristic equation is interpretted as explained in Method-2 of Case II.

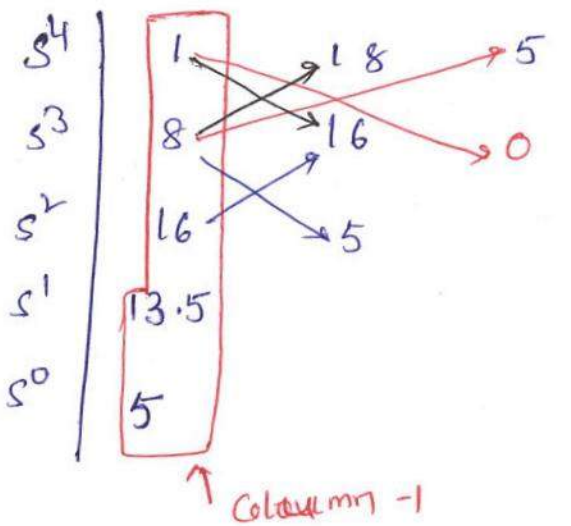
Problem 1:

Using Routh Criterion, Determine the stability of the system represented by the characteristic equation $s^4+8s^3+18s^2+16s+5=0$. Comment on the location of the roots of characteristic equation.

Sol: The characteristic equation of the system is

$s^4+8s^3+18s^2+16s+5=0$.

The given characteristic equation is 4th order equation and so it has 4 roots. Since the highest power of s is even number. from the first row of routh array using the co-efficients of even powers of s and form the second row using the coefficients of odd powers of s.



$$\frac{8\times18-16\times1}{8} = \frac{144-16}{8} = \frac{128}{8} = 16$$

$$\frac{8\times5-1\times0}{8} = \frac{40-0}{8} = \frac{40}{8} = 5$$

$$\frac{16\times16-8\times5}{16} = \frac{256-40}{16} = 13.5$$

Comment:

On examining the elements of first column of routh array it is observed that all the elements are +ve and there is no sign

Change. Hence all the roots are lying left half of S-plane and the System is stable.

Result: 1. Stable System.

2. All four roots are lying on the left half of S-plane.

Problem: 2

Construct Routh array and determine the stability of the System represented by the characteristic equation $9s^5-20s^4+10s^3-s^2-9s-10=0$. Comment on the location of roots of characteristic equation.

Sol: The characteristic polynomial of the system is

$$9s^5-20s^4+10s^3-s^2-9s-10=0.$$

On examining the Coefficients of the Characteristic Polynomial, it is found that some of the Coefficients are negative and so some roots will lie on the right of s-plane. Hence the System is unstable. The Routh array Can be Constructed to find the number of roots lying on right half of S-plane.

The given polynomial is 5th order equation and so it has 5 roots. Since the highest power of s is odd number, form the first row of array using the Coefficients of odd power of s and form the second row using the Coefficients of even power of s.

Given Characteristic equation

$$9s^5 - 20s^4 + 10s^3 - s^2 - 9s - 10 = 0$$

s^5	9	10	... Row1
s^4	-20	-1	-10 ... Row2
s^3	9.55	-13.5	
s^2	-29.27	-10	
s^1	-16.76	0	
s^0	-10		

$$\frac{(-20\times10)-(9\times-1)}{-20} = \frac{-200+9}{-20} = \frac{-191}{-20} = 9.55$$

$$\frac{(-20\times-9)-(9\times-10)}{-20} = \frac{180+90}{-20} = \frac{-270}{20} = -13.5$$

$$\frac{(9.55\times-1)-(13.5\times20)}{9.55} = \frac{-9.55-270}{9.55} = -29.27$$

$$\frac{(9.55\times(10))-(13.5\times29.27)}{-29.27} = \frac{95.5-395.145}{-29.27} = \frac{-299.645}{-29.27} = +10.23$$

$$\frac{(-16.76\times-10)-0}{-16.76} = -10.$$

By examining the elements of 1st Column of routh array, it is observed that there are three sign changes and so three roots are lying on the right half of S-plane and the remaining two roots are lying on the left half of Splane.

Result:-

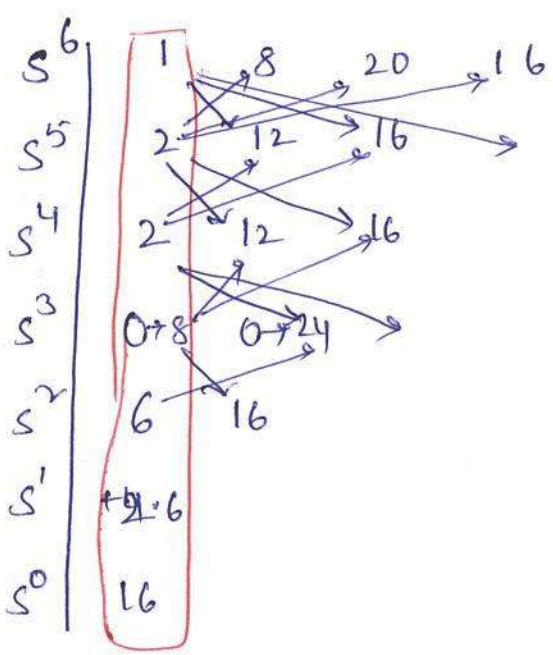
1. The System is unstable.
2. Three roots are lying on the right half and two roots are lying on the left half of S-plane.

Problem: 3

Pg Construct Routh array and determine the stability of the system whose characteristic equation is $s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$. Also determine the number of roots lying on right half of s-plane, left half of s-plane and on imaginary axis.

Sol: Given Characteristic equation is

$$s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0.$$



Row1 ... Row2 ... Row3 ... Row4 ... Row5 ... Row6 ... Row7.

$\frac{(2\times8)+(1\times12)}{2} \Rightarrow \frac{16-12}{2} \Rightarrow 2$

$\frac{(2\times20)-(1\times16)}{2}$

$\frac{40-16}{2} = \frac{24}{2} = 12$

$\frac{(2\times16)-(1\times0)}{2}$

$= \frac{32-0}{2} = 16$

$\frac{(2\times12)-(2\times12)}{2} = 0$

On examining the elements of 1st Column of Routh array it is observed that there is no sign change.

$\frac{(6\times24)-(8\times16)}{6}$

$\frac{144-128}{6}$

$\frac{16}{6} = 4$

$\frac{(16\times2.6)-0}{2.6}$

$\frac{(12\times8)-(2\times24)}{8}$

$= \frac{96-48}{8}$

$= \frac{48}{8} = 6$

$\frac{(16\times8)-(2\times0)}{8}$

$= \frac{128}{8}$

$= 16$

Step1: Determine Auxillary polynomial

$A = 2s^4+12s^2+16=0$

Step2 differentiate with respect to s

$\frac{dA}{ds} = 8s^3+24s$

Step3 The row is replaced by the coefficients.

The Row with all zeros indicate the possibility of roots on imaginary axis. Hence the system is limitedly or marginally stable.

The auxillary polynomial is

$s^4 + 6s^2 + 8$.

Let $s^2 = x \Rightarrow x^2 + 6x + 8$

$$\frac{-6 \pm \sqrt{36 - 32}}{2} \Rightarrow \frac{-6 \pm \sqrt{4}}{2}$$

$$\Rightarrow \frac{-6 + \sqrt{4}}{2}, \quad \frac{-6 - \sqrt{4}}{2}$$

$$\Rightarrow \frac{-6+2}{2} \Rightarrow -2 \; ; \quad \frac{-6-2}{2} = -4$$

$\therefore x = -2$ & -4

$s = \pm\sqrt{x}$	$s = \pm\sqrt{x}$
$s = \pm\sqrt{-2}$	$= \pm\sqrt{-4}$
$= \pm j\sqrt{2}$	$= \pm j2$

The roots of auxillary polynomial are also roots of Characteristic equation. Hence 4 roots are lying on imaginary axis and the remaining two roots are lying on the left half of S-plane.

Result: 1) The system is limitedly (or) Marginally stable.

2) 4 roots on imaginary axis and two roots are lying on the left half of s-plane.

Problem: b+ :

Construct Routh array and determine the stability of the system represented by the characteristic equation $s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$. Comment on the location of

the roots.

Sol: Given characteristic equation is

$$s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0.$$

s^5	1	2	3
s^4	1	2	5
s^3	$0 \to \epsilon$	-2	
s^2	$\frac{2\epsilon+2}{\epsilon}$	$\frac{5\epsilon-0}{\epsilon}$	
s^1	$-\frac{(5\epsilon^2+4\epsilon+1)}{2\epsilon+2}$		
s^0	5		

-- Row1
-- Row2
-- Row3
-- Row4
--- Row5
-- Row6.

Replace 0 by ϵ.

$$\frac{\frac{2\epsilon+2}{\epsilon}2 - 5\epsilon}{\frac{2\epsilon+2}{\epsilon}}$$

$$\Rightarrow \frac{-4(\epsilon+1) - 5\epsilon^2}{2\epsilon+2}$$

$$= \frac{-4\epsilon - 4 - 5\epsilon^2}{2\epsilon+2}$$

$$= -\frac{(5\epsilon^2+4\epsilon+1)}{2\epsilon+2}$$

Let $\epsilon \to 0$. we get.

s^5	1	2	3
s^4	1	2	5
s^3	0	-2	
s^2	α	5	
s^1	-2		
s^0	5		

On observing the elements of first column of routh array. it is found that there are two sign changes. Hence two roots are lying on the right half of s-plane and the system is unstable. The remaining three roots are lying on the left half of s-plane.

Result:- 1) The system is unstable

2) Two roots are lying on right half of s-plane and three roots are lying on the left half of s-plane.

* PQ

Problem 5:

The characteristic polynomial of a system is $s^7+9s^6+24s^5+24s^4+24s^3+24s^2+23s+15=0$. Determine the location of roots on s-plane and hence the stability of the system.

Sol: The given characteristic equation.

$$s^7+9s^6+24s^5+24s^4+24s^3+24s^2+23s+15=0$$

s^7	1	24	24	23
s^6	9	24	24	15
s^5	21.3	21.3	21.3	
s^4	15	15	15	
s^3	0→4	0→2		
s^2	0.5	1		
s^1	-6			
s^0	1			

Row 1 ---

Row 2 ---

$$\frac{(9\times24)-(24\times1)}{9} = 21.3$$

$$\frac{(9\times23)-(1\times15)}{9} = 21.3$$

$$\frac{(21.3\times24)-(9\times21.3)}{21.3}$$

Two roots of the polynomial are lying on the right half of the s-plane and the remaining two are on the left side of the s-plane

$A = s^4+s^2+1$

$s^2 = x$

$= x^2+x+1$

$$\frac{-1\pm\sqrt{1-4}}{2}$$

$$\frac{-1\pm\sqrt{3}j}{2}$$

$$x = -\frac{1}{2}\pm\sqrt{3}j$$

$s = \sqrt{x}$

$= \sqrt{-\frac{1}{2}+\sqrt{3}j}$

Auxillary polynomial

$A = s^4+s^2+1=0$

$x^2 = s$

$s^2+s+1=0$

$$-1\pm\sqrt{1-4}$$

$$\frac{dA}{ds} = 4s^3+2s$$

The roots of auxillary equation are also the roots of characteristic polynomial. The two roots lying on the right half of s-plane are indicated by two sign changes in the first Column of routh array. The remaining 5 roots are lying on the left half of the s plane. No roots are lying on imaginary axis.

Result: 1) The system is unstable.

2) Two roots are lying on Left half of S-plane
five " " " " right " " " .

Problem: 6

The Characteristic polynomial of a system is $s^7 + 5s^6 + 9s^5 + 9s^4 + 4s^3 + 20s^2 + 36s + 36 = 0$. Determine the Location of roots on the S-plane and hence the stability of the system.

Sol:

s^7	1	9	4	36
s^6	5	9	20	36
s^5	7.5	0	28.8	0
s^4	9	0.8	36	0
s^3	-0.66	-1.2		
s^2	15.5	36	0	
s^1	-0.33			
s^0	36			

$\frac{45-9}{5} = \frac{36}{5} = 7.5$

$\frac{20-20}{5} = 0$

$\frac{180-36}{5} = 28.8$

$\frac{67.5-0}{7.5} = 9$

$\frac{150-144}{7.5} = 0.8$

$\frac{270-0}{7.5} = 36$

$\frac{0-6}{9} = -0.66$

$\frac{259.2-270}{9} = -1.2$

$\frac{-0.528+10.8}{-0.66} = 15.5$

$\frac{23.76-0}{-0.66} = 36$

$\frac{-23.76+18.6}{15.5} = -0.33$

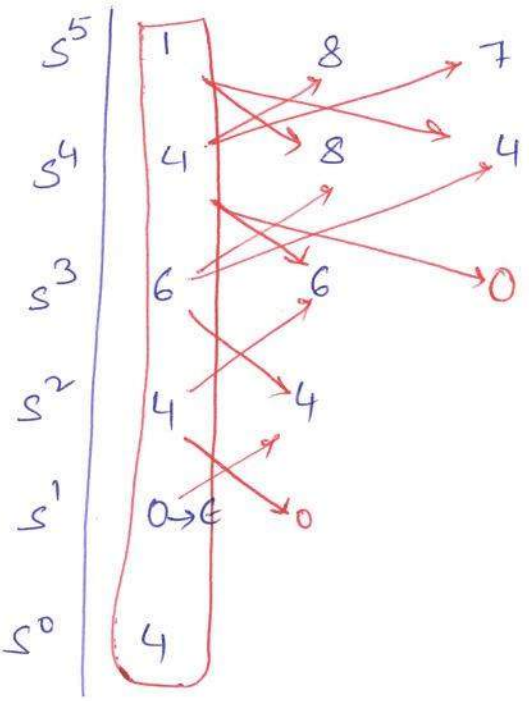
$\frac{-11.88+0}{-0.33} = 36$

4 sign changes
3 roots – LHS
4 roots – RHS
Unstable system.

Problem 7:

Use the routh stability criterion to determine the location of roots on the s-plane and hence the stability for the system represented by the characteristic equation $s^5+4s^4+8s^3+8s^2+7s+4=0$.

Sol.



$$\frac{32-8}{4}=\frac{24}{4}=6$$

$$\frac{28-4}{4}=\frac{24}{4}=6$$

$$\frac{48-24}{6}=\frac{24}{6}=4$$

$$\frac{24-0}{6}=\frac{24}{6}=4$$

$$\frac{24-24}{4}=0.$$

$$\frac{(\epsilon\times4)-(4\times0)}{\epsilon}=\frac{4\epsilon}{\epsilon}=4$$

When $\epsilon\to0$ there is no sign change in the first column of routh array. But we have a row of all zeroes and so there is a possibility of roots on imaginary axis. Hence the auxillary polynomial is. $A(s)=4s^2+4=0$

$$s^2+1=0$$
$$s=\sqrt{-1}\Rightarrow\pm j1$$

Comment:

The roots of $A(s)$ are $\pm j1$ lying on imaginary axis. Hence two roots of given characteristic equation lying on imaginary axis. remaining three are on left side. so the system is marginally stable.

Problem 8:

Determine the range of k for stability of unity feedback system whose open loop transfer function is

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

Sol: The closed loop Transfer function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

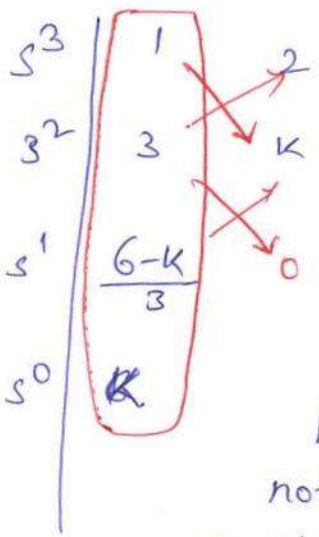
$$= \frac{\frac{K}{s(s+1)(s+2)}}{1+\frac{k}{s(s+1)(s+2)}} \Rightarrow \frac{k}{s(s+1)(s+2)+K}$$

The characteristic equation.

$$s(s+1)(s+2)+k = 0.$$

$$s(s^2+3s+2)+k = 0$$

$$s^3+3s^2+2s+k = 0.$$



$$\frac{6-K}{3}$$

$$\frac{\frac{(6-K)K}{3}}{\frac{(6-K)}{3}} = K$$

For the system to be stable - these should not be any sign change in the first column.

∴ k should be positive.

for the system to be stable $k > 0$.

∴ $\frac{6-K}{3} > 0$ ∴ The range of k for the system to be stable $0 < k < 6$.

Problem 8:

A feedback system has open loop transfer function of $G(s) = \frac{k \cdot e^{-s}}{s(s^2+5s+9)}$. Determine the maximum value of k for stability of closed loop system.

Sol: Generally Control systems have very low bandwidth which implies that it has very low frequency range of operation. Hence for low frequency ranges the term e^{-s} can be replaced by $(1-s)$ $[\because e^{-sT} \approx (1-sT)]$

$$G(s) = \frac{k \cdot e^{-s}}{s(s^2+5s+9)} = \frac{k(1-s)}{s(s^2+5s+9)}$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$= \frac{k(1-s)/s(s^2+5s+9)}{1+\frac{k(1-s)}{s(s^2+5s+9)}}$$

$$= \frac{k(1-s)}{s(s^2+5s+9)+k(1-s)}$$

$\therefore$ The characteristic equation.

$$s(s^2+5s+9)+k(1-s)=0.$$

$$s^3+5s^2+9s+k-ks=0.$$

$$s^3+5s^2+(9-k)s+k=0.$$

$$s^3+5s^2+(9-k)s+k=0.$$

s^3	1	$(9-k)$
s^2	5	k
s^1	$\frac{45-6k}{5}$	0
s^0	k.	

$$\frac{(9-k)5-k}{5}$$

$$= \frac{45-5k-k}{5}$$

$$= \frac{45-6k}{5}$$

$$\frac{\left(\frac{45-6k}{5}\right)k - 0}{\left(\frac{45-6k}{5}\right)}$$

To get the system stable 'k' should be positive and $k>0$.

$$\frac{45-6k}{5} > 0.$$

$$45-6k>0 \Rightarrow -6k<-45$$

$$k < 7.5$$

$$0<k<7.5$$

Comment

For the stability of the system k is in the range of $0<k<7.5$.

pg

* Problem 9:

Find the stability of the following system with characteristic equation using R-H. Criteria. $2s^4+s^3+3s^2+5s+10=0$

s^4	2	3	10
s^3	1	5	0
s^2	-7	10	
s^1	6.42	0	
s^0	10.		

$$\frac{3-10}{1} = -7$$

$$\frac{10-0}{1} = 10$$

$$\frac{-35-10}{-7} = \frac{-45}{-7} = 6.42$$

$$\frac{(6.42)\cdot 10}{6.42}$$

Comment: By observing the first Coloumn of Routh array, there are two sign changes. so the two roots are lie on the Right half of the s-plane remaining two roots are on left half of the s-plane, no roots on imaginary axis. so the system unstable.

* PQ

Problem 10:

Using Routh Criterion determine the stability of the system with characteristic equation is given by

$9s^5 - 20s^4 + 8s^3 - 8s^2 - 6s + 5 = 0.$

Sol:

s^5	9	8	-6	
s^4	-20	-8	+5	
s^3	4.4	-3.75	0	
s^2	-25.04	5	0	
s^1	13.2	0		
s^0	5			

$\frac{-160+72}{-20} = 4.4$

$\frac{120-45}{-20} = -3.75$

$\frac{-35.2-75}{4.4} = \frac{-110.2}{4.4} = -25.04$

$\frac{22-0}{4.4} = 5$

$\frac{93.9-425.9}{-25.04}$

$= \frac{-332}{-25.04} = 13.2$

$\frac{1277.76-0}{(13.2)\,96.8-0}$

Comment: By observing the first Column of routh array there are 4 sign changes, so 4 roots are lies on the Right half of the s-plane remaining only one root is on the left half of the s-plane. So the system is unstable.

Root locus Technique:

Construction Rules:

1. The root locus is Symmetrical about the real axis
2. The root locus always start from the open loop poles and terminates on either finite openloop zero (or) infinity.
3. The number of branches terminating on infinite $p-z$. (No. of poles - No of zeros).
4. A point on the real axis lies on the locus if the no. of open loop poles & zeros on the real axis to the right half of the s-plane is odd.
5. The angle of Asymptotes.

   if $p > z$ $\quad \phi_A = \dfrac{(2q+1)\,180^\circ}{p-z}$ $\quad q = 0, 1, 2 \ldots p-z$.

   if $z > p$ $\quad \phi_A = \dfrac{(2q+1)180^\circ}{z-p}$ $\quad q = 0, 1, 2 \ldots z-p$.

6. Centroid:

   The asymptotes meet the real axis at Centroid.

   $$\text{Centroid} = \frac{\text{Sum of real parts of poles} - \text{Sum of real parts of zero}}{\text{number of poles} - \text{number of zeros}}$$

7. Breakaway & Breakin is Calculated by solving $dk/ds = 0$.

8. The angle of departure from an open loop pole is given by $\phi_p = 180^\circ(2q+1) + \phi$; $q = 0, 1, 2 \dots$

9. The angle of arrival at an open loop zero is given by $\phi_z = 180^\circ(2q+1) \mp \phi$; $q = 0, 1, 2 \dots$

10. The intersection of root locus branches with the imaginary axis can be determined by use of routh criterion.

Root Locus technique:

Rules:

Rule1: The root locus is symmetrical about the real axis.

Rule2: Each branch of the root locus originates from an openloop zero corresponding to $k=\alpha$. The number of branches of the root locus terminating on infinity is equal to $(n-m)$ i.e., the number of open loop poles minus the number of finite zeros

Rule3: Segments of the real axis having an odd number of real axis openloop poles plus zeros to their right are parts of the root locus.

Rule4: The $(n-m)$ root locus branches that tend to infinity, do so along straight line asymptotes making angles with the real axis given by

$$\phi_A = \frac{\pm 180^\circ [2q+1]}{(n-m)} \quad ; \quad q = 0, 1, 2 \cdots (n-m).$$

Rule5: The point of intersection of the Asymptotes with the real axis is at $s=\sigma_A$ where $\sigma_A = \frac{\text{Sum of poles} - \text{sum of zeroes}}{n-m}$

Rule6: The breakaway and break in points of the root locus are determined from the roots of the equation $\frac{dk}{ds}=0$. If r numbers of branches of root locus meet at a point, then they break away at an angle of $\pm 180^\circ/r$.

Rule 7: The angle of departure from a Complex open loop pole is given by

$$\phi_p = \pm 180^\circ (2q+1) + \phi \; ; \quad q = 0, 1, 2 \cdots$$

where ϕ is the net angle Contribution at the pole by all other open loop poles and zeros, Similarly the angle of arrival at a Complex open loop zero is given by $\phi_z = \pm 180^\circ (2q+1) + \phi \; ; \; q = 0, 1, 2 \cdots$

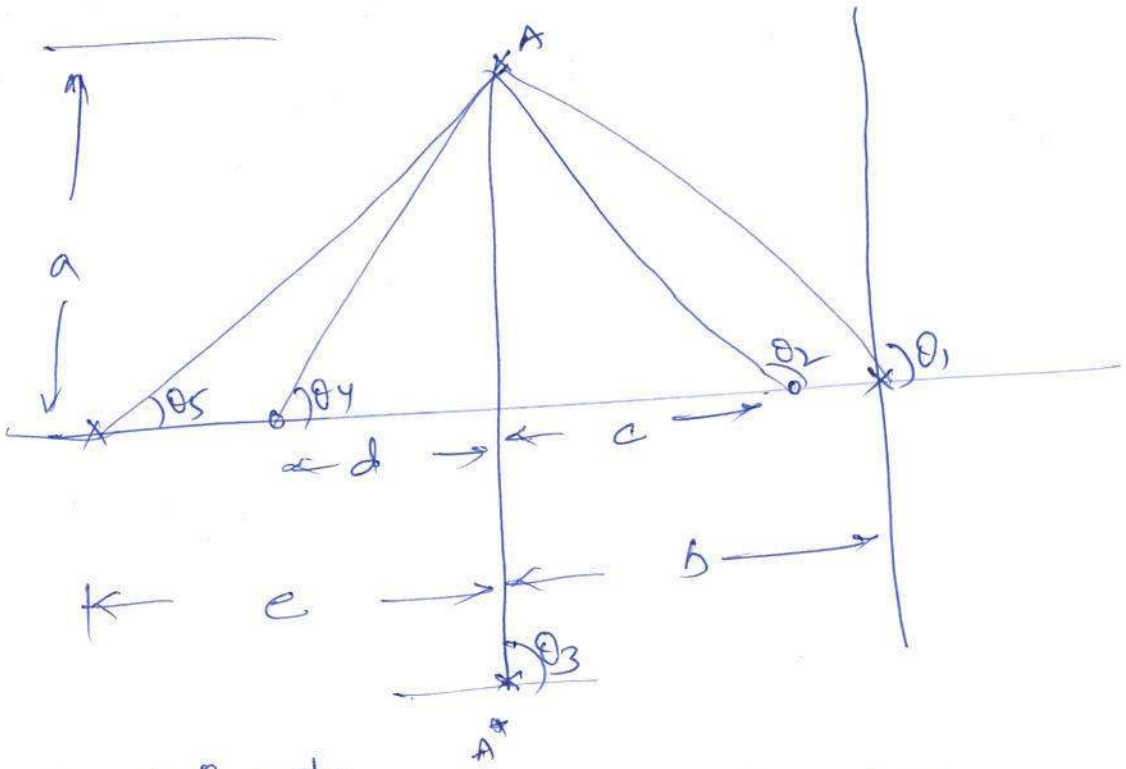
where ϕ is the net angle Contribution at the zero by all other open loop poles and zeros.

Rule 8: The intersection of root locus branches with the imaginary axis Can be determined by use of Routh criterion, or by letting $s = j\omega$ in the Characteristic equation and equating the real part and imaginary part to zero, to solve for ω and K. The value of ω is the intersection point on imaginary axis and K is the value of gain at the intersection point.

Rule 9: The open loop gain K at any point $s = s_a$ on the root locus is given by

$$K = \frac{\prod_{i=1}^{n} |s_a + p_i|}{\prod_{i=1}^{m} |s_a + z_i|}$$

$$= \frac{\text{product of vector lengths from open loop poles to the point } s_a}{\text{Product of vector lengths from open loop zeros to the point } s_a.}$$



$\theta_1 = 180^\circ - \tan^{-1}\frac{a}{b}$

$\theta_2 = 180^\circ - \tan^{-1}\frac{a}{c}$

$\theta_3 = 90^\circ$

$\theta_4 = \tan^{-1}\frac{a}{d}$

$\theta_5 = \tan^{-1}\frac{a}{e}$

Angle of departure (from complex pole A)

$A = 180^\circ - ($(sum of angles of vector to the complex pole A from other poles)

$+ ($(sum of angle of vectors to the complex pole A from zeros)$)$

$= 180^\circ - (\theta_1 + \theta_3 + \theta_5) + (\theta_2 + \theta_4)$.

$A^* = -(A)$.

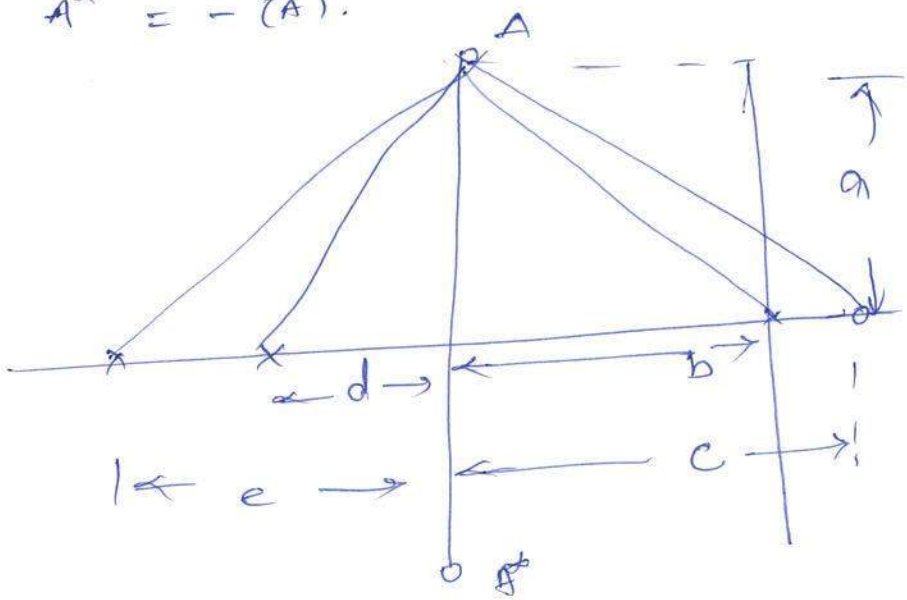
$\theta_1 = 180^\circ - \tan^{-1}\frac{a}{c}$

$\theta_2 = 180^\circ - \tan^{-1}\frac{a}{b}$

$\theta_3 = 90^\circ$

$\theta_4 = \tan^{-1}\frac{a}{d}$

$\theta_5 = \tan^{-1}\frac{a}{e}$



~~Angle of arrival at~~ a Complex zero A $= 180^\circ -$ (Sum of angles of vectors to the Complex zero A from all other zeros)

+ (Sum of angles of vectors to the Complex zero A from poles).

$= 180^\circ - (\theta_1 + \theta_3) + (\theta_2 + \theta_4 + \theta_5)$

$A^* = -(A)$.

Procedure for Constructing Root locus:

Step1: Locate the poles and zeros of $G(s).H(s)$ on the s-plane. The root locus branch start from open loop poles and terminate at zeros.

Step2: Determine the root locus on real axis

Step3: Determine the asymptotes of root locus branches and meeting point of asymptotes with real axis.

Step4: Find the breakaway and breakin points.

Step5: If there is a Complex pole then determine the angle of departure from the Complex pole. If there is a Complex zero then determine the angle of arrival at the Complex zero.

Step6: Find the points where the roots loci may cross the imaginary axis.

Step7: Take a series of test point in the broad neighbourhood of the origin of the s-plane and adjust the test point to satisfy angle Criterion. Sketch the rootlocus by joing the test point by smooth curve.

Step8: The value of gain k at any point on the locus can be determined from magnitude Condition. The magnitude Condition is given by

$$\text{Gain } k \text{ at a point } (s=s_a) = \frac{\text{Product of length of vectors from poles to the point } (s=s_a)}{\text{Product of length of vectors from finite zeros to the point } (s=s_a)}$$

Problem 1:

A unity feedback Control System has an open loop transfer function $G(s) = \frac{k}{s(s^2+4s+13)}$. Sketch the root locus

Sol: Step 1: To locate poles and zeros.

The poles of open loop transfer function are the roots of the equation, $s(s^2+4s+13)=0$.

The roots of the quadratic are

$$s = \frac{-4 \pm \sqrt{4^2-(4\times13)}}{2} = -2 \pm j3$$

$\therefore$ The poles are $0, -2+j3$ and $-2-j3$.

The poles are marked by x (cross)

Step 2: To find the root locus on real axis

There is only one pole on real axis at the origin. Hence if we choose any test point on the negative real axis then to the right of that point the total number of real poles and zeros is one, which is an odd number. Hence the entire negative real axis will be part of root locus. The root locus on real axis is shown as bold line in fig.

Step 3: To find angles of asymptotes and centroid.

Since there are 3 poles, the number of root locus branches are three. There is no finite zero. Hence all the three root locus branches ends at zeros at infinity. The number of asymptotes required are three

Angles of asymptotes = $\frac{\pm 180^\circ(2q+1)}{n-m}$

Here $n=3$, and $m=0$.

If $q=0$, angles = $\pm \frac{180^\circ}{3} = \pm 60^\circ$

If $q=1$, Angles = $\pm \frac{180^\circ \times 3}{3} = \pm 180^\circ$

If $q=2$, Angles = $\pm \frac{180^\circ \times 5}{3} = \pm 300^\circ$

If $q=3$, angles = $\pm \frac{180^\circ \times 7}{3} = \pm 420^\circ$

$$\text{Centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m}$$

$$= \frac{0-2+j3-2-j3-0}{3}$$

$$= \frac{-4}{3} = -1.33.$$

The Centroid is marked on real axis and from the Centroid the angles of asymptotes are marked using protractor. The Asymptotes are drawn as dotted lines in the fig.

Step 4: To find the breakaway and breakin points.

The closed loop transfer function $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

$$\frac{C(s)}{R(s)} = \frac{\frac{K}{s(s^2+4s+13)}}{1+\frac{K}{s(s^2+4s+13)}}$$

$$= \frac{K}{s(s^2+4s+13)+K}$$

The characteristic equation is $s(s^2+4s+13)+k=0$

$$s^3+4s^2+13s+k=0$$

$$\therefore k=-s^3-4s^2-13s$$

On differentiating the equation of k with respect to s we get $\frac{dk}{ds}=-(3s^2+8s+13)$

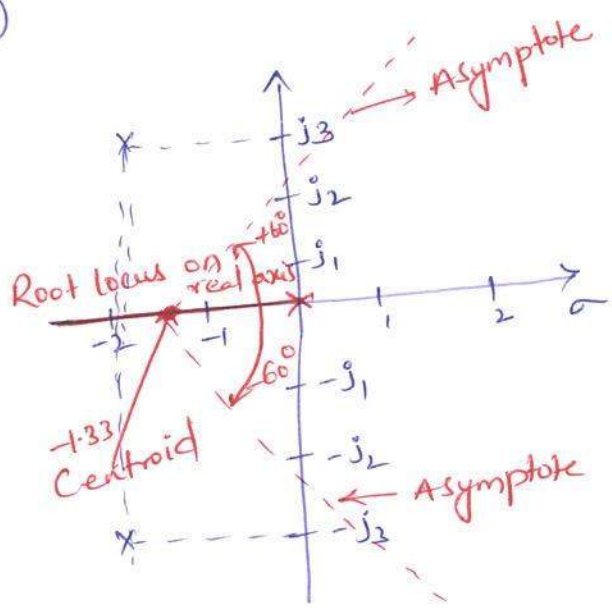
put $\frac{dk}{ds}=0$

$$\therefore -(3s^2+8s+13)=0$$

$$3s^2+8s+13=0$$

$$\therefore s=\frac{-8\pm\sqrt{8^2-(4\times13\times13)}}{2\times3}$$

$$=-1.33\pm j1.6$$



Check for k

when $s=(-1.33+j1.6)$, the value of k is given by

$$k=-(s^3+4s^2+13s)$$

$$=-\left[(-1.33+j1.6)^3+4(-1.33+j1.6)^2+13(-1.33+j1.6)\right]$$

$\neq$ positive and real

Also when $s=-1.33-j1.6$ the value of k is not equal to real and positive.

Since the value of k for $s=(-1.33\pm j1.6)$ are not real and positive, the points are not an actual breakaway or breakin points. The root locus has neither breakaway nor breakin point.

Step5: To find the angle of departure.

Let us Consider the Complex pole A shown in fig. Draw vectors from all other poles to the poles A as shown in fig. Let the angles of these vectors be θ_1 and θ_2.

Here $\theta_1 = 180^\circ - \tan^{-1}(3/2) = 123.7^\circ$

$\theta_2 = 90^\circ$

Angle of departure from the Complex pole

$A = 180^\circ - (\theta_1 + \theta_2)$

$= 180^\circ - (123.7^\circ + 90^\circ)$

$= -33.7^\circ$.

The Angle of departure at Complex pde A^* is -ve of the angle departure at Complex pole A.

∴ Angle of departure at pole $A^* = 33.7^\circ$ (Mark the angle of departure at Complex poles using protractor).

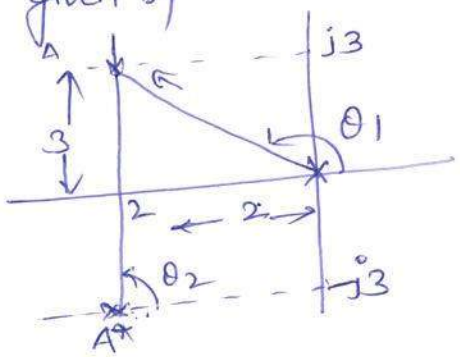
Step6: To find the Crossing point on imaginary axis

The characteristic equation is given by

$s^3 + 4s^2 + 13s + k = 0$

put $s = j\omega$

$(j\omega)^3 + 4(j\omega)^2 + 13(j\omega) + k = 0$



$$-j\omega^3 - 4\omega^2 + 13j\omega + k = 0$$

On equating imaginary part to zero

$$-\omega^3 + 13\omega = 0$$

$$-\omega^3 = -13\omega$$

$$\omega = \pm\sqrt{13}$$

$$= \pm 3.6$$

On equating real part to zero

$$-4\omega^2 + k = 0$$

$$k = 4\omega^2$$

$$= 4(13) = 52.$$

The crossing point of root locus is $\pm j3.6$. The value of k at this crossing point is $k = 52$ (This is the limiting value of k for the stability of the system).

The root locus has three branches one branch starts at the pole at origin and travel through negative real axis to meet the zero at infinity. The other two root locus branches starts at complex poles, crosses the imaginary axis at $\pm j3.6$ travel parallel to Asymptotes to meet the zeros at infinity.

Problem-2:

Sketch the root locus of unity feedback system with open loop transfer function as k varied from 0 to ∞.

$$G(s)\,H(s) = \frac{K}{S(S+2)(S+3)}$$

Sol:

Step1: Locate the poles and zeros of $G(s)\cdot H(s)$ on the S-plane. (To Locate poles and zeros)

$$\left[\because \frac{G}{1+GH}\right]$$

$$1 + G(s)H(s) = 0$$

$$1 + \frac{K}{S(S+2)(S+3)} = 0.$$

$$S(S+2)(S+3) + K = 0$$

$S=0$, $S=-2$, $S=-3$ at $K=0$. ⟶ poles.

⟶ No zeros

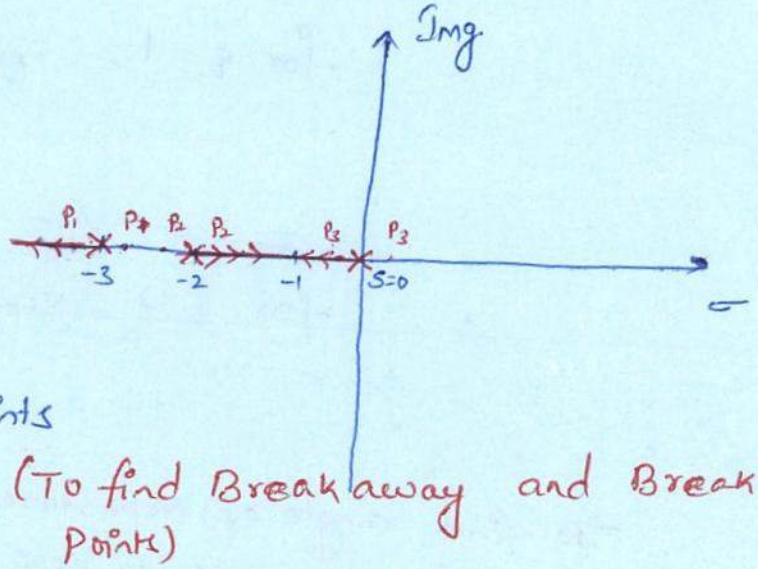
Step2: Root locus starts from poles and terminates at zeros (To find the root locus on real axis)

Poles: $S=0, -2, -3$ at $K=0$

Terminate at ∞ i.e., $K=\infty$ (Since no zeros).

Step3:

$P > Z$; $3 > 0$ → 3 branches



Step4: Break away points

$$\frac{dG(s)}{ds} = 0.$$

(To find Break away and Break in Points)

$$G(s) = \frac{K}{S(S+2)(S+3)} \quad ; \quad H(s) = 1$$

$$\frac{dG(s)}{ds} = \frac{K(3s^2+10s+6)}{[S(S+2)(S+3)]^2} = 0 \qquad \left[\because \frac{u}{v} = \frac{v\,du + u\,dv}{v^2}\right.$$

$$= \frac{3s^2+10s+6}{[S(S+2)(S+3)]^2} = 0$$

$$3s^2+10s+6 = 0$$

$$S = \frac{-10 \pm \sqrt{100-72}}{6}$$

$$\frac{-10 \pm \sqrt{28}}{6} \Rightarrow \frac{-10 \pm 5.29}{6}$$

$$\Rightarrow -0.79, -2.54$$

Steps:- Determine the Asymptotes on real axis

(To find Asymptotes + centroid).

Centroid: $\quad x = \frac{\Sigma m - \Sigma m}{n-m}$

$$= \frac{0+(-2)+(-3)-0}{3}$$

$$= \frac{-5}{3}.$$

$$= -1.667$$

Angle of Asymptotes $\quad \theta = \frac{(2q+1)\times 180^\circ}{p-z}$, $q = 0, 1, 2 \cdots p-z-1$

$p-z-1$

$3-0-1 \Rightarrow 2$

for $q=0$

$$\theta = \frac{((2\times 0)+1)\times 180^\circ}{3-0}$$

$$= \frac{180}{3} = 60^\circ$$

for $q=1$

$$\theta = \frac{((2\times 1)+1)\times 180^\circ}{3-0}$$

$$= \frac{3\times 180^\circ}{3} = 180^\circ$$

for $q=2$

$$\theta = \frac{((2\times 2)+1)\times 180^\circ}{3-0}$$

$$= \frac{900}{3} = 300^\circ$$

To find angle of departure + arrival

(There is no pole or zero as complex).

Step 6: Intersection of root locus branches with imaginary axis. (To find crossing point of imaginary axis).

Method-1

Put $s=j\omega$

$s^3+5s^2+6s+K=0$

$(j\omega)^3+5(j\omega)^2+6j\omega+K=0$

$-j\omega^3-5\omega^2+6j\omega+K=0$

Equating Img. to 0

$-j\omega^3+6j\omega=0$

$-j\omega^3=-6j\omega$

$\omega^2=6$

$\omega=\pm\sqrt{6}$

Equating real to 0

$-5\omega^2+K=0$

$K=6\omega^2=6\times6=36.$

$\boxed{K=36}$

Method-2

R-H. Criterion.

$s^3+5s^2+6s+K=0.$

s^3	1	6
s^2	5	K
s^1	$\frac{30-K}{5}$	0
s^0	K	

Column 1

for the system to be stable when $K>0$

$\frac{30-K}{5}>0$

$30-K>0$

$30-K=0$

Gain → $\boxed{K=30}$

Using 'K' $s^2 \rightarrow$ row

$5s^2+K=0$

$5s^2+30=0$

$s^2=\pm 6$

$\boxed{s=\pm 2.5j}$ → Intersection with the imaginary axis.

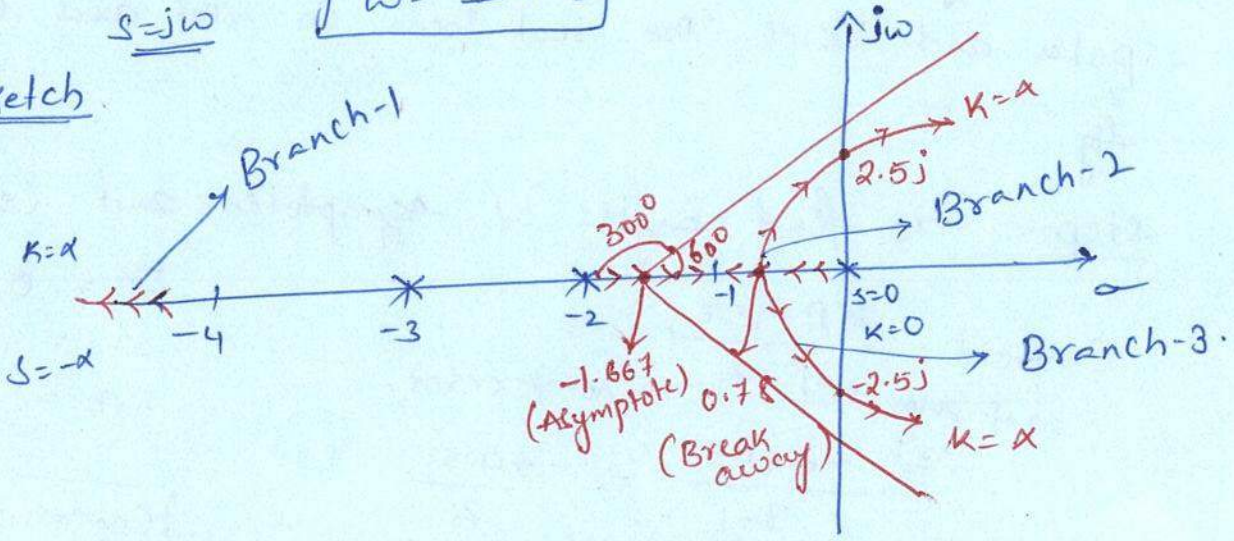
$\therefore s=\sigma+j\omega$

$s=0+j\omega$

$s=j\omega$

$\boxed{\omega=\pm 2.5j}$

Sketch



Problem 2:

Step 6: Sketch the root locus for the unity feedback system whose openloop transfer function is $G(s)H(s) = \frac{K(s+1.5)}{s(s+1)(s+5)}$

Sol:

Step 1: To locate poles and zeros

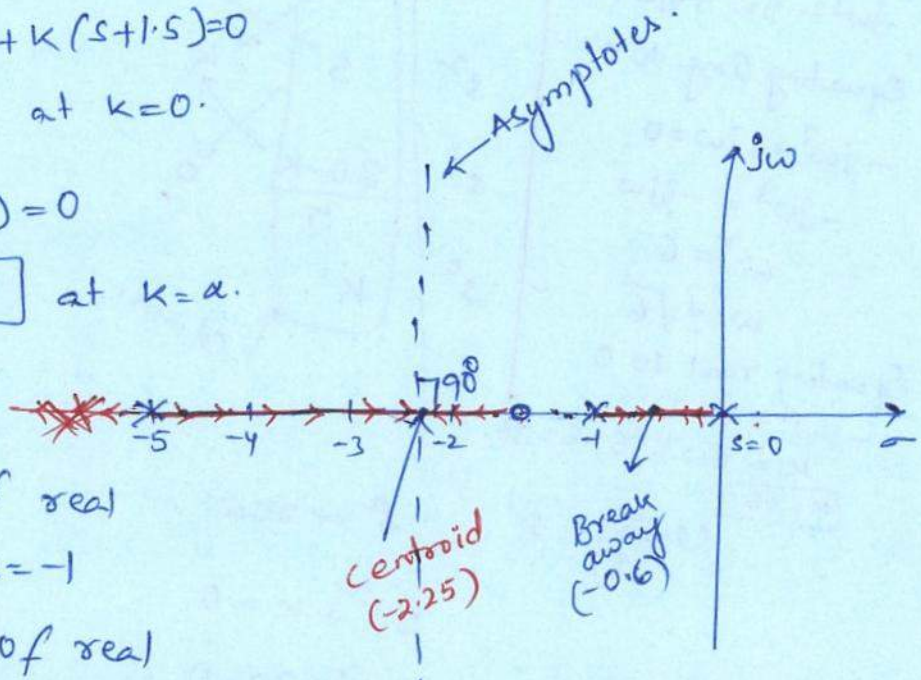
$$1 + G(s)H(s) = 0$$

$$s(s+1)(s+5) + K(s+1.5) = 0$$

Poles → $\boxed{s = 0, -1, -5}$ at $K = 0$.

Zeros → $K(s+1.5) = 0$

$\boxed{s = -1.5}$ at $K = \infty$.



Step 2: To find root locus on real axis

The branch of real axis b/w $s=0$ & $s=-1$ and the branch of real axis b/w $s=-1.5$ and $s=-5$ will be a part of root locus. Because if we choose a test point in this segment then to the right of this point we have odd number of real poles and zeros. The root locus on real axis shown in fig.

Step 3: To find angles of Asymptotes and Centroid.

$$\text{Centroid} = \frac{\Sigma p - \Sigma z}{\text{no of poles} \leftarrow p - z \rightarrow \text{no. of zeros}}$$

$$= \frac{-6 + 1.5}{3 - 1} \Rightarrow \frac{-4.5}{2} \rightarrow -2.25$$

$\Sigma p = 0 + (-1) + (-5) = -6$

$\Sigma z = -1.5$

$\therefore \boxed{\text{Centroid} = -2.25}$

Angle of Asymptotes $= \pm \frac{(180^\circ)(2q+1)}{p-z}$; $q = 0, 1, \cdots p-z-1$

$3-1-1 = 1$

for $q=0 \Rightarrow \theta = \pm \frac{180^\circ(1)}{3-1}$ $q=0$ only.

$= \pm \frac{180^\circ}{2}$

$\boxed{\theta = \pm 90^\circ}$

Step 4: To find the breakaway and break in points.

$\frac{dG(s)}{ds} = 0.$ $\quad G(s) = \frac{K(s+1.5)}{s(s+1)(s+5)}$

$\frac{d}{ds}\left[\frac{K(s+1.5)}{s(s+1)(s+5)}\right]$ since $H(s)=1$.

$= \frac{K(s+1.5)(3s^2+12s+5) - (s^3+6s^2+5s)K}{[s(s+1)(s+5)]^2} = 0.$

$= K(3s^3+12s^2+4.5s^2+5s+18s+7.5) - (s^3+6s^2+5s)K = 0$

$= K[(3s^3+16.5s^2+23s+7.5) - s^3 - 6s^2 - 5s] = 0$

$= 2s^3 + 10.5s^2 + 18s + 7.5 = 0.$

$= s^3 + 5.25s^2 + 9s + 3.75 = 0.$

$s = -0.6, \quad -2.3 \pm j0.89.$

for $\boxed{s = -0.6}$, $K = 1.17$.

For $s = -0.6$, the value of k is positive and real and so it is actual breakaway point. it can be shown in fig. for $s = -2.3 \pm j0.86$ the value of k is not +ve and real and so they can not be breakaway points.

Step5 :- To find angle of departure

Since there are no Complex pole or zero we need not find angle of departure or arrival.

Step6 :- To find crossing point of imaginary axis.

$$S(S+1)(S+5)+k(S+1.5)=0.$$

$$s^3+6s^2+5s+ks+1.5k=0.$$

put $s=jw$

$$(jw)^3+6(jw)^2+5(jw)+k(jw)+1.5k=0.$$

$$-jw^3-6w^2+j5w+jkw+1.5k=0.$$

Imaginary

$$-w^3+5w+kw=0.$$

$$-w^3=-(5+k)w$$

$$w^2=5+k$$

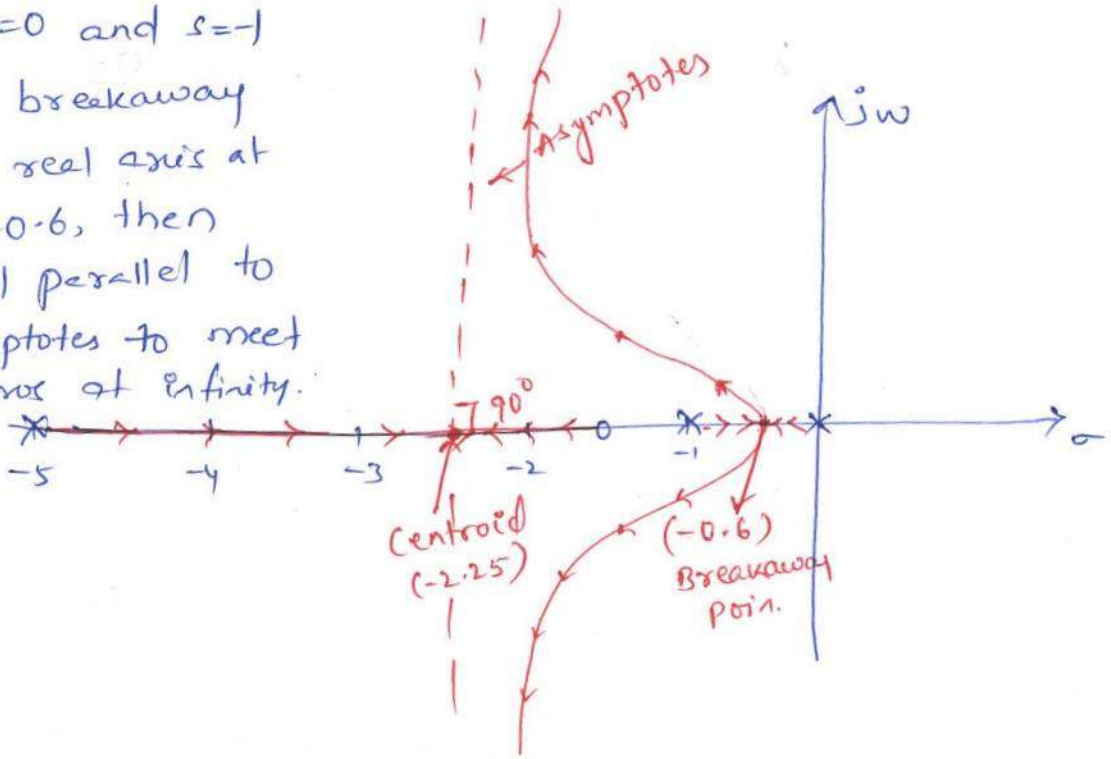
real

$$-6w^2+1.5k=0$$

$$-6(5+k)+1.5k=0$$

$$k=-6.67.$$

Since the value of k is negative there is no Crossing point on imaginary axis. The root locus has three branches. on branch starts at $S=-5$ and ends at finite zero. $S=1.5$. The other two root locus starts at $S=0$ and $S=-1$ and breakaway from real axis at $S=-0.6$, then travel parallel to asymptotes to meet the zeros at infinity.



Problem 4:

The open loop transfer function of a unity feedback system is given by $G(s) = \frac{k(s+9)}{s(s^2+4s+11)}$. sketch the root-locus of the system.

Sol: Step1: To locate poles and zeros

Poles: $s(s^2+4s+11)+k(s+9)=0$.

$s^3+4s^2+11s+ks+9k=0$.

$s(s^2+4s+11)=0$ at $k=0$.

$s^2+4s+11=0$

$s = -2 \pm j2.64$

$\boxed{s = 0,\ -2+j2.64,\ -2-j2.64}$

$\frac{G}{1+GH} = T.F$

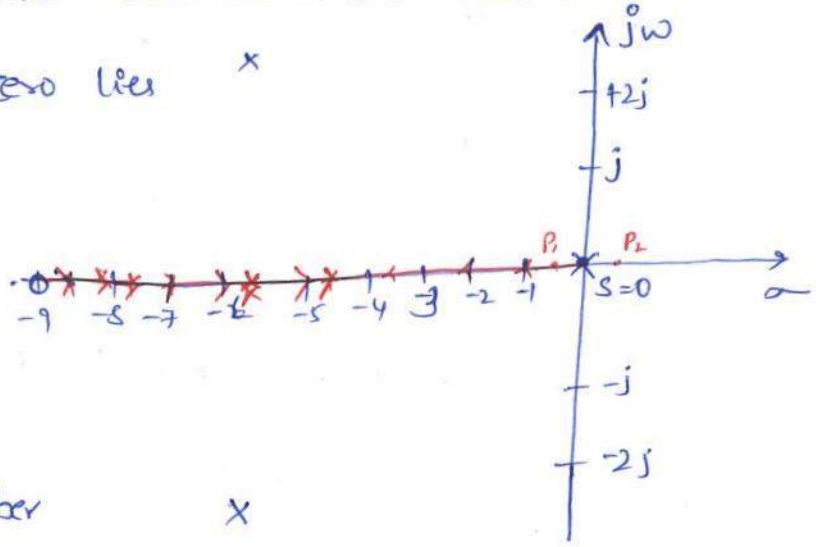
$\frac{\frac{k(s+9)}{s(s^2+4s+11)}}{1+\frac{k(s+9)}{s(s^2+4s+11)}} = 0$

$\frac{k(s+9)}{s(s^2+4s+11)+k(s+9)} = 0$.

Zeros: $k(s+9)=0$.

$\boxed{s=-9}$

Step2: To find the root locus on real axis



One pole and one zero lies on real axis. Choose a test point to the left of $s=0$, then to the right of this point, the total number of poles and zeros in one which is an odd number. Hence the portion of the real axis from $s=0$ to -9 will be a part of root locus. If we choose a test

point to the left of $s=-9$ then to the right of this point the total number of pole and zeros is two, which an even number. Hence the real axis from $s=-9$ to $-\infty$ will not be a part of root locus. The root locus on real axis is shown as bold line.

Step3: To find angles of asymptotes and Centroid.

Since there are 3 poles, the number of rootlocus branches are three.

Centroid: $\frac{\Sigma p - \Sigma z}{p-z}$

$\Sigma p = 0 - 2 + j2.69 - 2 - j2.69$

$= -4$

$\Sigma z = -9$

$= \frac{-4+9}{3-1}$

$= \frac{5}{2} = 2.5$

Centroid = 2.5

Asymptotes: $\theta = \frac{\pm 180^\circ (2q+1)}{p-z}$ $\quad q = 0, 1, \cdots p-z-1$

$3-1-1$

1

$q = 0 \quad \theta = \pm \frac{180^\circ}{2} = \pm 90^\circ$

$q = 1 \quad \theta = \frac{\pm 180^\circ (3)}{2} \pm 270^\circ$ or $\mp 90^\circ$

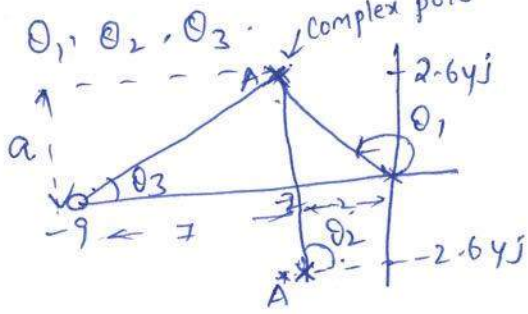
The centroid is marked and from the centroid, the angles of Asymptotes are marked using a protractor. The Asymptotes are drawn in fig as dotted line.

Step4: To find the break away and breakin points. from the location of poles and zero and from the knowledge of typical sketches of rootlocus, it can be

Concluded that there is no possibility of breakaway or breakin points.

Step 5: To find the angle of departure

Let us Consider the Complex pole A. Draw a vectors from all other poles and zero to the pole A. Let the angles of these vectors $\theta_1, \theta_2, \theta_3$.



$$\theta_1 = 180^- \tan^{-1}\left(\frac{2.64}{2}\right)$$

$$= 127.1^\circ$$

$$\theta_2 = 90^\circ$$

$$\theta_3 = \tan^{-1}\frac{2.64}{7} = 20.7^\circ.$$

Angle of departure from the Complex pole A $= 180 - (\theta_1 + \theta_2) + \theta_3$

$$\phi = 180 - (127.1^\circ + 90^\circ) + 20.7^\circ$$

$$= -16.4^\circ.$$

The Angle of departure at the Complex pole A^* is negative of the angle of departure at Complex pole A

$\therefore$ Angle of departure at pole $A^* = -(-16.4) = 16.4^\circ$.

Step 6: TO find the Crossing point of imaginary axes.

put $s = j\omega$ in a characteristic eq.

$$s(s^2+4s+11)+k(s+9)=0.$$

$$s^3+4s^2+11s+ks+9k=0$$

$$(j\omega)^3+4(j\omega)^2+11(j\omega)+k(j\omega)+9k=0$$

$$-j\omega^3-4\omega+j11\omega+jk\omega+9k=0.$$

Imaginary coefficients equate to 0.

$-w^3 + 11w + kw = 0$.

$-w^3 = -(11w + kw)$

$= -w(11 + k)$

$w^2 = 11 + k$

~~$(2.25)^2 - 11 = k$~~ $\Rightarrow \frac{9}{4}k = 11 + k$

~~$k = -5.93$~~ $\cdots \frac{9}{4}k - k = 11$

$1.25k = 11$

$k = \frac{11}{1.25}$

$= 8.8$

real coefficients

$-4w^2 + 9k = 0$.

$-4w^2 = -9k$

~~$w^2 = 9/4 \Rightarrow 2.25$~~

$w^2 = +\frac{9k}{4}$

$w^2 = \frac{9 \times 8.8}{4}$

$w^2 = 19.8$

$w = \sqrt{19.8}$

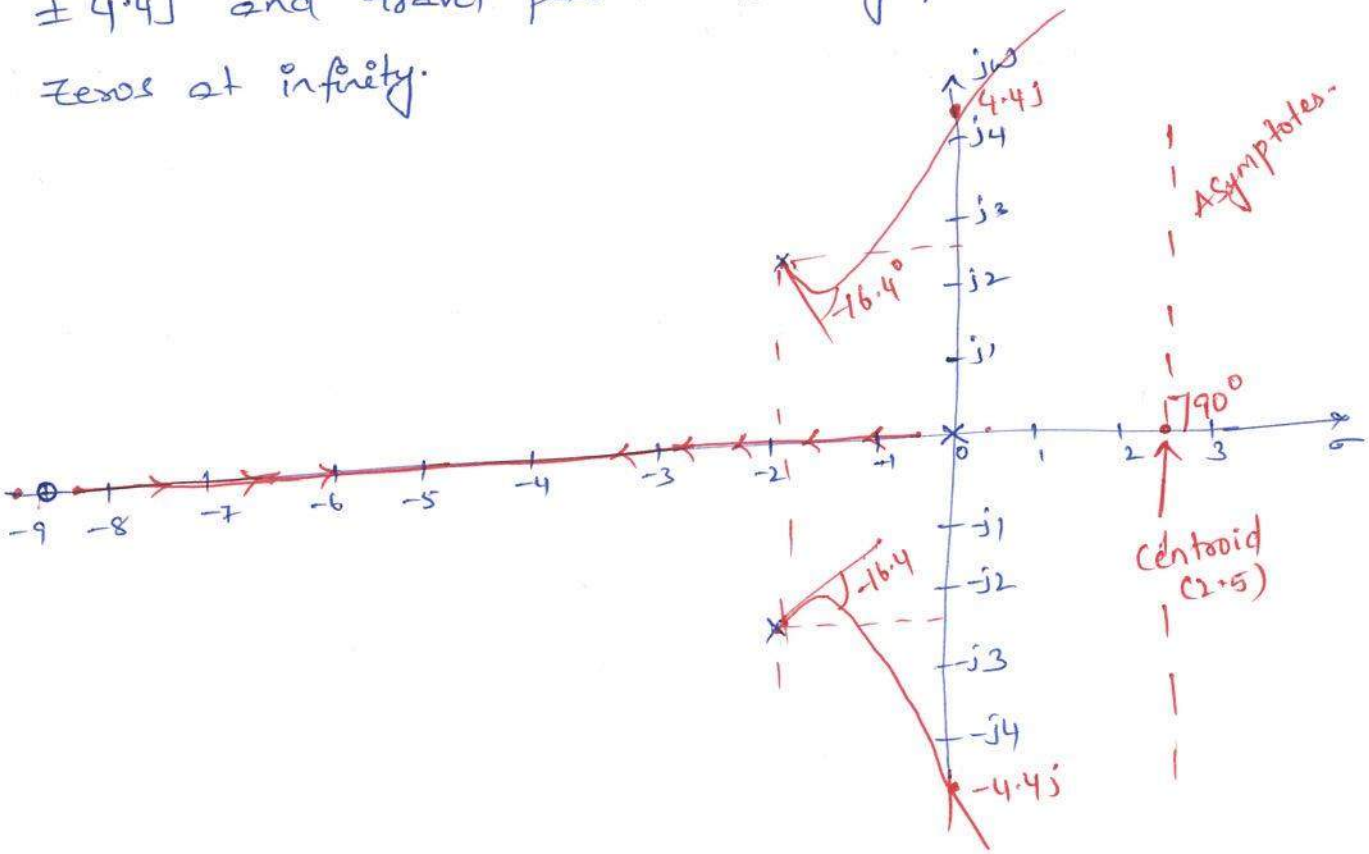
$= \pm 4.4$

$s = jw$

$\boxed{s = \pm 4.4j}$

The Crossing point of root locus is $\pm j4.4$. The value of k at this Crossing point is $k = 8.8$.

The complete root locus has three branches. one branch start at pole at origin and travel through negative real axis to meet the zero at $s = -9$. The other two root locus branches starts at Complex poles Crosses the imaginary axis at $\pm 4.4j$ and travel parallel to asymptotes to meet the zeros at infinity.



Problem 5:

Sketch the root locus of the unity feedback system with $G(s) = \frac{K}{s(s+2)(s^2+2s+4)}$

Step1: poles & zeros.

$1 + G(s)H(s) = 0$

$1 + \frac{K}{s(s+2)(s^2+2s+4)} = 0$

$s(s+2)(s^2+2s+4)+K=0$

$s(s^3+2s^2+4s+2s^2+4s+8)+K=0$

$s^4+2s^3+4s^2+2s^3+4s^2+8s+K=0$

$s^4+4s^3+8s^2+8s+K=0$

$s=0,\ s=-2,\ -1\pm1.732j$

$S = 0, -2, -1+1.73j, -1-1.73j$ → poles

zeros = 0

No zeros

Step2: To find root locus on real axis

Step3: Centroid: $\frac{\Sigma p - \Sigma z}{p-z}$ $q=0,$

$= \frac{0-2-1+1.73j-1-1.73j}{3}$

$= \frac{-4}{3}$

Centroid = −1.33

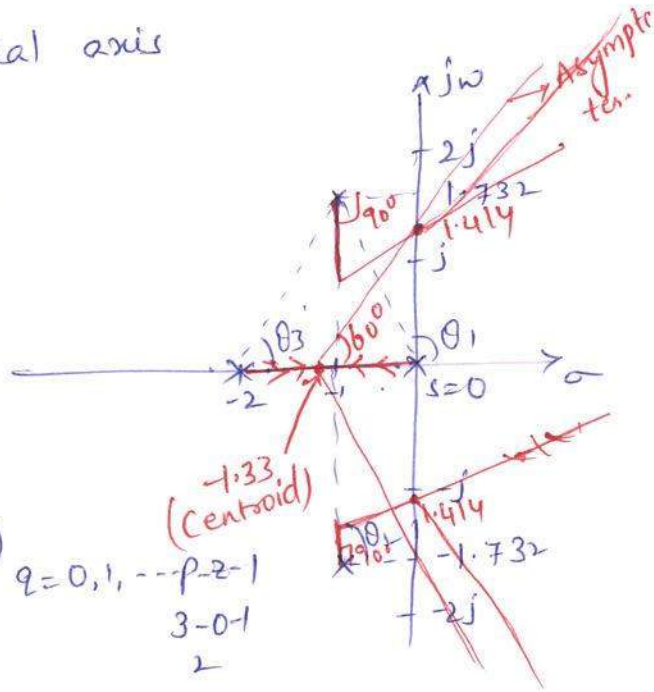
Asymptotes $= \frac{\pm 180^\circ(2q+1)}{p-z}$ $q=0,1,\cdots p-z-1$

$3-0-1$

2

for $q=0 = \frac{\pm 180^\circ(1)}{3-0} \Rightarrow \pm 60^\circ$

$q=1 = \frac{\pm 180^\circ(3)}{3} = \pm 180^\circ$



$$q=2 \Rightarrow \pm \frac{60^\circ (5)}{3} = 300^\circ$$

4. Breakaway points

$$\frac{d}{ds}G(s) = 0$$

$$\frac{d}{ds}\left[\frac{k}{s(s+2)(s^2+2s+4)}\right]$$

$$\frac{d}{ds}\left\{\frac{k}{s^4+4s^3+8s^2+8s}\right\} = 0 \qquad \frac{du}{dv} = \frac{v\,du - u\,dv}{v^2}$$

$$\frac{(s^4+4s^3+8s^2+8s)0 - k(4s^3+12s^2+16s+8)}{(s^4+4s^3+8s^2+8s)^2} = 0$$

$$-k(4s^3+12s^2+16s+8) = 0$$

$$4s^3+12s^2+16s+8 = 0$$

$$s^3+3s^2+4s+2 = 0$$

$$s =$$

5. Angle of departure.

$$= 180^\circ - (\theta_1+\theta_2+\theta_3)$$

$$= 180^\circ - (120+90+60)$$

$$= 90^\circ$$

where

$$\theta_1 = 180^\circ - \tan^{-1}\left(\frac{1.732}{1}\right)$$

$$= 120^\circ$$

$$\theta_2 = 90^\circ$$

$$\theta_3 = \tan^{-1}\left(\frac{1.732}{1}\right)$$

$$= 60^\circ$$

6. Crossing point on imaginary axis

$$s = jw.$$

$$s^4 + 4s^3 + 8s^2 + 8s + k = 0$$

$$(jw)^4 + 4(jw)^3 + 8(jw)^2 + 8(jw) + k = 0.$$

$$\omega^4 + 4j\omega^3 - 8\omega^2 + j8\omega + k = 0$$

Imaginary

$$-4j\omega^3 + 8j\omega = 0$$

$$-4j\omega^2 = -8$$

$$\omega^2 = 2 \Rightarrow \omega = \sqrt{2} = 1.414$$

$$s = j\omega$$

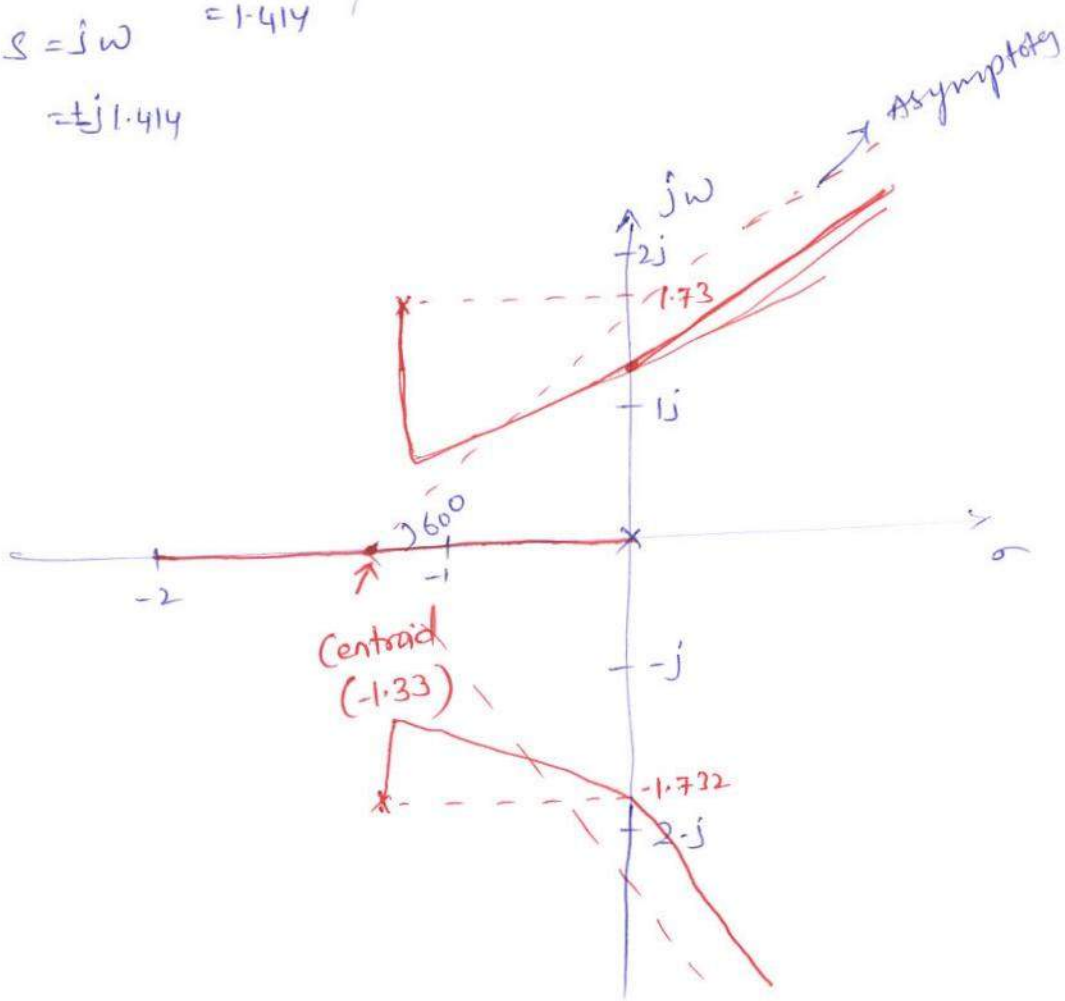
$$= \pm j1.414$$

Real

$$\omega^4 - 8\omega^2 + k = 0$$

$$4 - 16 + k = 0$$

$$k = +12$$



3.19 SHORT QUESTION AND ANSWER

Q3.1 What is time response?

The time response is the output of the closed loop system as a function of time. It is denoted by c(t). It is given by inverse Laplace of the product of input and transfer function of the system.

The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\,H(s)}$

Response in s-domain, $C(s) = \frac{R(s)\,G(s)}{1+G(s)\,H(s)}$

Response in time domain, $c(t) = L^{-1}[C(s)] = L^{-1}\left[\frac{R(s)\,G(s)}{1+G(s)\,H(s)}\right]$

Q3.2 What is transient and steady state response?

The transient response is the response of the system when the input changes from one state to another. The response of the system as $t \to \infty$ is called steady state response.

Q3.3 What is the importance of test signals?

The test signals can be easily generated in test laboratories and the characteristics of test signals resembles, the characteristics of actual input signals. The test signals are used to predetermine the performance of the system. If the response of a system is satisfactory for a test signal, then the system will be suitable for practical applications.

Q3.4 Name the test signals used in control system

The commonly used test input signals in control system are impulse, step, ramp, acceleration and sinusoidal signals.

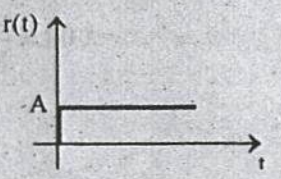
Q3.5 Define step signal.

The step signal is a signal whose value changes from 0 to A and remains constant at A for $t > 0$. The mathematical representation of step signal is

$r(t) = A\,u(t)$

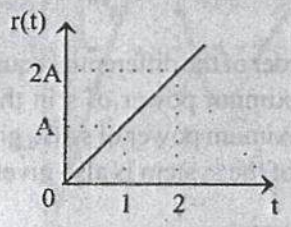
Where, $u(t) = 1, t \geq 0$

$u(t) = 0, t < 0$



Q3.6 Define ramp signal.

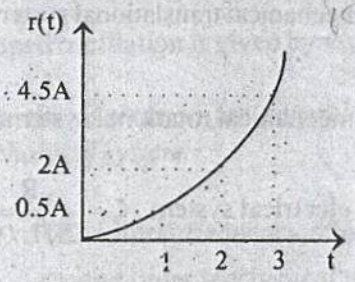
A ramp signal is a signal whose value increases linearly with time from an initial value of zero at t = 0. It is mathematically represented as



$$r(t) = At, \; t \geq 0$$
$$= 0, \; t < 0$$

Q3.7 Define parabolic signal.

It is a signal in which the instantaneous value varies as square of the time from an initial value of zero at t = 0. It is mathematically represented as



$$r(t) = \frac{At^2}{2}, \; t \geq 0$$
$$= 0, \; t < 0$$

Q3.8 What is an impulse signal?

A signal which is available for very short duration is called impulse signal. Ideal impulse signal is a unit impulse signal which is defined as a signal having zero values at all time except at t = 0. At t = 0 the magnitude becomes infinite. It is denoted by δ(t) and mathematically expressed as

$$\delta(t) = 0 \text{ for } t \neq 0 \quad \text{and} \quad \lim_{t_1 \to 0} \int_{-t_1}^{+t_1} \delta(t)\, dt = 1$$

Q3.9 What is weighting function?

The impulse response of system is called weighting function. It is given by inverse laplace transform of system transfer function.

Q3.10 Define pole.

The pole of a function, F(s) is the value at which the function, F(s) becomes infinite, where F(s) is a function of complex variable **s.**

Q3.11 Define zero.

The zero of a function, F(s) is the value at which the function, F(s) becomes zero, where F(s) is a function of complex variable s.

Q3.12 What is the order of a system?

The order of the system is given by the order of the differential equation governing the system. It is also given by the maximum power of s in the denominator polynomial of transfer function. The maximum power of s also gives the number of poles of the system and so the order of the system is also given by number of poles of the transfer function.

Q3.13 Define damping ratio.

The damping ratio is defined as the ratio of the actual damping to critical damping.

Q3.14 Give the expression for damping ratio of mechanical and electrical system.

The damping ratio of second order mechanical translational system, $\zeta = \frac{B}{2\sqrt{MK}}$

The damping ratio of second order mechanical rotational system, $\zeta = \frac{B}{2\sqrt{JK}}$

The damping ratio of second order electrical system, $\zeta = \frac{R}{2\sqrt{L/C}}$

Q3.15 How the system is classified depending on the value of damping?

Depending on the value of damping, the system can be classified into the following four cases

Case 1 : Undamped system, $\zeta = 0$

Case 2 : Underdamped system, $0 < \zeta < 1$

Case 3 : Critically damped system, $\zeta = 1$

Case 4 : Over damped system, $\zeta > 1$.

Q3.16 What will be the nature of response of a second order system with different types of damping?

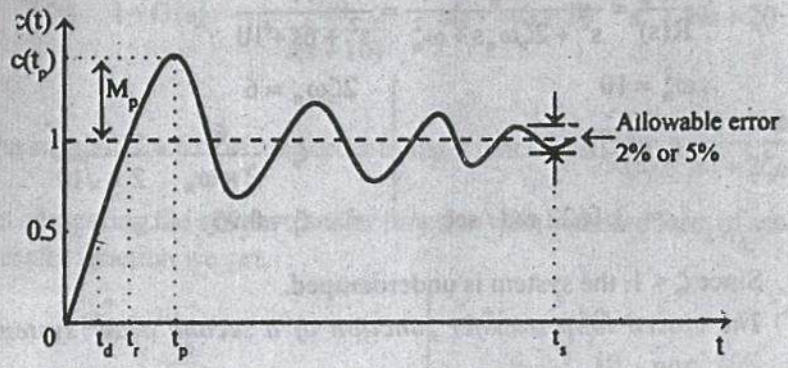
For undamped system the response is oscillatory.

For underdamped system the response is damped oscillatory.

For critically damped system the response is exponentially rising.

For overdamped system the response is exponentially rising but the rise time will be very large.

Q3.17. *Sketch the response of a second order underdamped system.*



Q3.18. *What is damped frequency of oscillation?*

In underdamped system the response is damped oscillatory. The frequency of damped oscillation is given by $\omega_d = \omega_n\sqrt{1-\zeta^2}$

Q3.19. *Give the expression for natural frequency of oscillations of electrical and mechanical system.*

The natural frequency of oscillation of second order mechanical translational system $\Big\}\ \omega_n = \sqrt{\frac{K}{M}}$

The natural frequency of oscillation of second order mechanical rotational system $\Big\}\ \omega_n = \sqrt{\frac{K}{J}}$

The natural frequency of oscillation of second order electrical system, $\Big\}\ \omega_n = \frac{1}{\sqrt{LC}}$

Q3.20. *The closed loop transfer function of second order system is* $\frac{C(s)}{R(s)} = \frac{10}{s^2+6s+10}$. *What is the type of damping in the system?.*

Let us compare the given transfer function with the standard form of second order transfer function

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{10}{s^2 + 6s + 10}$$

$\therefore \omega_n^2 = 10$ | $2\zeta\omega_n = 6$

$\omega_n = \sqrt{10}$ | $\zeta = \frac{6}{2 \times \omega_n} = \frac{6}{2 \times \sqrt{10}}$

$= 3.1622$ rad / sec | $\therefore \zeta = 0.95$

Since $\zeta < 1$, the system is underdamped.

Q3.21 *The closed loop transfer function of a second order system is given by* $\frac{200}{s^2 + 20s + 200}$. *Determine the damping ratio and natural frequency of oscillation.*

Let us compare the given transfer function to standard form of second order transfer function

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{200}{s^2 + 20s + 200}$$

$\therefore \omega_n^2 = 200$ | $2\zeta\omega_n = 20$

$\omega_n = \sqrt{200} = 14.14$ rad / sec | $\zeta = \frac{20}{2 \times \omega_n} = \frac{20}{2 \times 14.14}$

$\therefore \zeta = 0.707$

Damping ratio, $\zeta = 0.707$

Natural frequency of oscillation, $\omega_n = 14.14$ rad/sec.

Q3.22 *A second order system has a damping ratio of 0.6 and natural frequency of oscillation is 10 rad/sec. Determine the damped frequency of oscillation.*

Damped frequency of oscillation, $\omega_d = \omega_n\sqrt{1-\zeta^2} = 10\sqrt{1-(0.6)^2}$

$= 10 \times 0.8 = 8$ rad / sec

Q3.23 *The open loop transfer function of a unity feedback system is* $G(s) = \frac{20}{s(s+10)}$.

What is the nature of response of closed loop system for unit step input.

The closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{20/s(s+10)}{1+\frac{20}{s(s+10)}} = \frac{20}{s(s+10)+20} = \frac{20}{s^2+10s+20}$$

The standard form of second order transfer function is $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$.

On comparing the system transfer function with standard form of second order transfer function we get,

$\omega_n^2 = 20$

$\therefore \omega_n = \sqrt{20} = 4.47$ rad / sec

$2\zeta\omega_n = 10$

$\zeta = \frac{10}{2\times\omega_n} = \frac{10}{2\times 4.47} = 1.12$

Since damping ratio, $\zeta > 1$, the system is overdamped and the response will be exponentially rising.

Q3.24 List the time domain specifications.

The time domain specifications are :

(i) Delay time
(ii) Rise time
(iii) Peak time
(iv) Maximum overshoot
(v) Settling time.

Q3.25 Define delay time.

It is the time taken for response to reach 50% of the final value, for the very first time.

Q3.26 Define rise time.

It is the time taken for response to raise from 0 to 100% for the very first time. For underdamped system, the rise time is calculated from 0 to 100%. But for overdamped system it is the time taken by the response to raise from 10% to 90%. For critically damped system, it is the time taken for response to raise from 5% to 95%.

Q3.27 Define peak time.

It is the time taken for the response to reach the peak value for the very first time (or) It is the time taken for the response to reach peak overshoot, M_p.

Q3.28 Define peak overshoot.

It is defined as the ratio of the maximum peak value measured from final value to final value. Let final value = $c(\infty)$, Maximum value = $c(t_p)$

$$\text{Peak overshoot, } M_p = \frac{c(t_p) - c(\infty)}{c(\infty)}$$

Q3.29 Define settling time.

It is defined as the time taken by the response to reach and stay within a specified error and the error is usually specified as % of final value. The usual tolerable error is 2% or 5% of the final value.

Q3.30 The damping ratio of a system is 0.75 and the natural frequency of oscillation is 12 rad sec. Determine the peak overshoot and the peak time.

$$\text{Peak overshoot, } M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} = e^{\frac{-0.75\pi}{\sqrt{1-(0.75)^2}}}$$

$$\%M_p = 0.028 \times 100 = 2.8\%$$

$$\text{Damped frequency of oscillation, } \omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$= 12\sqrt{1-(0.75)^2} = 7.94 \text{ rad / sec.}$$

$$\text{Peak time, } t_p = \frac{\pi}{\omega_d} = \frac{\pi}{7.94} = 0.396 \text{ sec.}$$

Q3.31 The damping ratio of system is 0.6 and the natural frequency of oscillation is 8 rad sec. Determine the rise time.

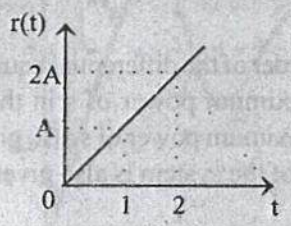
$$\text{Rise time, } t_r = \frac{\pi - \theta}{\omega_d}$$

$$\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = \tan^{-1}\frac{\sqrt{1-(0.6)^2}}{0.6} = 53.13^\circ$$

$$= \frac{53.13}{180} \times \pi \text{ rad} = 0.927 \text{ rad}$$

Q3.6 Define ramp signal.

A ramp signal is a signal whose value increases linearly with time from an initial value of zero at t = 0. It is mathematically represented as



$$r(t) = At, \; t \geq 0$$
$$= 0, \; t < 0$$

Q3.7 Define parabolic signal.

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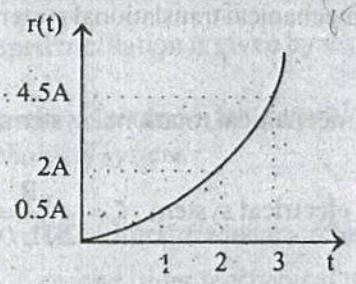


$$r(t) = \frac{At^2}{2}, \; t \geq 0$$
$$= 0, \; t < 0$$

Q3.8 What is an impulse signal?

A signal which is available for very short duration is called impulse signal. Ideal impulse signal is a unit impulse signal which is defined as a signal having zero values at all time except at t = 0. At t = 0 the magnitude becomes infinite. It is denoted by δ(t) and mathematically expressed as

$$\delta(t) = 0 \text{ for } t \neq 0 \quad \text{and} \quad \lim_{t_1 \to 0} \int_{-t_1}^{+t_1} \delta(t)\, dt = 1$$

Q3.9 What is weighting function?

The impulse response of system is called weighting function. It is given by inverse laplace transform of system transfer function.

Q3.10 Define pole.

The pole of a function, F(s) is the value at which the function, F(s) becomes infinite, where F(s) is a function of complex variable **s**.

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 8\sqrt{1-(0.6)^2} = 6.4 \text{ rad / sec}$$

$$\therefore t_r = \frac{\pi - 0.927}{6.4} = 0.34 \text{ sec}$$

Q3.32 *What is type number of a system? What is its significance?*

The type number is given by number of poles of loop transfer function at the origin. The type number of the system decides the steady state error.

Q3.33 *Distinguish between type and order of a system.*

(i) Type number is specified for loop transfer function but order can be specified for any transfer function. (open loop or closed loop transfer function).

(ii) The type number is given by number of poles of loop transfer function lying at origin of s-plane but the order is given by the number of poles of transfer function.

Q3.34 *For the system with following transfer function, determine type and order of the system.*

(i) $G(s)\,H(s) = \dfrac{K}{s(s+1)(s^2+6s+8)}$ (ii) $G(s)\,H(s) = \dfrac{20(s+2)}{s^2(s+3)(s+0.5)}$

(iii) $G(s)\,H(s) = \dfrac{(s+4)}{(s-2)(s+0.25)}$ (iv) $G(s)\,H(s) = \dfrac{10}{s^3(s^2+2s+1)}$

Ans: (i) Type - 1, order - 4
(ii) Type - 2, order - 4
(iii) Type - 0, order - 2
(iv) Type - 3, order - 5.

Q3.35 *What is steady state error?*

The steady state error is the value of error signal e(t), when t tends to infinity. The steady state error is a measure of system accuracy. These errors arise from the nature of inputs, type of system and from non-linearity of system components.

Q3.36 *What are static error constants?*

The K_p, K_v and K_a are called static error constants. These constants are associated with steady state error in a particular type of system and for a standard input.

Q3.37 *Define positional error constant.*

The positional error constant $K_p = \underset{s\to 0}{Lt}\ G(s)\,H(s)$. The steady state error in type - 0 system when the input is unit step is given by $\frac{1}{1+K_p}$.

Q3.38 *Define velocity error constant.*

The velocity error constant $K_v = \underset{s\to 0}{Lt}\ sG(s)\,H(s)$. The steady state error in type-1 system for unit ramp input is given by $1/K_v$.

Q3.39 *Define acceleration error constant.*

The acceleration error constant $K_a = \underset{s\to 0}{Lt}\ s^2\,G(s)\,H(s)$. The steady state error in type-2 system for unit parabolic input is given by $1/K_a$.

Q3.40 *A unity feedback system has a open loop transfer function of* $G(s) = \frac{10}{(s+1)(s+2)}$. *Determine the steady state error for unit step input.*

The steady state error for unit step input, $e_{ss} = \frac{1}{1+K_p}$.

Where $K_p = \underset{s\to 0}{Lt}\ G(s)\,H(s)$

For unity feedback system $H(s) = 1$

$$\therefore K_p = \underset{s\to 0}{Lt}\ G(s) = \underset{s\to 0}{Lt}\ \frac{10}{(s+1)(s+2)} = 5$$

$$\therefore e_{ss} = \frac{1}{1+5} = \frac{1}{6}$$

Q3.41 *A unity feedback system has a open loop transfer function of* $G(s) = \frac{25(s+4)}{s(s+0.5)(s+2)}$. *Determine the steady state error for unit ramp input.*

The steady state error for unit ramp input is $e_{ss} = \frac{1}{K_v}$,

Where $K_v = \underset{s\to 0}{Lt}\ s\,G(s)\,H(s)$. For unity feedback system $H(s) = 1$.

$$\therefore K_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s\left[\frac{25\,(s+4)}{s(s+0.5)\,(s+2)}\right] = \frac{25\times 4}{0.5\times 2} = 100$$

$$e_{ss} = \frac{1}{K_v} = \frac{1}{100} = 0.01$$

Q3.42 A unity feedback system has a open loop transfer function of $G(s) = \dfrac{20\,(s+5)}{s(s+0.1)\,(s+3)}$. *Determine the steady state error for parabolic input.*

The steady state error for unit parabolic input is $e_{ss} = \dfrac{1}{K_a}$

Where $K_a = \lim_{s\to 0} s^2 G(s)\,H(s)$. For unity feedback system $H(s) = 1$.

$$\therefore K_a = \lim_{s\to 0} s^2\left[\frac{20\,(s+5)}{s^2(s+0.1)\,(s+3)}\right] = \frac{20\times 5}{0.1\times 3} = \frac{100}{0.3} = 333.33$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{333.33} = 0.003$$

Q3.43 What are generalized error coefficients?

They are the coefficients of generalized error series. The generalized error series is given by

$$e(t) = C_0 r(t) + C_1 \dot{r}(t) + \frac{C_2}{2!}\ddot{r}(t) + \frac{C_3}{3!}\dddot{r}(t) + \ldots\ldots + \frac{C_n}{n!}r^{n}(t)\ldots\ldots$$

The coefficients $C_0, C_1, C_2 \ldots\ldots C_n$ are called generalized error coefficients or dynamic error coefficients.

The nth coefficient, $C_n = \lim_{s\to 0} \dfrac{d^n}{ds^n} F(s)$, Where $F(s) = \dfrac{1}{1+G(s)\,H(s)}$

Q3.44 Give the relation between generalized and static error coefficients.

The following expression shows the relation between generalized and static error coefficient

$$C_0 = \frac{1}{1+K_p}, \quad C_1 = \frac{1}{K_v}, \; C_2 = \frac{1}{K_a}$$

Q3.45 *Mention two advantages of generalized error constants over static error constants.*

(i) Generalized error series gives error signal as a function of time.

(ii) Using generalized error constants the steady state error can be determined for any type of input but static error constants are used to determine steady state error when the input is anyone of the standard input.

Q3.46 *What is the effect on system performance when a proportional controller is introduced in a system?*

The proportional controller improves the steady-state tracking accuracy, disturbance signal rejection and relative stability of the system. It also increases the loop gain of the system which results in reducing the sensitivity of the system to parameter variations.

Q3.47 *What is the disadvantage in proportional controller?*

The disadvantage in proportional controller is that it produces a constant steady state error.

Q3.48 *What is the effect of PI controller on the system performance?*

The PI controller increases the order of the system by one, which results in reducing, the steady state error. But the system becomes less stable than the original system.

Q3.49 *What is the effect of PD controller on the system performance?*

The effect of PD controller is to increase the damping ratio of the system and so the peak overshoot is reduced.

Q3.50 *Why derivative controller is not used in control systems?*

The derivative controller produces a control action based on rate of change of error signal and it does not produce corrective measures for any constant error. Hence derivative controller is not used in control systems.

3.20 EXERCISES

E3.1 What is the unit-step response of the system shown in fig E3.1

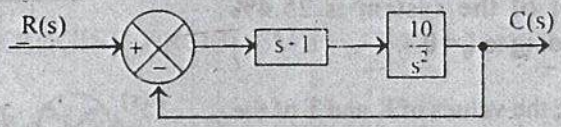


Fig E3.1

E3.2 Obtain the unit-step response of a unity-feedback system whose open-loop transfer function is $G(s) = 5(s+20)/s\,(s+4.59)\,(s^2+3.41s+16.35)$.

E3.3 The open loop transfer function of an unity feedback control system is given by $G(s) = 100/s\,(s+2)\,(s+5)$. For unit step input, find the time response of the closed loop system and determine % over shoot and the rise time.

E3.4 A Servomechanism has its moment of inertia $J = 10\times10^{-6}$ Kg-m^2, retarding friction, $B = 400\times10^{-6}$ N-m/(rad/sec) and elasticity coefficient, $K = 0.004$ N-m/rad. Find the natural frequency and damping factor of the system.

E3.5 For a second order system whose open loop transfer function $G(s) = 4/s(s+2)$, determine the maximum over shoot, the time to reach the maximum overshoot when a step displacement of 18° is given to the system. Find the rise time, time constant and the settling time for an error of 7%.

E3.6 Consider the unity feedback closed loop system where the forward transfer function is $G(s) = 25/s\,(s+5)$. Obtain the rise time, Peak time, Maximum overshoot and the settling time when the system is subjected to a unit-step input.

E3.7 Consider the system shown in fig E3.7, where $\zeta = 0.6$ and $\omega_n = 0.5$ rad/sec. Determine the rise time, peak time, maximum overshoot and settling time, when the system is subjected to a unit-step input

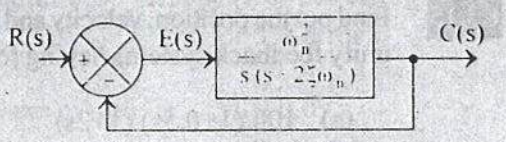


Fig E3.7.

E3.8 For the system shown in fig E3.8, determine the values of K and K_h so that the maximum overshoot in the unit step response is 0.2 and the peak time is 1 sec. With these values of K and K_h, obtain rise time and settling time.

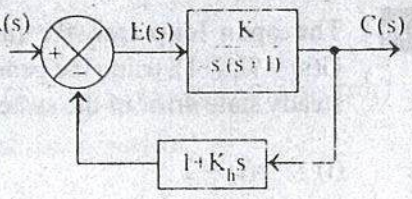


Fig E3.8.

E3.9 The system shown in fig E3.9 subjected to a unit-step input. Determine the values of K and T, where the Maximum overshoot of the system is 25.4% corresponding to $\zeta = 0.4$

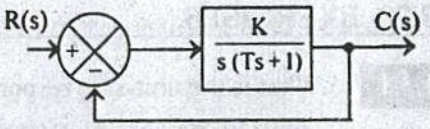
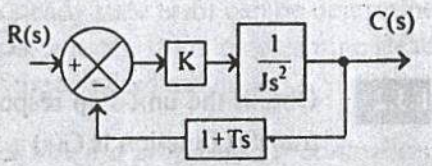


Fig E3.9.

E3.10 Determine the values of K and T of the closed-loop system shown in Fig E3.10 so that the maximum overshoot in unit-step response is 25% and the peak time is 2 sec. Assume that J=1 Kg-m^2.



FigE3.10.

E3.11 A unity-feedback system is characterized by the open-loop transfer function $G(s) = 1/s(0.5s+1)(0.2s+1)$. a) Determine the steady-state errors to unit-step, unit-ramp and unit parabolic inputs. b) Determine rise time, peak time, peak overshoot and settling time of the unit-step respone of the system.

E3.12 For a system whose $G(s) = 10/s(s+1)(s+2)$, find the steady state error when it is subjected to the input, $r(t) = 1+2t+1.5\,t^2$.

E3.13 A unity feedback system has $G(s)=1/s(1+s)$. The input to the system is described by $r(t) = 4+6t+2t^3$. Find the generalized error coefficients and steady state error.

E3.14 A unity feedback system has the forward path transfer function $G(s) = 10/(s+1)$. Find the steady state error and the generalised error coefficient for $r(t) = t$.

E3.15 Find out the position, velocity and acceleration error coefficients for the following unity feedback systems having forward loop transfer function G(s) as.

(a) $100/(1+0.5s)(1+2s)$ (b) $K/s(1+0.1s)(1+s)$

(c) $K/s^2(s^2+8s+100)$ (d) $K(1+s)(1+2s)/s^2(s^2+4s+20)$

E3.16 The open loop transfer function of a unity feedback control system is $G(s) = 9/(s+1)$, using the generalized error series determine the error signal and steady state error of the system when the system is excited by

(i) $r(t) = 2$ (ii) $r(t) = t$

(iii) $r(t) = 3t^2/2$ (iv) $r(t) = 1+2t+3t^2/2$

P 3.17 For unity feedback system having open loop transfer function as $G(s) = K(s+2)/s^2(s^2+7s+12)$, determine (i) type of system (ii) error constants K_p, K_v and K_a (iii) steady state error for parabolic input.

ANSWER FOR EXERCISE PROBLEMS

P 3.1 $c(t) = -1.1455\, e^{-8.87t} + 0.1455 e^{-1.13t} + 1$

P 3.2 $c(t) = 1 + \frac{3}{8}e^{-t}\cos 3t - \frac{17}{24}e^{-t}\sin 3t - \frac{11}{8}e^{-3t}\cos t - \frac{13}{8}e^{-3t}\sin t$

P 3.3 $c(t) = \left[1 - 0.186\, e^{-7.45t} - 0.88\, e^{0.225t}\cos(3.65t - 22°)\right]$

As t tends to infinity, c(t) tends to infinity and so the system is unstable. Therefore % over shoot and rise time are not defined.

E3.4 Natural frequency, $\omega_n = 20$ rad/sec, Damping-factor, $\zeta = 1$.

E3.5 Maximum overshoot = 0.16, when input is 18°, $M_p = 2.88°$

Peak time, $t_p = 1.81$sec. Rise time, $t_r = 1.21$ sec

Time constant, T=1 sec, Settling time for 7% error = 2.66 sec.

E3.6 Rise time, $t_r = 0.55$sec, %Peak overshoot, $M_p = 9.5\%$

Peak time, $t_p = 0.785$sec, Settling time, $t_s = 1.33$sec (for 2% error)
$= 1$ sec(for 5% error)

E3.7 Rise time, $t_r = 0.55$sec, Maximum overshoot, $M_p = 0.095$

Peak time, $t_p = 0.785$sec, Settling time, $t_s = 1$ sec (for 5% criterion)

E3.8 K=12.5, Rise time, $t_r = 0.65$sec

$K_h = 0.178$, Settling time, $t_s = 2.48$ sec (for 2% error)

$t_s = 1.86$ sec (for 5% error)

E3.9 K=1.42 T=1.09

E3.10 K=2.95 N-m T=0.471 sec

E3.11 (a) $e_{ss}\big|_{\text{unit step}} = 0$

$e_{ss}\big|_{\text{unit ramp}} = 1$

$e_{ss}\big|_{\text{unit parabola}} = \infty$

(b) Rise time, $t_r = 1.91$ sec

Peak time, $t_p = 2.79$ sec

Peak overshoot. $M_p = 0.1265$

Settling time, $t_s = 5.4$ sec

E3.12 The total steady state error is ∞.

E3.13 $C_0 = 0$ $C_1 = 1$ $e_{ss} = \infty$

$C_2 = 0$ $C_3 = -6$

E3.14 $C_o = 1/11$ $e_{ss} = \infty$.

$C_1 = 10/121$

E3.15

Question	K_p	K_v	K_a
(a)	100	0	0
(b)	∞	K	0
(c)	∞	∞	K/100
(d)	∞	∞	K/20

E3.16 (i) $e(t) = 0.2$; $e_{ss} = 0.2$

(ii) $e(t) = 0.1t + 0.09$; $e_{ss} = \infty$.

(iii) $e(t) = 0.15t^2 + 0.27\,t - 0.054$; $e_{ss} = \infty$

(iv) $e(t) = 0.15t^2 + 0.77\,t + 0.226$; $e_{ss} = c$

E3.17 (i) It is type 2 system

(ii) $K_p = \infty$.

$K_v = \infty$.

$K_a = K/6$

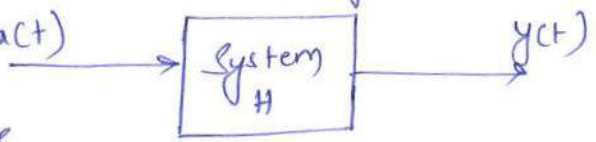
(iii) $e_{ss} = 6/K$

3. Frequency Response Analysis.

The frequency Response is the steady state response of a system when the input to the system is a sinusoidal signal.

Consider a linear time invariant (LTI) system, H.

Let x(t) be an input sinusoidal signal, the response (or) output y(t) is also a sinusoidal signal of same frequency but with different magnitude and phase angle.



The magnitude and phase relationship between the sinusoidal input and the steady state output of a system is termed the frequency response. In linear time invariant systems the frequency response is independent of the amplitude and phase of the input signal.

$x(t) = X \sin \omega t$

$y(t) = Y \sin(\omega t + \phi)$.

The frequency response of a system is normally obtained by varying the frequency of the input signal by keeping the magnitude of the input signal at a constant value.

In the system transfer function $T(s)$, if 's' is replaced by 'jw' then the resulting transfer function $T(jw)$ is called sinusoidal transfer function.

The frequency response of the system can be directly obtained from the sinusoidal transfer function $T(j\omega)$ of the system. The transfer function $T(j\omega)$ is a complex function of frequency. The magnitude and phase of $T(j\omega)$ are functions of frequency and can be evaluated for various values of frequency.

The frequency response can be evaluated for both open loop system and closed loop system.

Open loop transfer function, $G(j\omega) = |G(j\omega)| \angle G(j\omega)$

Loop transfer function $G(j\omega)H(j\omega) = |G(j\omega)H(j\omega)| \angle G(j\omega)H(j\omega)$

Closed loop transfer function $\frac{C(j\omega)}{R(j\omega)} = M(j\omega)$

$= |M(j\omega)| \angle M(j\omega)$

Advantages of frequency Response Analysis:

1. The Absolute and relative stability of the Closed loop system can be estimated from the knowledge of their open loop frequency response.
2. The practical testing of a systems can be easily carried with available sinusoidal signal generators and precise measurement equipements.
3. The transfer function of complicated systems can be determined experimentally by frequency response tests.

4. The design and parameter adjustment of the openloop transfer function of a system for specified closed loop performance is carried out more easily in frequency domain.

5. When the system is designed by use of the frequency response analysis, the effect of noise disturbance and parameters variations are relatively easy to visualize and incorporate corrective measures.

6. The frequency response analysis and design can be extended to certain non-linear control systems.

Frequency domain specifications:

The performance and characteristics of a system in frequency domain are measured interms of frequency domain specifications. The requirements of system to be designed are usually specified interms of these specifications.

The frequency domain specifications are:

1. Resonant peak, M_r
2. Resonant frequency, ω_r
3. Bandwidth
4. Cut-off rate
5. Gain Margin.
6. Phase Margin.

Resonant peak (M_r) :-

The maximum value of the magnitude of closed loop transfer function is called the resonant peak M_r. A large resonant peak corresponds to a large overshoot in transient response.

Resonant frequency (ω_r) :-

The frequency at which the resonant peak occurs is called resonant frequency ω_r. This is related to the frequency of oscillation in the step response and thus it is indicative of the speed of transient response.

Bandwidth (ω_b) :-

The bandwidth is the range of frequency's for which the system gain is more than $-3db$. The frequency at which the gain is $-3db$ is called Cut-off frequency. Bandwidth is usually defined for closed loop system and it transmits the signals whose frequencies are less than cut-off frequency. The Bandwidth is a measure of the ability of a feedback system to reproduce the input signal, noise rejection characteristics and rise time. A large bandwidth corresponds to a small rise time or fast response.

Cut-off Rate:

The slope of the large magnitude curve near the Cutt off frequency is called the cut-off rate. The cut-off rate indicates the ability of the system is to distinguish the signal from noise.

Gain margin, K_g:-

The Gain margin, K_g is defined as the reciprocal of the magnitude of open loop transfer function at phase Cross over frequency. The frequency at which the phase of open loop transfer function is 180° is called the phase cross over frequency ω_{pc}.

$$\text{Gain margin, } K_g = \frac{1}{|G(j\omega_{pc})|}$$

The Gain margin can be expressed as

$$K_g \text{ in db} = 20 \log K_g = 20 \log \frac{1}{|G(j\omega_{pc})|} = -20 \log |G(j\omega_{pc})|$$

Phase margin (γ)

The phase margin γ, is that amount of additional phase lag at the gain cross over frequency required to bring the system to the verge of instability. The gain cross over frequency ω_{gc}, is the frequency at which the magnitude of the openloop transfer function is unity.

(or) it is the frequency at which the db magnitude is zero.

The phase margin γ is obtained by adding 180° to the phase angle ϕ of the open loop transfer function at the gain cross over frequency.

Phase margin, $\gamma = 180^\circ + \phi_{gc}$

where $\phi_{gc} = \angle G(j\omega_{gc})$.

Estimation of frequency Domain specifications for a Second order system:

Resonant peak (M_r):

Consider the closed loop transfer function of second order system.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

The sinusoidal transfer function $T(j\omega)$ is obtained by letting $s = j\omega$.

$$\therefore T(j\omega) = \frac{C(j\omega)}{R(j\omega)} = \frac{\omega_n^2}{(j\omega)^2 + 2\xi\omega_n(j\omega) + \omega_n^2}$$

$$= \frac{\omega_n^2}{-\omega^2 + 2j\xi\omega\omega_n + \omega_n^2}$$

$$= \frac{\omega_n^2}{\omega_n^2\left(\frac{-\omega^2}{\omega_n^2} + j2\xi\frac{\omega}{\omega_n} + 1\right)}$$

$$= \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\xi\frac{\omega}{\omega_n}}$$

Let normalized frequency, $u = \frac{\omega}{\omega_n}$

$$\therefore T(j\omega) = \frac{1}{1-(u)^2+j2\xi u}$$

$$= \frac{1}{\{1-(u)^2+j2\xi u}$$

$$= \frac{1}{\sqrt{(1-u^2)^2+(2\xi u)^2}\ \angle \tan^{-1}\frac{2\xi u}{1-u^2}}$$

$$\left[\because x+iy = \sqrt{x^2+y^2}\ \angle \tan^{-1}\left(\frac{y}{x}\right)\right.$$

Let M = Magnitude of closed loop transfer function

α = phase of closed loop transfer function.

$$M = |T(j\omega)| = \left[\frac{1}{(1-u^2)^2+(2\xi u)^2}\right]^{1/2} \quad \left[\because |T(j\omega)| = \sqrt{T(j\omega)}\right.$$

$$= \left[(1-u^2)^2+(2\xi u)^2\right]^{-1\times\frac{1}{2}}$$

$$= \left[(1-u^2)^2+4\xi^2u^2\right]^{-1/2} \quad \text{---------- (1)}$$

$$\alpha = \angle T(j\omega) = -\tan^{-1}\frac{2\xi u}{1-u^2}$$

The Resonant peak is the maximum value of M.

The condition for maximum value of M can be obtained by differentiating the equation of M with respect to u and letting $\frac{dM}{du} = 0$, when $u = u_r$.

where $u_r = \frac{\omega_r}{\omega}$ = Normalized resonant frequency.

On differentiating eq (1).

$$\frac{dM}{du} = \frac{d}{du}\left[(1-u^2)^2 + 4\xi^2 u^2\right]^{-1/2}$$

$$= \frac{-1}{2}\left[(1-u^2)^2 + 4\xi^2 u^2\right]^{-3/2}\left[2(1-u^2)(-2u) + 8\xi^2 u\right] \quad \left[\because \frac{d}{du}uv = v\right.$$

$$= \frac{-\left[-4u(1-u^2) + 8\xi^2 u\right]}{2\left[(1-u^2)^2 + 4\xi^2 u^2\right]^{3/2}}$$

$$= \frac{4u(1-u^2) - 8\xi^2 u}{2\left[(1-u^2)^2 + 4\xi^2 u^2\right]^{3/2}}$$

Replace u by u_r in the above eq. and equate to zero

$$= \frac{4u_r(1-u_r^2) - 8\xi^2 u_r}{2\left[(1-u_r^2)^2 + 4\xi^2 u_r^2\right]^{3/2}} = 0.$$

$$= 4u_r(1-u_r^2) - 8\xi^2 u_r = 0.$$

$$= 4u_r - 4u_r^3 - 8\xi^2 u_r = 0.$$

$$4u_r^3 = 4u_r - 8\xi^2 u_r \Rightarrow 4u_r^3 = 4u_r(1-2\xi^2)$$

$$u_r^2 = 1 - 2\xi^2$$

The resonant peak occurs when $u_r = \sqrt{1-2\xi^2}$

Put this Condition in equation 'M'

$$\therefore M_r = \left[\frac{1}{(1-u^2)^2 + 4\xi^2 u^2}\right]^{1/2}\Bigg|_{u=u_r}$$

$$= \left[\frac{1}{(1-u_r^2)^2 + 4\xi^2 u_r^2}\right]^{1/2} = \left[\frac{1}{1-((1-2\xi^2))^2 + 4\xi^2(1-2\xi^2)^{1/2}}\right]^{1/2}$$

$$= \left[\frac{1}{4\xi^4 + 4\xi^2 - 8\xi^4}\right]^{1/2}$$

$$= \frac{1}{\left[4\xi^2 - 4\xi^4\right]^{1/2}} = \frac{1}{\left[4\xi^2(1-\xi^2)\right]^{1/2}}$$

$$= \frac{1}{2\xi\sqrt{1-\xi^2}}$$

∴ The resonant peak $\boxed{M_r = \frac{1}{2\xi\sqrt{1-\xi^2}}}$

Resonant frequency (ω_r):-

The normalized resonant frequency

$$u_r = \frac{\omega_r}{\omega_n} = \sqrt{1-2\xi^2}$$

The Resonant frequency $\omega_r = \omega_n\sqrt{1-2\xi^2}$.

Bandwidth (ω_b):

Let Normalized Bandwidth $u_b = \frac{\omega_b}{\omega_n}$

When $u = u_b$, the magnitude M, of the closed loop ~~transfer~~ system is $\frac{1}{\sqrt{2}}$ (or $-3db$).

Hence in the equation of M.

Put $u = u_b$ and equate to $1/\sqrt{2}$

$$M = \frac{1}{\sqrt{2}} = \frac{1}{\left[(1-u_b^2)^2 + 4\xi^2 u_b^2\right]^{1/2}}$$

On Squaring and Cross multiplying

$$(1-u_b^2)^2 + 4\xi^2 u_b^2 = 2$$

$$1 + u_b^4 - 2u_b^2 + 4\xi^2 u_b^2 = 2$$

$$u_b^4 - 2u_b^2(1-2\xi^2) - 1 = 0$$

Let $x = u_b^2$

$\therefore x^2 - 2(1-2\xi^2)x - 1 = 0$.

$$\therefore x = \frac{2(1-2\xi^2) \pm \sqrt{4(1-2\xi^2)^2+4}}{2}$$

$$= \frac{2(1-2\xi^2) \pm 2\sqrt{(1+4\xi^4-4\xi^2)+1}}{2}$$

By taking only the positive sign

$$x = 1 - 2\xi^2 + \sqrt{2-4\xi^2+4\xi^4}$$

But $u_b = \sqrt{x} = \left[1 - 2\xi^2 + \sqrt{2-4\xi^2+4\xi^4}\right]^{1/2}$

Also $u_b = \frac{\omega_b}{\omega_n}$

$\therefore$ Bandwidth $= \omega_b = \omega_n u_b$

$$= \omega_n \left[1 - 2\xi^2 + \sqrt{2-4\xi^2+4\xi^4}\right]^{1/2}.$$

Phase margin :-

The open loop transfer function of second order system. $G(s) = \frac{\omega_n^2}{s(s+2\xi\omega_n)}$.

The sinusoidal transfer function $G(j\omega)$ is obtained by letting $s = j\omega$

$$G(j\omega) = \frac{\omega_n^2}{j\omega(j\omega+2\xi\omega_n)}$$

$$= \frac{\omega_n^2}{-\omega^2 + j2\xi\omega_n\omega}$$

$$= \frac{\omega_n^2}{\omega_n^2\left(\frac{-\omega^2}{\omega_n^2} + j2\xi\frac{\omega}{\omega_n}\right)}$$

Let normalized frequency $u = \omega/\omega_n$.

On substituting $u = \omega/\omega_n$

$$G(j\omega) = \frac{1}{(-u^2 + j2\xi u)}$$

Magnitude of $G(j\omega) = |G(j\omega)| = \frac{1}{\sqrt{u^4 + 4\xi^2 u^2}}$

phase of $G(j\omega) = \angle G(j\omega) = -\tan^{-1}\frac{2\xi u}{-u^2}$

$$= -\tan^{-1}\left(\frac{-2\xi}{u}\right)$$

$$= \tan^{-1}\left(\frac{2\xi}{u}\right).$$

At the gain Cross-over frequency ω_{gc}, the magnitude of $G(j\omega)$ is unity.

Let normalized gain Cross over frequency $u_{gc} = \omega_{gc}/\omega_n$

Substitute u by u_{gc} in the equation for magnitude of $G(j\omega)$ and equate to unity

$\therefore$ At $= u = u_{gc}$.

$$|G(j\omega)| = 1 = \frac{1}{\sqrt{u_{gc}^4 + 4\xi^2 u_{gc}^2}}$$

$$\therefore u_{gc}^4 + 4\xi^2 u_{gc}^2 = 1$$

$$u_{gc}^4 + 4\xi^2 u_{gc}^2 - 1 = 0.$$

Let $x = u_{gc}^2$

$$\therefore x^2 + 4\xi^2 x - 1 = 0$$

$$x = \frac{-4\xi^2 \pm \sqrt{16\xi^4 + 4}}{2}$$

$$= -2\xi^2 \pm \sqrt{4\xi^4 + 1}$$

on taking only the positive sign.

$$\omega_{gc} = \sqrt{x} = \left[-2\xi^2 + \sqrt{4\xi^4+1}\right]^{1/2}$$

The phase Margin $\gamma = 180 + \angle G(j\omega)\Big|_{\omega=\omega_{gc},\ u=u_{gc}}$.

Substituting for $\angle G(j\omega)$

$$\gamma = 180^\circ + \tan^{-1}\frac{2\xi}{\omega_{gc}}$$

$$= 180^\circ + \tan^{-1}\left[\frac{2\xi}{\left(-2\xi^2+\sqrt{4\xi^4+1}\right)^{1/2}}\right]$$

Correlation between Time and Frequency Response:

The Correlation between time and frequency response Can be Correlated for second order system as:

Consider the standard form of transfer function of second order system.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

Where ξ = Damping factor.

ω_n - undamped natural frequency.

The Sinusoidal transfer function of the system is obtained by letting $s=j\omega$.

$$\frac{C(j\omega)}{R(j\omega)} = T(j\omega) = \frac{\omega_n^2}{(j\omega)^2+2\xi\omega_n(j\omega)+\omega_n^2}$$

$$= \frac{\omega_n^2}{-\omega^2+j2\xi\omega_n\omega+\omega_n^2}$$

$$= \frac{\omega_n^2}{\omega_n^2\left(\frac{-\omega^2}{\omega_n^2} + j2\xi\frac{\omega}{\omega_n} + 1\right)}$$

$$= \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\xi\left(\frac{\omega}{\omega_n}\right)}$$

$$= \frac{1}{(1-u^2) + j2\xi u}, \quad \text{where } u = \left(\frac{\omega}{\omega_n}\right) \text{ is the normalized frequency.}$$

Magnitude of closed loop system

$$M = |T(j\omega)| = \frac{1}{\sqrt{(1-u^2)^2 + (2\xi u)^2}}$$

phase of closed loop system

$$\alpha = \angle T(j\omega) = -\tan^{-1}\left(\frac{2\xi u}{1-u^2}\right).$$

The magnitude and phase angle characteristics for normalized frequency u, for certain values of ξ are shown in fig1 and fig2.

The frequency at which M has a peak value is known as the resonant frequency. The peak value of the magnitude is the resonant peak M_r, At this frequency the slope of the magnitude curve is zero.

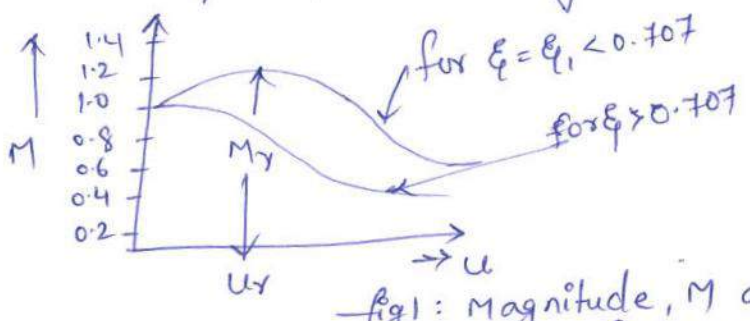


fig1: Magnitude, M as a function of u.

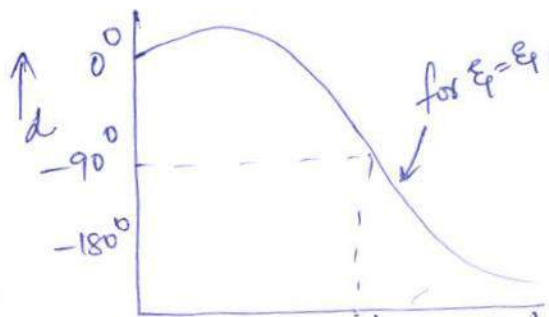


fig2: Phase α as function of u

Let ω_r be the resonant frequency and $u_r = \left(\frac{\omega_r}{\omega_n}\right)$ be the normalized resonant frequency.

The expression for resonant frequency ω_r can be obtained by differentiating M with respect to ω and equating $\frac{dM}{du}$ to zero.

The M_r and the corresponding phase α_r can be obtained by substituting the expression for ω_r in the equation of M and α.

$$\therefore M_r = \frac{1}{2\xi\sqrt{1-\xi^2}} \quad [\because \text{from desivation of Resonant peak}].$$

$$u_r = \sqrt{1-2\xi^2}$$

when $\xi = 0 \Rightarrow \omega_r = \omega_n\sqrt{1-2\xi^2}$

$$= \omega_n$$

$$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}} = \infty.$$

Hence it is clear that as ξ tends to zero, ω_r approaches ω_n and M_r approaches infinity.

for $0 < \xi < \frac{1}{\sqrt{2}}$, the resonant frequency always has a value less than ω_n and the resonant peak has a value greater than one.

for $\xi > \frac{1}{\sqrt{2}}$, the Condition $\left(\frac{dM}{du}\right) = 0$, will not be Satisfied for any real value of ω.

Hence when $\xi > \frac{1}{\sqrt{2}}$ the Magnitude M decreases from $M=1$ at $u=0$ with increasing u. It follows that for $\xi > \frac{1}{\sqrt{2}}$, there is no resonant peak and the greatest Value of M equals one.

The frequency at which M has a value of $1/\sqrt{2}$ is of special significane and is Called the Cutt-off frequency ω_c. The signal frequencies above Cutkoff are greatly attenuated on passing through a System.

For feedback Control System, the range of frequencies Over which $M \geq 1/\sqrt{2}$ is defined as bandwidth ω_b, the bandwidth ω_b is equal to Cut-off frequency ω_c.

The normalized bandwidth

$$u_b = \frac{\omega_b}{\omega_n}$$

$$u_b = \left[1 - 2\xi^2 + \sqrt{2 - 4\xi^2 + \xi^4}\right]^{1/2}$$

$\therefore u_b$ is a function of ξ alone. The graph between ξ and u_b shown in fig.

The expression for the damped frequency of oscillation ω_d and peak overshoot M_p of the step response for $0 \leq \xi \leq 1$ are:

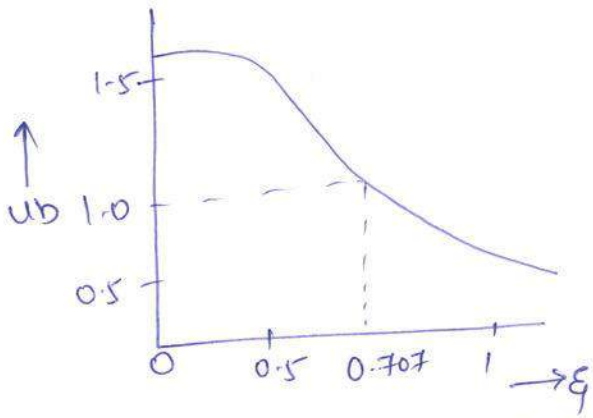


fig: Normalized bandwidth as a function of ξ.

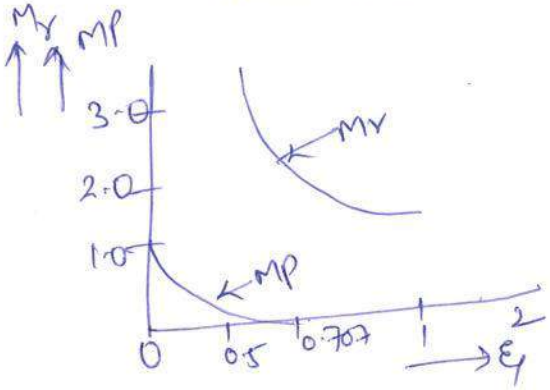
Damped frequency

$$\omega_d = \omega_n \sqrt{1-\xi^2}$$

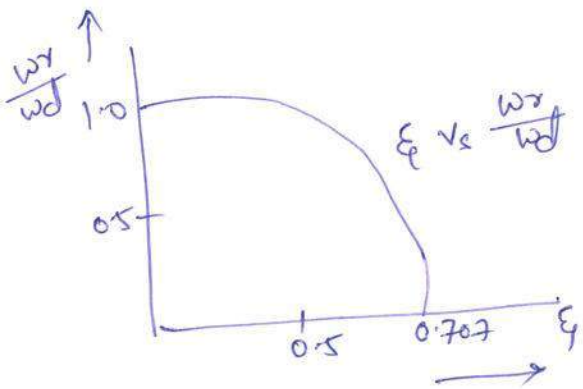
peak overshoot

$$M_p = e^{-\xi\pi/\sqrt{1-\xi^2}}$$

Comparision of the equation of M_r and M_p reveals that both are functions of only ξ.



figa: M_r and M_p as a function of ξ



figb: ω_r/ω_d as a function of ξ.

The above graph shows that a system with a given value of M_r exhibit a corresponding value of M_p if subjected to a step input. For $\xi > \frac{1}{\sqrt{2}}$ the resonant peak M_r does not exist and the correlation breakdown. the step response oscillations are well damped & M_p negligible.

The Comparision of the equation of ω_r and ω_d reveals that there exists a definite correlation between them. The sketch of ω_r/ω_d with respect to ζ is shown in fig a and fig b.

Bodeplot:

The Bodeplot is a frequency response plot of the transfer function of a system. A Bodeplot consists of two graphs: 1) Mangnitude of sinusoidal transfer function versus $\log\omega$.

2) phase angle of sinusoidal transfer function versus $\log\omega$.

✓ The Bode pot can be drawn for both open loop and closed loop transfer function.

✓ The standard representation of logesthmic magnitude of open loop transfer function of $G(j\omega)$ is $20\log_{10}|G(j\omega)|$ The unit used in this representation of magnitude is the decible. (db).

✓ The main advantage of the bodeplot is that multiplication of magnitude can be converted into addition.

Consider the openloop transfer function

$$G(s) = \frac{K(1+sT_1)}{s(1+sT_2)(1+sT_3)}$$

$$G(j\omega) = \frac{K(1+j\omega T_1)}{j\omega(1+j\omega T_2)(1+j\omega T_3)}$$

$$= \frac{K\angle 0^\circ \sqrt{1+\omega^2T_1^2}\ \angle \tan^{-1}\omega T_1}{\omega \angle 90^\circ \sqrt{1+\omega^2T_2^2}\ \angle \tan^{-1}\omega T_2 \sqrt{1+\omega^2T_3^2}\ \angle \tan^{-1}\omega T_3}.$$

The magnitude of $G(j\omega) = |G(j\omega)|$

$$= \frac{K\sqrt{1+\omega^2T_1^2}}{\omega\sqrt{1+\omega^2T_2^2}\sqrt{1+\omega^2T_3^2}}$$

The phase angle of the $G(j\omega) = \angle G(j\omega)$

$$= \tan^{-1}\omega T_1 - 90^\circ - \tan^{-1}\omega T_2 - \tan^{-1}\omega T_3.$$

The magnitude or $G(j\omega)$ can be expressed in db.

$$|G(j\omega)| \text{ in db} = 20\log|G(j\omega)|$$

$$= 20\log\left[\frac{K\sqrt{1+\omega^2T_1^2}}{\omega\sqrt{1+\omega^2T_2^2}\sqrt{1+\omega^2T_3^2}}\right]$$

$$= 20\log\left[\frac{K}{\omega}\times\sqrt{1+\omega^2T_1^2}\times\frac{1}{\sqrt{1+\omega^2T_2^2}}\times\frac{1}{\sqrt{1+\omega^2T_3^2}}\right]$$

$$= 20\log\frac{K}{\omega} + 20\log\sqrt{1+\omega^2T_1^2} + 20\log\frac{1}{\sqrt{1+\omega^2T_2^2}}$$

$$+ 20\log\frac{1}{\sqrt{1+\omega^2T_3^2}}$$

$$= 20\log\frac{K}{\omega} + 20\log\sqrt{1+\omega^2T_1^2} - 20\log\sqrt{1+\omega^2T_2^2}$$

$$- 20\log\sqrt{1+\omega^2T_3^2} \quad \text{---(i)}$$

By the above eq it is clear that when the magnitude is expressed in db. the multiplication is converted to addition. Hence in magnitude plot, the

magnitudes of individual factors of $G(j\omega)$ can be added.

Basic factors of $G(j\omega)$

The basic factors that frequently occur in a typical transfer function $G(j\omega)$ are

1. Constant gain, k
2. integral part (factor) $\frac{k}{j\omega}$ or $\frac{k}{(j\omega)^n}$
3. Derivative factor, $k(j\omega)$ or $k(j\omega)^n$
4. first order factor in denominator. $\frac{1}{1+j\omega T}$ or $\frac{1}{(1+j\omega T)^m}$
5. first order factor in numerator. $(1+j\omega T)$ or $(1+j\omega T)^m$
6. Quadratic factor in denominator $\left[\frac{1}{1+2\xi(j\omega/\omega_n)+(j\omega/\omega_n)^2}\right]$
7. Quadratic factor in numerator $\left[1+2\xi\left(\frac{j\omega}{\omega_n}\right)+\left(\frac{j\omega}{\omega_n}\right)^2\right]$.

Q: Constant Gain K

Let $G(s)=K$

1. $\therefore G(j\omega)=K \quad [\because s=j\omega]$
2. $M_{db}=20\log[G(j\omega)]$
   $=20\log|K|$
   for $K=0 \rightarrow M_{db}$ is 0
   for $K>0 \rightarrow M_{db}$ is +ve
   for $K<0 \rightarrow M_{db}$ is -ve
3. $\phi=\tan^{-1}\left[\frac{0}{K}\right]$
   [$\because$ No imaginary part available]

procedure to draw a Bode plot

1. Replace s by $j\omega$ to Convert open loop transfer function into frequency domain.
2. Calculate magnitude in db.
   $M_{db}=20\log|G(j\omega)H(j\omega)|$
3. Find phase angle $\varphi=\tan^{-1}\left[\frac{\text{Imaginary part}}{\text{Real part}}\right]$
4. vary the ω from minimum to maximum & find M & φ and draw magnitude & phase plot.

$\therefore \phi = \tan^{-1}(0)$

$= 0^\circ$

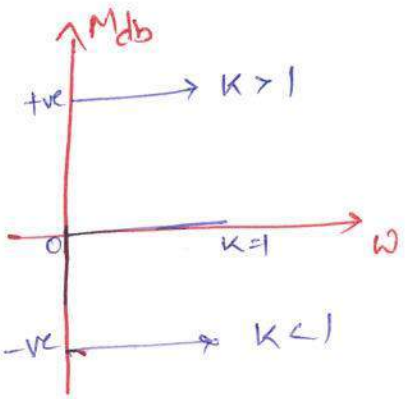


fig: Magnitude plot

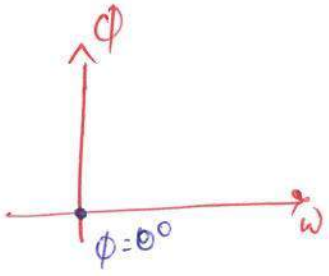


fig: phase plot.

Q: Integral factor

$G(s) = k/s$ or $k/(s)^n$

1. $G(s) = \frac{k}{s}$

Let $s = j\omega$

$G(j\omega) = \frac{k}{j\omega}$

$|G(j\omega)| = \frac{k}{j\omega}$

2. $M_{db} = 2\log|G(j\omega)|$

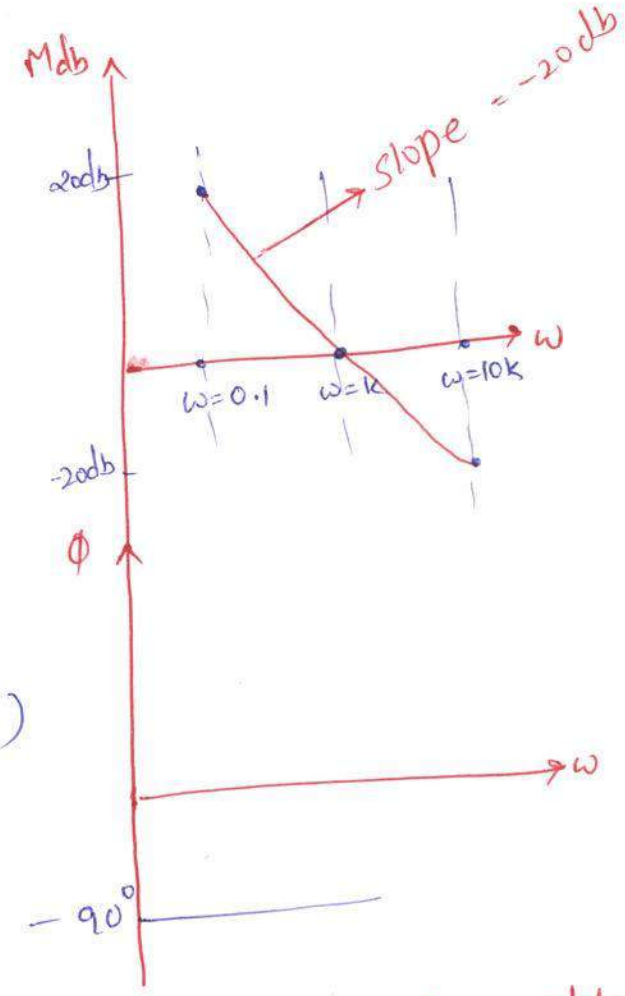
$= 2\log\left|\frac{k}{j\omega}\right|$

3. $\phi = \angle G(j\omega) = \tan^{-1}\left(\frac{k/\omega}{0}\right)$

[∵ No real part available)

$\phi = \tan^{-1}(\infty)$

$= -90^\circ$



Magnitude & phase plot.

Tabular form

w	0.1k	k	10k
Mdb	20db	0db	-20 db

Q: Derivative - factor

$G(s) = ks$

1. $G(j\omega) = k(j\omega)$

$$|G(j\omega)| = |k\omega|$$

2. $M_{db} = 20 \log_{10} |G(j\omega)|$

$= 20 \log_{10} |k\omega|$

when $\omega = 0.1/k$ $\quad M_{db} = -20db$

$\omega = 1/k$ $\quad M_{db} = 0\ db$

$\omega = 10/k$ $\quad M_{db} = 20db$

3. $\phi = \tan^{-1}\left(\frac{k\omega}{0}\right)$

$= 90^\circ.$

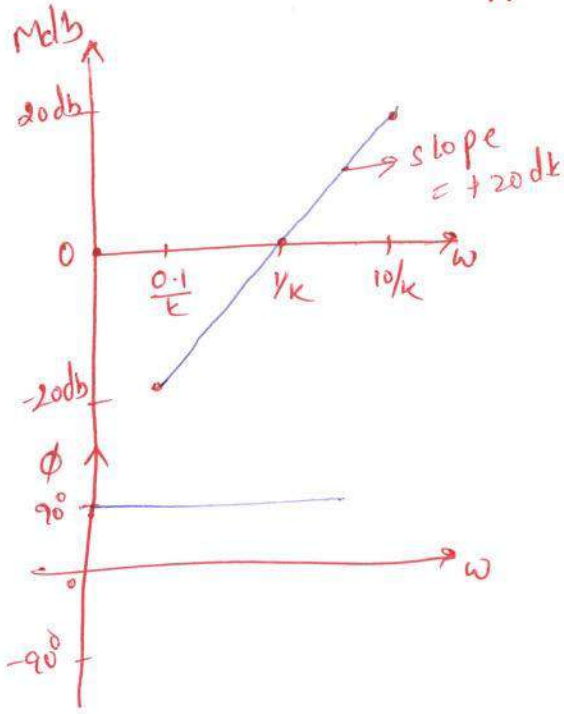


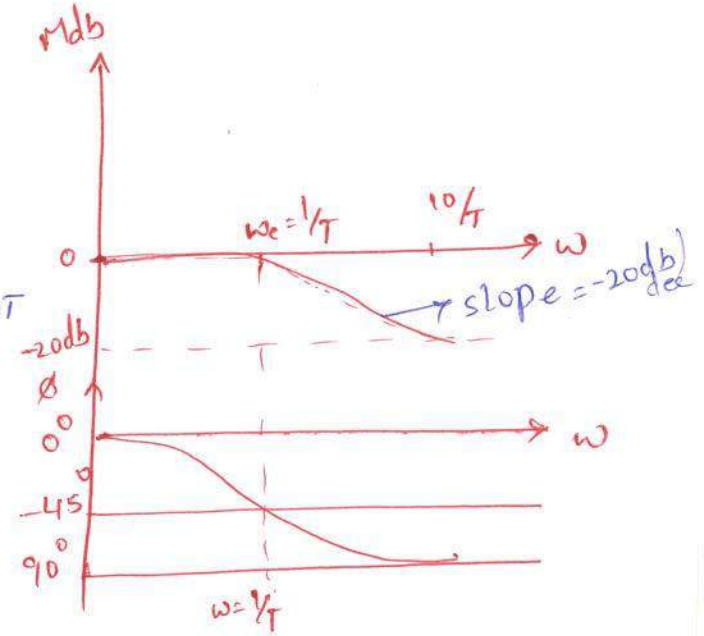
fig: Magnitude & phase plots.

①: First order factor in Denominator:

$$G(s) = \frac{1}{1+sT}$$

1. $G(j\omega) = \frac{1}{1+j\omega T}$

$$|G(j\omega)| = \frac{1}{\sqrt{1+\omega^2T^2}} \quad \angle -\tan^{-1}\omega T$$



2. $M_{db} = 20 \log |G(j\omega)|$

$= 20 \log \frac{1}{\sqrt{1+\omega^2T^2}}$

$= -20 \log \sqrt{1+\omega^2T^2}$

At very low frequencies $\omega T \ll 1$

$M_{db} = -20 \log \sqrt{1+\omega^2T^2} \approx 0.$

At very high frequencies $\omega T \gg 1$

$$M_{db} = -20\log\sqrt{1+\omega^2T^2} \approx -20\log\sqrt{\omega^2T^2} = -20\log\omega T$$

at $\omega = 1/T$ $\quad M_{db} = -20\log 1 = 0$

at $\omega = \frac{10}{T}$ $\quad M_{db} = -20\log 10 = -20db$

The actual Magnitude at the Corner frequency $\omega_c = 1/T$

$$M_{db} = -20\log\sqrt{1+1} = -3db.$$

phase angle $\quad \phi = \angle G(j\omega) = -\tan^{-1}\omega T$

At the Corner frequency $\omega = \omega_c = 1/T$

$$\phi = -\tan^{-1}\omega T$$
$$= -\tan^{-1}1 = -45^0.$$

At $\omega \to 0$, $\phi \to 0$ and $\omega \to \infty$, $\phi \to 90^0$.

Q: First order factor in the numerator:

$$G(s) = 1+sT$$

1. $G(j\omega) = 1+j\omega T$

$$|G(j\omega)| = |1+j\omega T|$$
$$= \sqrt{1+\omega^2T^2}$$

2. $M_{db} = 20\log|G(j\omega)|$

$$= 20\log\sqrt{1+\omega^2T^2}$$

for $0 < \omega < 1/T$ $\quad M_{db} = 0$

for $\frac{\phi}{T} < \omega < \infty$ $\quad M_{db} = 20db$

3 $\phi = \tan^{-1}(\omega T/1)$

at $\omega = 1/T \Rightarrow \phi = +45^0$.

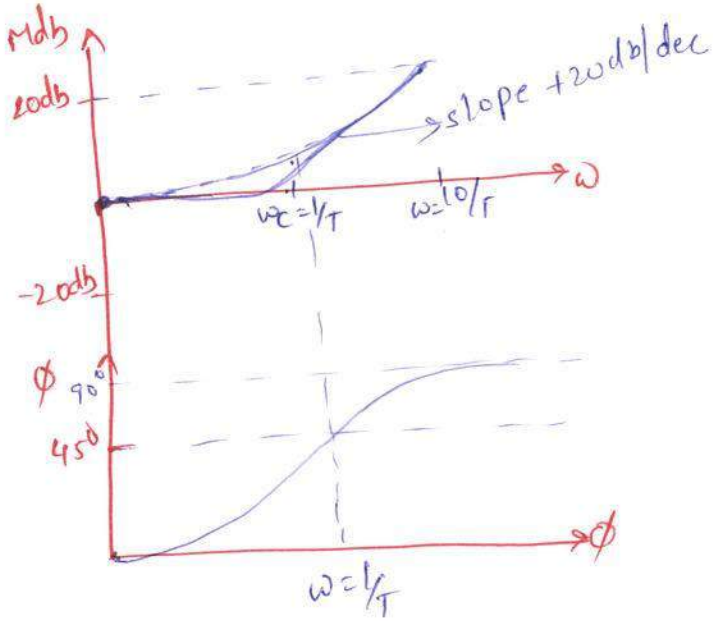


fig: Magnitude and phase plot.

Q: Quadractic factor in the Denominator.

$$G(s) = \frac{1}{1+2\xi\frac{s}{\omega_n}+\left(\frac{s}{\omega_n}\right)^2}$$

$$|G(j\omega)| = \frac{1}{1+j2\xi\frac{\omega}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2}$$

$$= \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2+j2\xi\frac{\omega}{\omega_n}}$$

$$= \frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+4\xi^2\frac{\omega^2}{\omega_n^2}}}$$

$$M_{db} = 20\log|G(j\omega)|$$

$$= 20\log\left[\frac{1}{\sqrt{1-\frac{\omega^2}{\omega_n^2}+4\xi^2\frac{\omega^2}{\omega_n^2}}}\right]$$

$$= -20\log\sqrt{1-\frac{\omega^2}{\omega_n^2}+4\xi^2\frac{\omega^2}{\omega_n^2}}$$

$$= -2\log\sqrt{1-\frac{\omega^2}{\omega_n^2}\left(2-4\xi^2\right)+\frac{\omega^4}{\omega_n^4}}$$

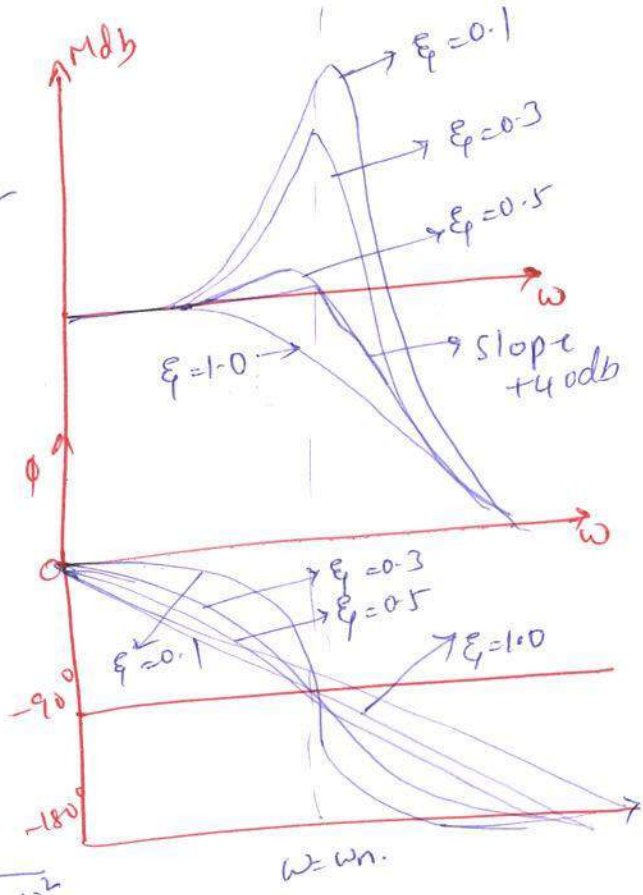


fig:- Magnitude and phase plot.

At very low frequencies when $\omega \ll \omega_n$

$$M_{db} = -20\log\sqrt{1-\frac{\omega^2}{\omega_n^2}\left(2-4\xi^2\right)+\frac{\omega^4}{\omega_n^4}}$$

$$= -20\log 1 = 0.$$

At very high frequencies when $\omega \gg \omega_n$

$$M_{db} = -20\log\sqrt{1-\frac{\omega^2}{\omega_n^2}\left(2-4\xi^2\right)+\frac{\omega^4}{\omega_n^4}}$$

$$= -20\log\sqrt{\frac{\omega^4}{\omega_n^4}} \Rightarrow -20\log\left(\frac{\omega^2}{\omega_n^2}\right)$$

$$M_{db} = -40\log\frac{\omega}{\omega_n}$$

At $\omega = \omega_n$ $M_{db} = -40 \log 1 = 0db$

At $\omega = 10\omega_n$ $M_{db} = -40 \log 10 = -40db$.

$$\phi = \angle G(j\omega) = -\tan^{-1}\left(\frac{2\xi \frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}}\right)$$

At $\omega = \omega_n$ $\phi = -\tan^{-1}\frac{2\xi}{0}$

$= -\tan^{-1}\alpha = -90^0$

$\omega \to 0$, $\phi = 0$ and As $\omega \to \infty$

$\phi \to -180^0$.

Q: Quadratic in the numerator:

$$G(s) = 1 + 2\xi\left(\frac{s}{\omega_n}\right) + \left(\frac{s}{\omega_n}\right)^2.$$

$$G(j\omega) = 1 + 2\xi\left(\frac{\omega}{\omega_n}\right) + \left(\frac{j\omega}{\omega_n}\right)^2$$

$$= \sqrt{1 - \left(\frac{\omega}{\omega_n}\right)^2 + 4\xi^2\left(\frac{\omega}{\omega_n}\right)^2}$$

$$\phi = \tan^{-1}\frac{2\xi \frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}.$$

The phase angle varies from 0 to 180^0, as ω varied from 0 to ∞.

At the corner frequency the phase angle is $+90^0$ and independent of ξ.

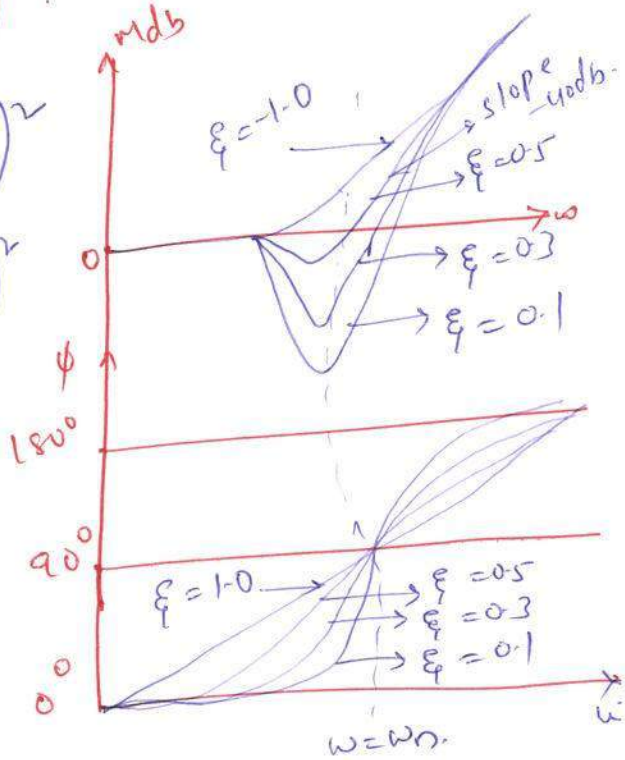


fig: Magnitude and phase plot.

Procedure for plotting the magnitude plot:

1. Convert the transfer function into Bode form or time Constant form
2. List the Corner frequencies in increasing order and prepare a table as.

Term	Corner frequency rad/sec	slope db/dec	Change in slope db/dec

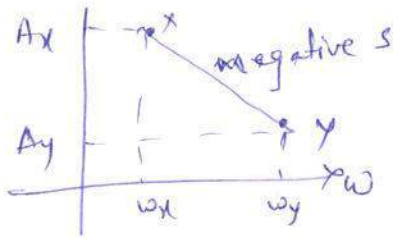
3. Choose an arbitary frequency ω_1 which is lesser than the lowest corner frequency. Calculate the db magnitude of K or $K/(j\omega)^n$ (or) $K/, K(j\omega)^n$ at ω_1 and at the lowest corner frequency.

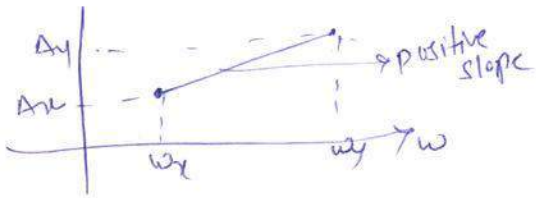
4. Then Calculate the gain (db) at every corner frequency one by one by using the formula

Gain at ω_y = Change in gain from ω_x to ω_y + Gain at ω_x

A_y = (slope from ω_x to ω_y $\times$ log (ω_y/ω_x) + Gain at ω_x.

5. Choose an arbitary frequency ω_h which is greater than the highest corner frequency. Calculate the gain at ω_h by using the formula in step4.





6. In a Semilog graph sheet mark the required range of frequency on x-axis and the range of db magnitude on y-axis after Choosing proper units.

7. Mark all the points obtained in steps, 3, 4, 5 on the graph and join the points by straight lines. Mark the slope at every part of the graph.

Procedure for phase plot of Bode plot:

The phase plot is an exact plot obtained with exact phase angles of $G(j\omega)$ computed for various values of ω and is then tabulated. The choice of frequencies are preferably the frequencies chosen for magnitude plot. usually the magnitude plot and phase plot are drawn in a single semilog sheet on a common frequency scale. Take another Y axis in the graph where the magnitude plot is drawn and in this Y axis, mark the desired range of phase angles after choosing proper units.

from the tabulated values of ω and phase angles mark all the points on the graph, join the points by smooth curve.

Determination of Gain margin and phase margin from Bodeplot:

The Gain margin in db is given by the negative of db magnitude of $G(j\omega)$ at the phase Crossover frequency, ω_{pc}.

The ω_{pc} is the frequency at which phase of $\angle G(j\omega) = 180^\circ$. if the db magnitude of $G(j\omega)$ at ω_{pc} is negative then gain margin is +ve and vice-versa.

Let ϕ_{gc} be the phase angle of $G(j\omega)$ at gain cross over frequency ω_{gc}. The ω_{gc} is the frequency at which the db magnitude of $G(j\omega)$ is zero. Now the phase margin γ is given by $\gamma = 180^\circ + \phi_{gc}$

if ϕ_{gc} is less negative than -180°, then phase margin is positive and vice-versa.

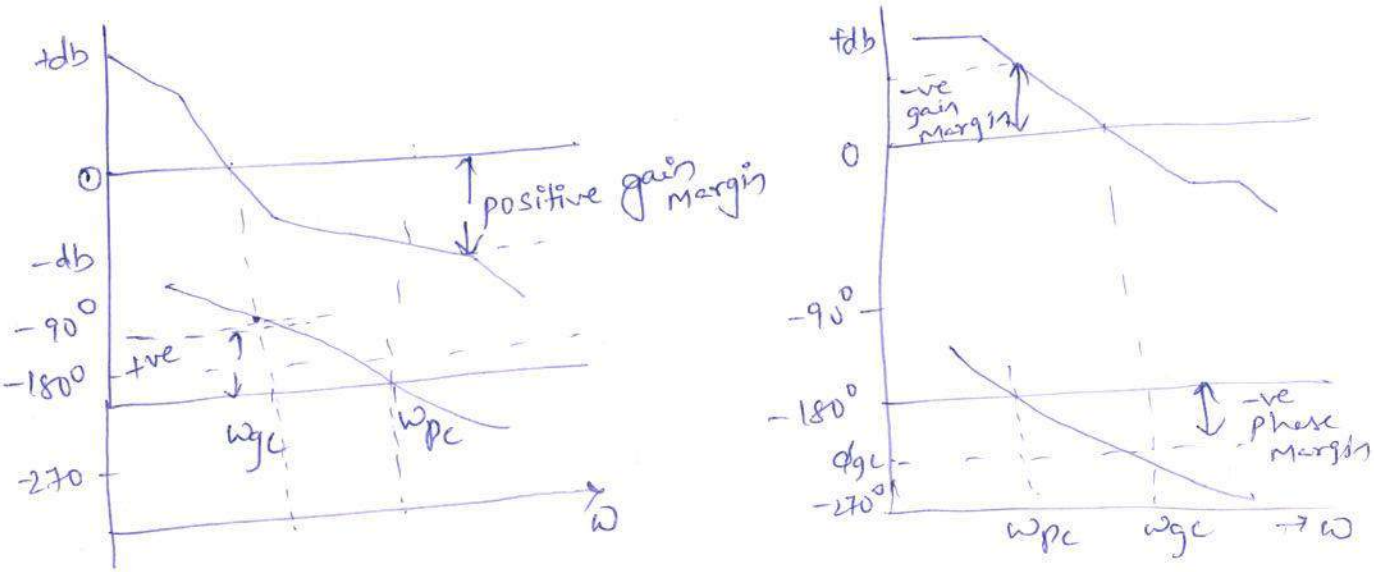


fig: Bode plot showing phase Margin (γ) and gain Margin (kg).

Problem :1

Sketch the Bodeplot for the following transfer function and determine the system gain k for the gain cross over frequency to be 5 rad/sec.

$$G(s) = \frac{ks^2}{(1+0.2s)(1+0.02s)}.$$

Sol: Step1:

The sinusoidal transfer function $G(j\omega)$ is obtained by replacing s by $j\omega$ in the given s-domain transfer function.

$$\therefore G(j\omega) = \frac{K(j\omega)^2}{(1+0.2j\omega)(1+0.02j\omega)}$$

Let $K=1$ $\therefore G(j\omega) = \dfrac{(j\omega)^2}{(1+0.2j\omega)(1+0.02j\omega)}$

Magnitude plot:

The Corner frequencies are:

$\omega_{c_1} = \dfrac{1}{0.2} = 5$ rad/sec

$\omega_{c_2} = \dfrac{1}{0.02} = 50$ rad/sec.

The various terms of $G(j\omega)$ are listed in Table:1. in the increasing order of their Corner frequency. Also the table shows the slope contributed by each term and the change in slope at the Corner frequency.

Step2:

Term	Corner frequency	slope db/dec	Change in slope db/dec
$(j\omega)^2$		+40	
$\dfrac{1}{1+j0.2\omega}$	$\omega_{c_1} = \dfrac{1}{0.2} = 5$	-20	40-20 = 20
$\dfrac{1}{1+j0.02\omega}$	$\omega_{c_2} = \dfrac{1}{0.02} = 50$	-20	20-20 = 0.

Step 3:

Choose a low frequency ω_l such that $\omega_l < \omega_{c_1}$ and Choose a high frequency ω_h such that $\omega_h > \omega_{c_2}$

Let $\omega_l = 0.5$ rad/sec , $\omega_h = 100$ rad/sec

Let $A = |G(j\omega)|$ in db.

Step4:

Let us Calculate A at ω_l, ω_{c_1}, ω_{c_2} and ω_h.

At $\omega = \omega_l$, $A = 20 \log |j\omega|^2$

$= 20 \log (\omega)^2$

$= 20 \log (0.5)^2 = -12$ db.

At $\omega = \omega_{c_1}$, $A = 20 \log |j\omega|^2$

$= 20 \log (\omega)^2$

$= 20 \log (5)^2 = 28$ db.

At $\omega = \omega_{c_2}$, $A = [$ slope from ω_{c_1} to $\omega_{c_2} \times \log \frac{\omega_{c_2}}{\omega_{c_1}}]$

$+ A$ (at ω_{c_1})

$= 20 \times \log \frac{50}{5} + 28$ $[\because \log_{10} 10 = 1]$

$= 48$ db

Step5:

At $\omega = \omega_h$, $A = [$ slope from ω_{c_2} to $\omega_h \times \log \frac{\omega_h}{\omega_{c_2}}]$

$+ A$ (at ω_{c_2}).

$= 0 \times \log \frac{100}{50} + 48.$

$= 48$ db

Let the points a, b, c and d be the points Corresponding to frequencies ω_l, ω_{c_1}, ω_{c_2} and ω_h respectively on the magnitude plot.

step 6 & 7:

In a semilog graph sheet choose a scale of 1 unit = 10 db on y-axis. The frequencies are marked in decades from 0.1 to 100 rad/sec on logarithmic scales in x-axis. Fix the points a, b, c and d on the graph. Join the points by straight lines and mark the slope on the respective region.

Phase plot:

The phase angle of $G(j\omega)$ as a function of ω is given by $\phi = \angle G(j\omega) = 180^\circ - \tan^{-1}(0.2\omega) - \tan^{-1}(0.02\omega)$

The phase angle of $G(j\omega)$ are calculated for various values of ω.

ω rad/sec	$\tan^{-1}0.2\omega$ deg	$-\tan^{-1}0.02\omega$ deg	$\phi = \angle G(j\omega)$ deg.
0.5	5.7	0.6	$173.7 \approx 174$
1	11.3	1.1	$167.6 \approx 168$
5	45	5.7	$129.3 \approx 130$
10	63.4	11.3	$105.3 \approx 106$
50	84.3	45	$50.7 \approx 50$
100	87.1	63.4	$29.5 \approx 30$.

On the semilog sheet choose a scale of 1 unit = 20° on the y axis on the right side of semilog sheet.

$$G(s) = \frac{Ks^2}{(1+0.2s)(1+0.02s)}$$

Fig 4.1.1.: Bode plot of transfer function. $G(j\omega) = \dfrac{K(j\omega)^2}{(1+j0.2\omega)(1+j0.02\omega)}$

Mark the calculated phase angle on the graph sheet. Join the points by smooth curve.

Calculation of k:

The gain cross over frequency is 5 rad/sec At $\omega = 5$ rad/sec the gain is 28 db. If gain cross over frequency is 5 rad/sec then at that frequency the db gain should be zero. Hence to every point of magnitude plot a db gain of -28 db should be added. The addition of -28 db shifts the plot downwards. The corrected magnitude plot is obtained by shifting the plot with $k=1$ by 28 db downwards. The magnitude correction is independent of frequency. Hence the magnitude of -28 db is contributed by the term k. The value of k is calculated by equating $20 \log k$ to -28 db.

$$\therefore 20 \log k = -28 \text{ db}$$

$$\log k = \frac{-28}{20}$$

$$\therefore k = 10^{(-28/20)}$$

$$= 0.0398.$$

Note:

The frequency $\omega = 5$ rad/sec is a corner frequency. Hence in the exact plot the db gain at $\omega = 5$ rad/sec will be 3 db/sec less than the approximate plot. Therefore for exact plot the $20 \log k$ will contribute a gain of -25 db

$$\therefore 20 \log k = -25 \text{ db} \Rightarrow \log k = \frac{-25}{20} \quad \therefore k = 10^{-25/20}$$

$$= 0.0562.$$

Problem 2:

Sketch the bode plot for the following transfer function and determine phase margin and gain margin.

$$G(s) = \frac{75(1+0.2s)}{s(s^2+16s+100)}.$$

Sol: The sinusoidal transfer function $G(j\omega)$ is obtained by replacing s by $j\omega$ in the given s-domain transfer function after converting it to bode form or time constant form.

Given that $G(s) = \frac{75(1+0.2s)}{s(s^2+16s+100)}$.

On comparing the quadratic factor of $G(s)$ with standard form of Quadratic factor, we can estimate ξ and ω_n.

$$\therefore s^2+16s+100 = s^2+2\xi\omega_n s+\omega_n^2$$

$$\omega_n^2 = 100 \qquad 2\xi\omega_n = 16$$

$$\omega_n = 10 \qquad \therefore \xi = \frac{16}{2\omega_n}$$

$$= \frac{16}{2\times 10}$$

$$= 0.8$$

$$G(s) = \frac{75(1+0.2s)}{s\times 100\left(\frac{s^2}{100}+\frac{16s}{100}+1\right)}$$

$$= \frac{0.75(1+0.2s)}{s(1+0.01s^2+0.16s)}$$

$$G(j\omega) = \frac{0.75\,(1+0.2j\omega)}{j\omega\,(1+0.01(j\omega)^2+0.16j\omega)}$$

$$= \frac{0.75\,(1+0.2j\omega)}{j\omega\,(1-0.01\omega^2+j0.16\omega)}$$

Magnitude plot:

The Corner frequencies are

$\omega_{c_1} = \frac{1}{0.2} = 5$ rad/sec

$\omega_{c_2} = \omega_n = 10$ rad/sec.

Term	Corner frequency rad/sec	slope db/dec	Change in slope db/dec
$\frac{0.75}{j\omega}$		-20	
$1+0.2j\omega$	$\omega_{c_1} = \frac{1}{0.2} = 5$	20	$-20+20=0$
$\frac{1}{1-0.01\omega^2+j0.16\omega}$	$\omega_{c_2} = \omega_n = 10$	-40	$0\ominus 40 = -40$

Choose a low frequency ω_l such that $\omega_l < \omega_{c_1}$ and Choose a high frequency ω_h such that $\omega_h > \omega_{c_2}$

Let $\omega_l = 0.5$ rad/sec, $\omega_h = 20$ rad/sec.

Let $A = |G(j\omega)|$ in db.

Let us Calculate A at ω_l, ω_{c_1}, ω_{c_2} and ω_h.

At $\omega = \omega_L$, $A = 20 \log \left| \frac{0.75}{j\omega} \right|$

$= 20 \log \frac{0.75}{0.5} = 3.5$ db.

At $\omega = \omega_{c_1}$, $A = 20 \log \left| \frac{0.75}{j\omega} \right|$

$= 20 \log \frac{0.75}{5} = -16.5$ db.

At $\omega = \omega_{c_2}$, $A = \left[\text{slop from } \omega_{c_1} \text{ to } \omega_{c_2} \times \log \frac{\omega_{c_2}}{\omega_{c_1}} \right] + A_{(\text{at } \omega = \omega_{c_1})}$

$= 0 \times \log \frac{10}{5} + (-16.5)$

$= -16.5$ db.

At $= \omega = \omega_h \Rightarrow A = \left[\text{slope from } \omega_{c_2} \text{ to } \omega_h \times \log \frac{\omega_h}{\omega_{c_2}} \right] + A_{(\text{at } \omega = \omega_{c_2})}$

$= -40 \times \log \frac{20}{10} + (-16.5)$

$= -28.5$ db.

Let the points a, b, c and d be the points corresponding to frequencies ω_L, ω_{c_1}, ω_{c_2} and ω_h respectively on the magnitude plot. In a semilog graph sheet choose a scale of 1 unit = 5 db on y axis. The frequencies are marked in decades from 0.1 to 100 rad/sec on logarithmic scales in x-axis. Fix the points a, b, c and d on the graph. Joint the points by straight lines and mark the slope on the respective region.

$$G(s) = \frac{75(1+0.2s)}{s(s^2+16s+100)}$$

Fig 4.2.1: Bode plot of open loop transfer function, $G(j\omega) = \frac{75(1+j0.2\omega)}{j\omega(1-0.01\omega^2+j0.16\omega)}$

phase plot :

The phase angle of $G(jw)$ as a function of w is given by

$$\phi = \angle G(jw) = \tan^{-1} 0.2w - 90^\circ - \tan^{-1} \frac{0.16w}{1-0.01w^2} \quad \text{for } w \leq w_n$$

$$\phi = \angle G(jw) = \tan^{-1} 0.2w - 90^\circ - \tan^{-1} \frac{0.16w}{1-0.01w^2} - 180^\circ \quad \text{for } w > w_n.$$

w rad/sec	$\tan^{-1} 0.2w$ deg	$\tan^{-1} \frac{0.16w}{1-0.01w^2}$ deg	$\phi = G(jw)$ deg.
0.5	5.7	4.6	$-88.9 \approx -88$
1	11.3	9.2	$-87.9 \approx -88$
5	45	46.8	$-91.8 \approx 92$
10	63.4	90	$116.6 \approx 116$
20	75.9	$46.8 + 180 = 133.2$	147.
50	84.3	$-18.4 + 180 = 161.6$	$-167.3 \approx -168$
100	87.1	$-9.2 + 180 = 170.8$	$-173.7 \approx 174$

On the same semilog sheet choose a scale of 1 unit = 20° on the y-axis on the right side of semilog sheet. Mark the calculated phase angle on the graph sheet. Join the points by a smooth curve.

The magnitude plot and the phase plot.

we find that the phase angle at gain crossover frequency $(w_{gc}) = \phi_{gc} = 88^\circ$

$\therefore$ phase margin $\gamma = 180 + \phi_{gc}$

$= 180^\circ - 88^\circ = 92^\circ$

Here gain margin $= = +\infty$, The phase plot crosses -180° only at infinity. The $|G(jw)|$ at infinity is $-\infty$ db. Hence gain margin is $+\infty$.

Problem 3:

Plot the Bode diagram for the following transfer function and obtain the gain and phase cross over frequencies. $G(s) = \frac{10}{s(1+0.4s)(1+0.1s)}$

Sol: The sinusoidal transfer function of $G(j\omega)$ is obtained by replacing s by $j\omega$ in the given transfer function.

$$G(s) = \frac{10}{s(1+0.4s)(1+0.1s)}$$

$$G(j\omega) = \frac{10}{j\omega(1+j0.4\omega)(1+j0.1\omega)}$$

Magnitude plot.

Term	Cornerfrequencies	Slop in db/dec	Change Slope
$\frac{10}{j\omega}$		-20	-20
$\frac{1}{1+j0.4\omega}$	$\omega_{c_1} = 2.5$	-20	-40
$\frac{1}{1+j0.1\omega}$	$\omega_{c_2} = 10$	-20	-60

Choose ω_l & ω_h ; $\omega_l < \omega_{c_1}$ & $\omega_h > \omega_{c_2}$

Let $\omega_l = 0.1$ rad sec , $\omega_h = 50$ rad/sec.

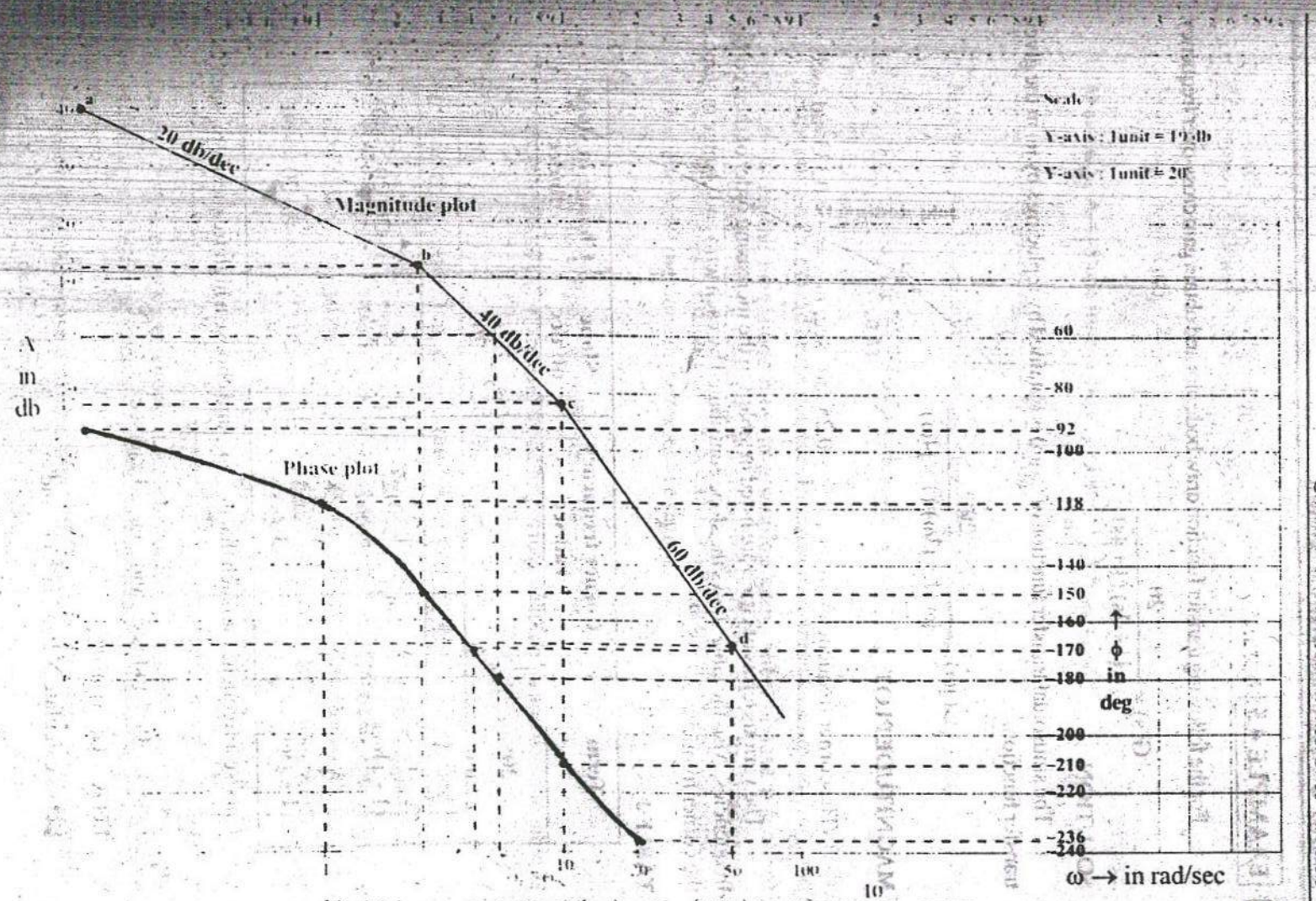


Fig ... plot of transfer function. $G(j\omega) = \frac{10}{j\omega(1+j0.4\omega)(1+j0.1\omega)}$

$G(s) = \frac{10}{s(1+0.4s)(1+0.1s)}$

At $\omega = \omega_l$

$$A\big|_{\omega=\omega_l} = 20 \log_{10} |G(j\omega)|$$

$$= 20 \log_{10} \left(\frac{10}{\omega}\right)$$

$$= 20 \log_{10} \left(\frac{10}{0.1}\right)$$

$$= 40 db$$

At $\omega = \omega_{c_1}$

$$A\big|_{\omega=\omega_{c_1}} = 20 \log_{10} |G(j\omega)|$$

$$= 20 \log_{10} \left(\frac{10}{\omega}\right)$$

$$= 20 \log_{10} \left(\frac{10}{2.5}\right)$$

$$= 12 db.$$

At $\omega = \omega_{c_2}$

$$A\big|_{\omega=\omega_{c_2}} = \left[\text{slope from } \omega_{c_1} \text{ to } \omega_{c_2} \times \log \left(\frac{\omega_{c_2}}{\omega_{c_1}}\right)\right] + A_{(\omega=\omega_{c_1})}$$

$$= \left[-40 \times \log \left(\frac{10}{2.5}\right)\right] + 12 db.$$

$$= -12 db.$$

At $\omega = \omega_h$

$$A\big|_{\omega=\omega_h} = \left[\text{slope from } \omega_{c_2} \text{ to } \omega_h \times \log \left(\frac{\omega_h}{\omega_{c_2}}\right)\right] + A_{(\omega=\omega_h)}$$

$$= \left[-60 \times \log \left(\frac{50}{10}\right)\right] \oplus 12$$

$$= -54 db.$$

Phase plot

$$\phi = \angle G(j\omega) = -90 - \tan^{-1}\left(\frac{0.4}{1}\right) - \tan^{-1}\left(\frac{0.1}{1}\right)$$

ω	$\tan^{-1} 0.4$	$\tan^{-1} 0.1$	$\phi = \angle G(j\omega)$
0.1	2.29	0.57	-92
1	21.80	5.71	-118
2.5	45.0	14.0	-150
4	57.99	21.8	-170
10	75.96	45.0	-210
20	82.87	63.43	-236.

Problem 4:

The function $GH(s) = 5(1+2s) / (1+4s)(1+0.25s)$

Draw the bode plot.

Sol:

$$G(s) = \frac{5(1+2s)}{(1+4s)(1+0.25s)}$$

$$G(j\omega) = \frac{5(1+j2\omega)}{(1+j4\omega)(1+j0.25\omega)}$$

Term	Cornerfrequencies	slope in db/dec	Change in slope
5		0	-
$\frac{1}{(1+j4\omega)}$ ($\frac{1}{1+sT}$, $T=4$)	$\omega_{c_1} = 0.25$	-20	-20
$(1+j\omega 2)$	$\omega_{c_2} = 0.5$	+20	0 (-20+20)
$\frac{1}{1+j0.25\omega}$	$\omega_{c_3} = \frac{1}{0.25} = 4$	-20	-20

Let $\omega_l < \omega_{c_1}$, $\omega_h > \omega_{c_2}$

$\omega_l = 0.1$ rad/sec & $\omega_h = 10$ rad/sec

$$A\big|_{\omega=\omega_l} = 20\log_{10}|G(j\omega)|$$

$$= 20\log_{10}|5|.$$

$$= +14db$$

$$A\big|_{\omega=\omega_{c_1}} = 20\log_{10}|G(j\omega)|$$

$$= 20\log_{10}|5|$$

$$= 14db.$$

$$A\big|_{\omega=\omega_{c_1}} = \left[\text{slope from } \omega_{c_1} \text{ to } \omega_{c_2} \times \log\frac{\omega_{c_2}}{\omega_{c_1}}\right] + A(\omega=\omega_{c_1})$$

$$= \left[-20 \times \log\left(\frac{0.5}{0.25}\right)\right] + 14$$

$$= +8db.$$

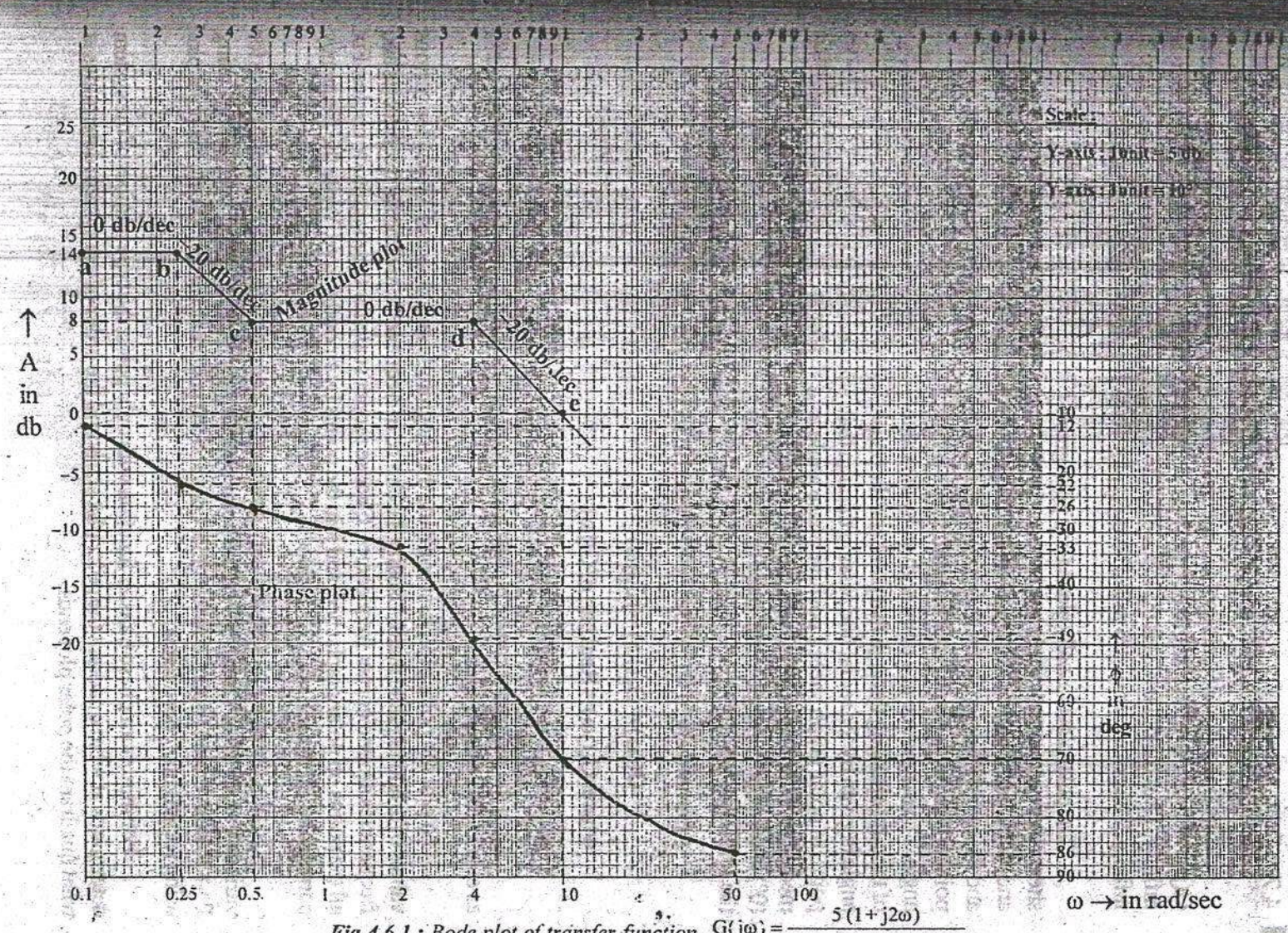


Fig 4.6.1.: Bode plot of transfer function, $G(j\omega) = \dfrac{5(1+j2\omega)}{(1+j4\omega)(1+j0.25\omega)}$

$$G(s) = \frac{5(1+2s)}{(1+4s)(1+0.25s)}$$

At $\omega = \omega_{c_3}$

$$A/_{\omega=\omega_{c_3}} = \left[\text{slope from } \omega_{c_2} \text{ to } \omega_{c_3} \times \log \frac{\omega_{c_3}}{\omega_{c_2}}\right] + A\,(\omega=\omega_{c_2})$$

$$= \left[0 \times \log\left(\frac{4}{0.5}\right)\right] + 8$$

$$= 8 \text{ db.}$$

At $\omega = \omega_h$

$$A/_{\omega=\omega_h} = \left[\text{slope from } \omega_{c_3} \text{ to } \omega_h \times \log \frac{\omega_h}{\omega_{c_3}}\right] + A\,(\omega=\omega_h)$$

$$= \left[-20 \times \log \frac{10}{4}\right] + [8]$$

$$= 0 \text{ db.}$$

Phase plot.

$$\phi = \angle G(j\omega) = -\tan^{-1}\left(\frac{4\omega}{1}\right) - \tan^{-1}\left(\frac{0.25\omega}{1}\right) + \tan^{-1}\left(\frac{2\omega}{1}\right)$$

ω	$\tan^{-1}(4\omega)$	$\tan^{-1}(2\omega)$	$\tan^{-1}(0.25\omega)$	ϕ
0.1	11.3	21.8	1.43	-12
0.25	26.56	45.0	3.5	-22
0.5	45.0	63.43	7.1	-26
2	75.96	82.87	26.56	-33
4	82.87	86.42	45.0	-49
10	87.13	88.56	68.19	-70
50	89.42	89.71	85.42	-86

H/w for the following transfer function Draw bode plot and obtain gain Cross over frequency

$$G(s) = \frac{20}{s(1+3s)(1+4s)}$$

Problem : 5

Given $G(s) = \dfrac{K e^{-0.2s}}{s(s+2)(s+8)}$. Find k so that the system is stable with (a) gain margin equal to 6 db and (b) phase margin equal to 45°.

Sol:

$$G(s) = \frac{k \cdot e^{-0.2s}}{s(s+2)(s+8)}$$

$$= \frac{k \cdot e^{-0.2s}}{s \times 2\left(1+\frac{s}{2}\right) 8\left(1+\frac{s}{8}\right)}$$

$$= \frac{k\, e^{-0.2s}}{16 s \left(1+\frac{s}{2}\right)\left(1+\frac{s}{8}\right)} \Rightarrow \frac{0.0625\, k\, e^{-0.2s}}{s(1+0.5s)(1+0.125s)}$$

Let $k=1$, $s=j\omega$

$$G(j\omega) = \frac{0.0625\, e^{-0.2j\omega}}{(j\omega)(1+j0.5\omega)(1+j0.125\omega)}$$

Magnitude plot

Term	Corner frequency	slope in db/dec	change in slop db/dec
$\frac{0.0625}{j\omega}$		-20	
$\frac{1}{1+j0.5\omega}$	$\omega_{c_1} = 2$	-20	-20-20 = -40
$\frac{1}{1+j0.125\omega}$	$\omega_{c_2} = \frac{1}{0.125} = 8$	-20	-40-20 = -60

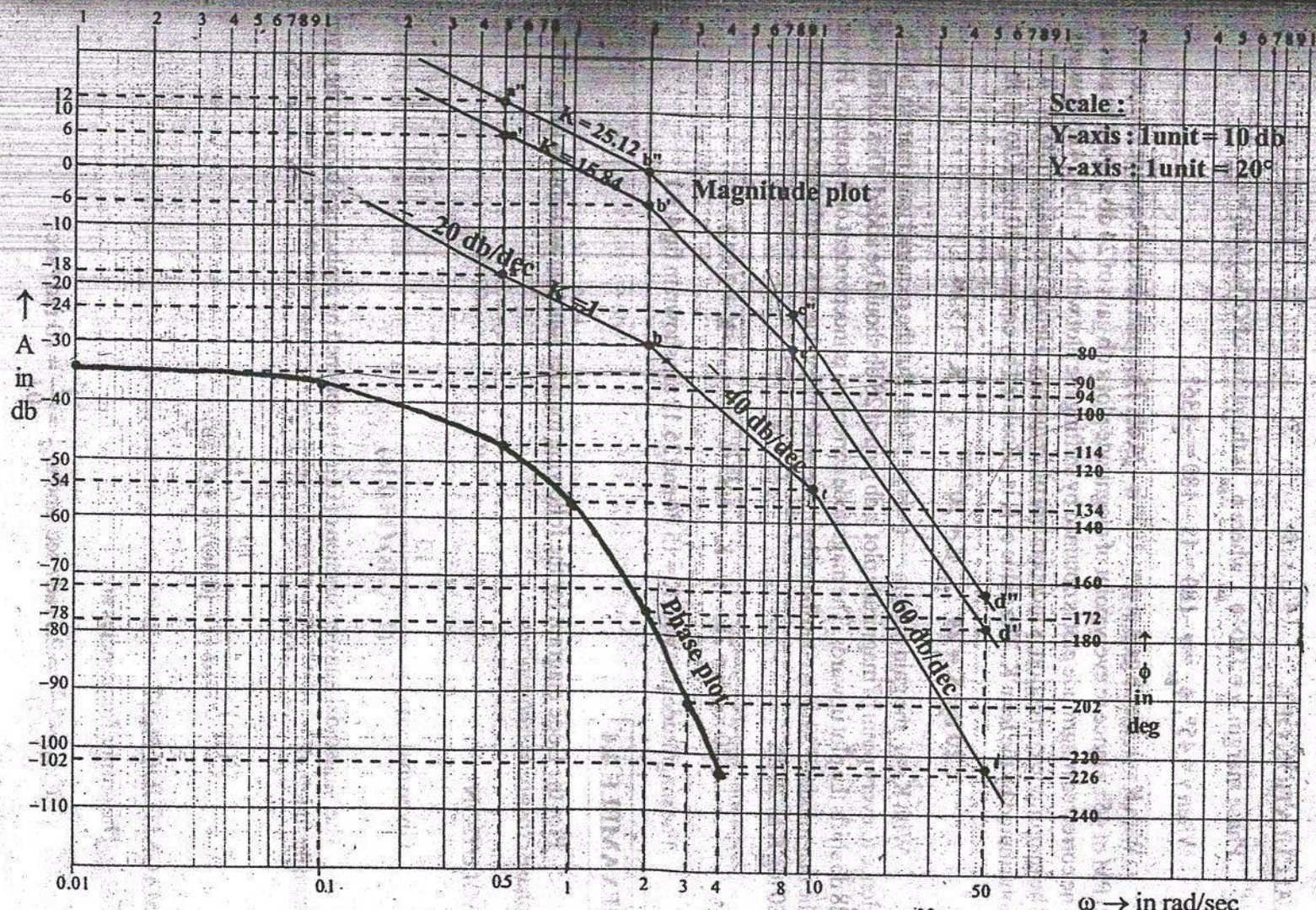


Fig 4.3.1: Bode plot of transfer function, $G(j\omega) = \dfrac{0.0625\,K\,e^{-j0.2\omega}}{j\omega\,(1+j0.5\omega)\,(1+j0.125\omega)}$

$$G(s) = \frac{K \cdot e^{-0.2s}}{s(s+2)(s+8)}$$

Choose $\omega_l < \omega_{c_1}$, $\omega_h > \omega_{c_2}$

Let $\omega_l = 0.5$ rad/sec , $\omega_h = 50$ rad/sec.

$$A\big|_{\omega=\omega_l} = 20\log_{10}|G(j\omega)|$$

$$= 20\log_{10}\left|\frac{0.0625}{\omega}\right|$$

$$= 20\log_{10}\left|\frac{0.0625}{0.5}\right|$$

$$= -18\,db.$$

$$A\big|_{\omega=\omega_{c_1}} = 20\log_{10}|G(j\omega)|$$

$$= 20\log_{10}\left|\frac{0.0625}{\omega}\right|$$

$$= 20\log_{10}\left|\frac{0.0625}{2}\right|$$

$$= -30\,db$$

$$A\big|_{\omega=\omega_{c_2}} = \left[\text{slope from } \omega_{c_2} \text{ to } \omega_{c_1} \times \log\left(\frac{\omega_{c_2}}{\omega_{c_1}}\right)\right] \dot{+} A_{(\omega=\omega_{c_1})}$$

$$= \left[-40 \times \log\left(\frac{8}{2}\right)\right] \dot{+} 30$$

$$= -54\,db$$

$$A\big|_{\omega=\omega_h} = \left[\text{slope from } \omega_h \text{ to } \omega_{c_2} \times \log\left(\frac{\omega_h}{\omega_{c_2}}\right)\right] + A_{(\omega=\omega_{c_2})}$$

$$= \left[-60 \times \log\left(\frac{50}{8}\right)\right] \dot{+} 54$$

$$= -102\,db.$$

Phase plot.

$$\phi = \angle G(j\omega) = -90^\circ - \tan^{-1}\left(\frac{0.5\omega}{1}\right) - \tan^{-1}\left(\frac{0.0125\omega}{1}\right) - 0.2\omega \times \frac{180}{\pi}$$

ω	$-0.2\omega(180/\pi)$	$\tan^{-1} 0.5\omega$	$\tan^{-1} 0.125\omega$	$\phi = \angle G(j\omega)$
0.01	-0.1145	0.28	0.0716	-90
0.1	-1.145	2.86	0.716	-94
0.5	-5.7	14	3.6	-114
1	-11.4	26	7.12	-134
2	-22.9	45	14	-172
3	-34.37	56.30	20.56	-202
4	-45.84	63.43	26.57	-226.

Calculation of k.

phase margin $\gamma = 180 + \phi_{gc}$

where ϕ_{gc} is the phase of $G(j\omega)$ at $\omega = \omega_{gc}$

when $\gamma = 45^0$ $\phi_{gc} = \gamma - 180^0$

$= 45 - 180 = -135^0$.

with $k=1$, the db gain at $\phi = -135^0$ is -24 db.

This gain should be made zero to have to phase margin 45^0. Hence to every point of magnitude plot a db gain of 24 db should be added. The corrected magnitude plot is obtained by shifting the plot with $k=1$ by 24 db upwards.

$\therefore 20 \log k = 24$

$k = 10^{24/20}$

$= 15.84$.

With $k=1$, the gain margin $-(-34) = 34$ db.

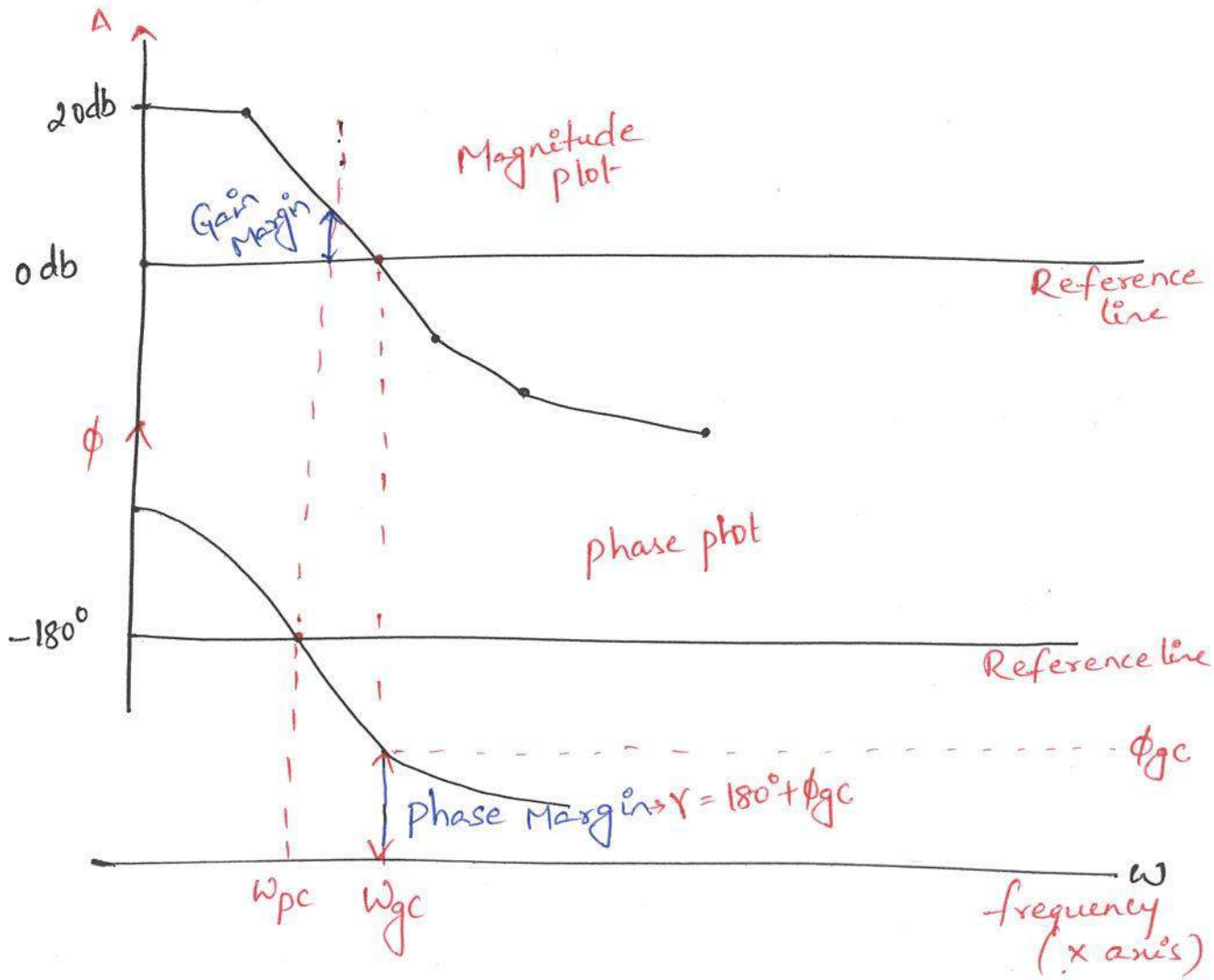
But the required gain margin is 6db. Hence to every point of magnitude plot a db gain of 28db should be added. This addition of 28db shifts the plot upwards.

$20 \log k = 28$

$k = 10^{28/20}$

$k = 25.12$

Calculation of phase and gain margins.



Polar plot

The polar plot of a sinusoidal transfer function G(jω) is a plot of the magnitude of G(jω) versus the phase angle of G(jω) on polar coordinates as ω is varied from 0 to ∞. The polarplot is also called Nyquist plot.

The polar plot is usually plotted on polar graph sheet. The polar graph sheet has concentric circles and radial lines. The circle represent the magnitude and the radial lines represent the phase angles.

Alternatively if $G(j\omega)$ can be expressed in rectangular coordinates as $G(j\omega) = G_R(j\omega) + j\,G_I(j\omega)$.

where $G_R(j\omega)$ = Real part of $G(j\omega)$

$G_I(j\omega)$ = Imaginary part of $G(j\omega)$.

-then polar plot can be plotted in ordinary graph sheet between $G_R(j\omega)$ and $G_I(j\omega)$ as ω is varied from 0 to ∞.

✓ To plot the polar plot, first compute the magnitude and phase of $G(j\omega)$ for various values of ω and tablate them.

✓ To plot the polar plot on ordinary graph sheet, Compute the magnitude and phase for various values of ω.

✓ Then convert polar coordinates to rectangular coordinates using P→R Conversion (polar to rectangular Conversion) in the Calculator. sketch the polar plot using rectangular coordinates.

Minimum phase system:

The minimum phase system are systems with all poles & zeros on the left half of s-plane

Non-minimum phase system:

The poles are either Left side (or) right side

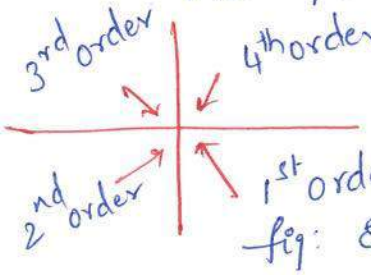


fig: End of polar plot.

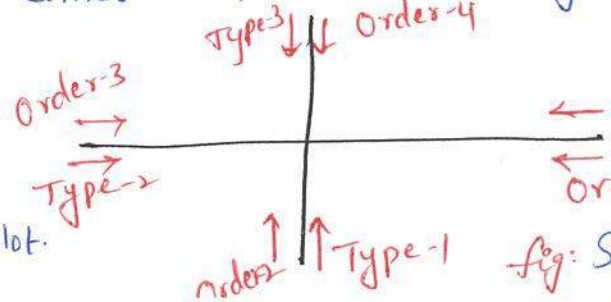


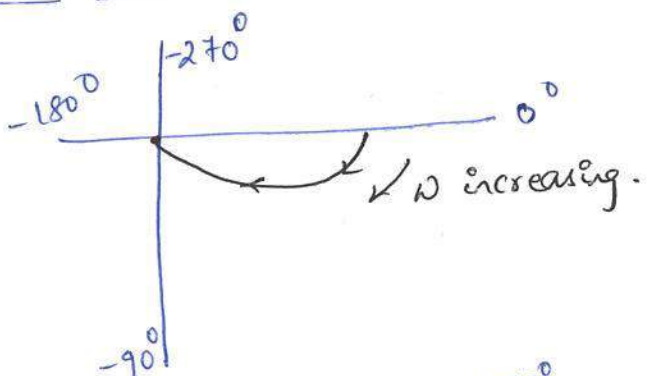
fig: Start of polar plot

Typical sketches of polar plot.

① Type 0, Order : 1

$$G(s) = \frac{1}{1+sT}$$

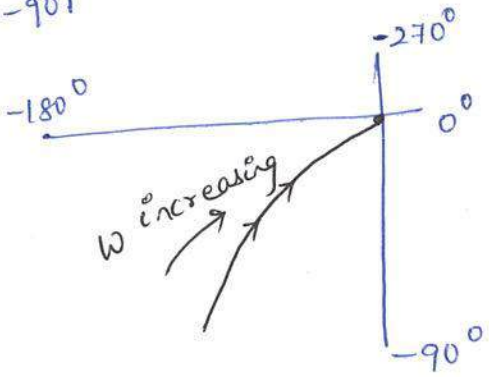
$$G(j\omega) = \frac{1}{1+j\omega T}$$



② Type 1, Order : 2

$$G(s) = \frac{1}{s(1+sT)}$$

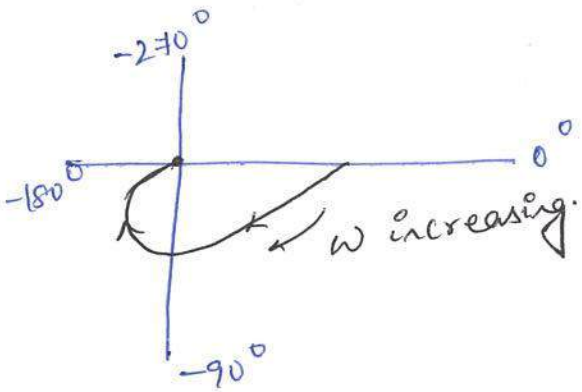
$$G(j\omega) = \frac{1}{j\omega(1+j\omega T)}$$



③ Type 0, order 2

$$G(s) = \frac{1}{(1+sT_1)(1+sT_2)}$$

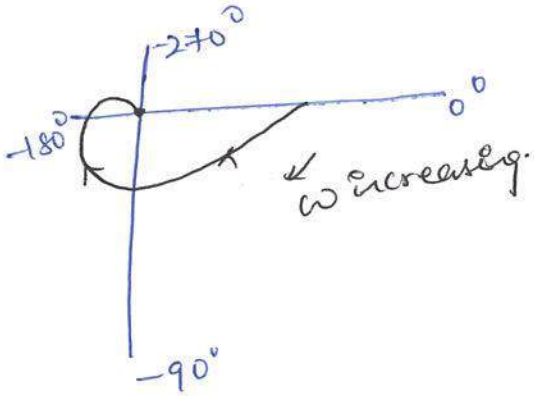
$$G(j\omega) = \frac{1}{(1+j\omega T_1)(1+j\omega T_2)}$$



④ Type 0, order 3

$$G(s) = \frac{1}{(1+sT_1)(1+sT_2)(1+sT_3)}$$

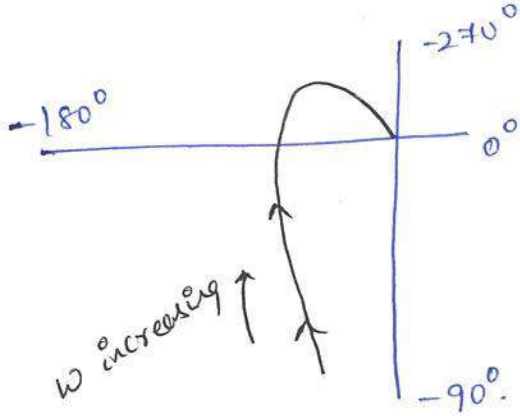
$$G(j\omega) = \frac{1}{(1+j\omega T_1)(1+j\omega T_2)(1+j\omega T_3)}$$



⑤ Type 1 : Order 3 :

$$G(s) = \frac{1}{s(1+sT_1)(1+sT_2)}$$

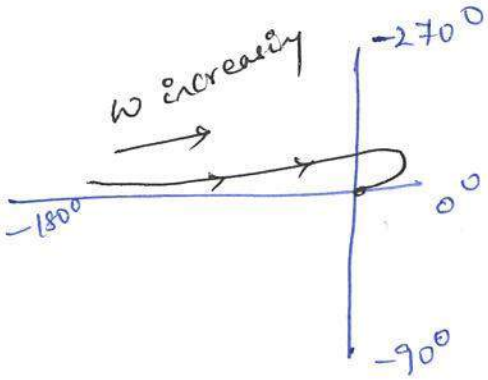
$$G(j\omega) = \frac{1}{j\omega(1+j\omega T_1)(1+j\omega T_2)}$$



⑥ Type 2 : order 4.

$$G(s) = \frac{1}{s^2(1+sT_1)(1+sT_2)}$$

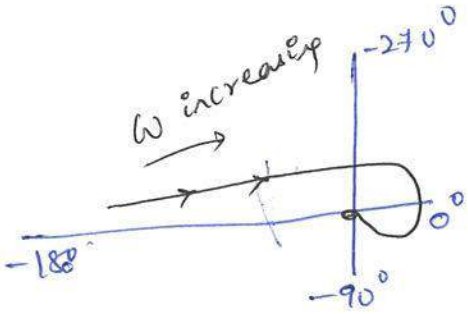
$$G(j\omega) = \frac{1}{(j\omega)^2(1+j\omega T_1)(1+j\omega T_2)}$$



⑦ Type 2 order 5

$$G(s) = \frac{1}{s^2(1+sT_1)(1+sT_2)(1+sT_3)}$$

$$G(j\omega) = \frac{1}{(j\omega)^2(1+j\omega T_1)(1+j\omega T_2)(1+j\omega T_3)}$$



Determination of Gain margin and phase margin from polar plot.

The gain margin is defined as the inverse of the magnitude of $G(j\omega)$ at phase cross over frequency. The phase cross over frequency at which the phase of $G(j\omega)$ is 180°.

The phase margin is defined as, phase margin $\gamma = 180° + \phi_{gc}$, where ϕ_{gc} is the phase angle of $G(j\omega)$ at gain cross over frequency. The gain cross over frequency is the frequency at which the magnitude of $G(j\omega)$ is unity.

Problem 1:

The open loop transfer function of a unity feedback system is given by $G(s) = \frac{1}{s(1+s)(1+2s)}$. sketch the polar plot and determine the gain margin and phase margin.

Sol:

$$G(s) = \frac{1}{s(1+s)(1+2s)}$$

put $s = j\omega$

$$G(j\omega) = \frac{1}{j\omega(1+j\omega)(1+j2\omega)}$$

$$\omega_{c_1} = \frac{1}{2} = 0.5 \text{ rad/sec} \quad ; \quad \omega_{c_2} = \frac{1}{1} = 1 \text{ rad/sec}.$$

Magnitude $|G(j\omega)| = \frac{1}{\sqrt{\omega^2}\left(\sqrt{1^2+\omega^2}\right)\left(\sqrt{1^2+4\omega^2}\right)}$ $\left[\because |x+iy| = x^2+y^2\right]$

$$= \frac{1}{\omega\sqrt{1+5\omega^2+4\omega^4}}$$

Phase $\angle G(j\omega) = -90^\circ - \tan^{-1}(\omega) - \tan^{-1}(2\omega)$

Using polar to rectangular Conversion, the polar Coordinates Converted to rectangular Coordinates.

ω rad/sec	0.35	0.4	0.45	0.5	0.6	0.7	1.0
$\lvert G(j\omega)\rvert$	2.2	1.8	1.5	1.2	0.9	0.7	0.3
$\angle G(j\omega)$	-144	-150	-156	-162	-171	-180	-198.

Table: Magnitude and phase of $G(j\omega)$ at various frequencies.

ω	0.35	0.4	0.45	0.5	0.6	0.7	1.0
$G_R(j\omega)$	-1.78	-1.56	-1.37	-1.14	-0.89	-0.7	-0.29
$G_I(j\omega)$	-1.29	-0.9	-0.61	-0.37	-0.14	0	0.09

Table: Real and imaginary part of $(G(j\omega))$ at various frequencies

Result: $k_g = 1.4286 \rightarrow$ Gain Margin $\Rightarrow k_g = \frac{1}{G(j\omega_{pc})}$

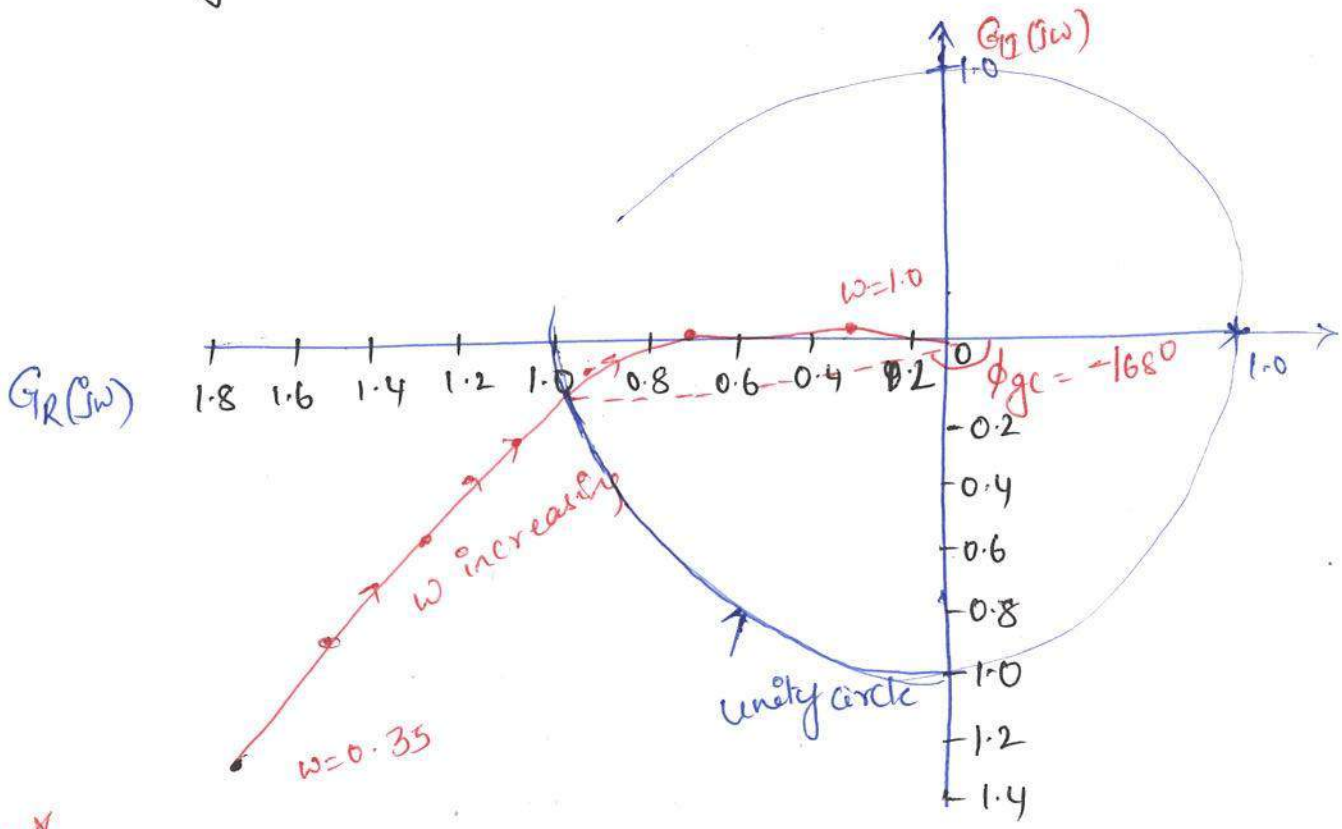
$= \frac{1}{0.7} = 1.4286$

$\gamma = 12^\circ \rightarrow$ Phase Margin. $\rightarrow 180^\circ - \phi_{gc}$

$180^\circ - (168^\circ)$

$= 12^\circ$

[$\because$ ϕ_{gc} = (Angle from orgin to (polar plot ~~touches the~~ crossing the unity circle)



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Problem 2:

The open loop transfer function of a unity feed back system is given by $G(s) = \frac{1}{s^2(1+s)(1+2s)}$. sketch the polar plot and determine the gain margin and phase margin.

Sol:

$$G(s) = \frac{1}{s^2(1+s)(1+2s)}$$

$s = j\omega$

$$G(j\omega) = \frac{1}{(j\omega)^2(1+j\omega)(1+2j\omega)}$$

$$|G(j\omega)| = \frac{1}{\omega^4\sqrt{1+\omega^2}\sqrt{1+4\omega^2}}$$

$G(s) = \frac{1}{s(1+s)(1+2s)}$

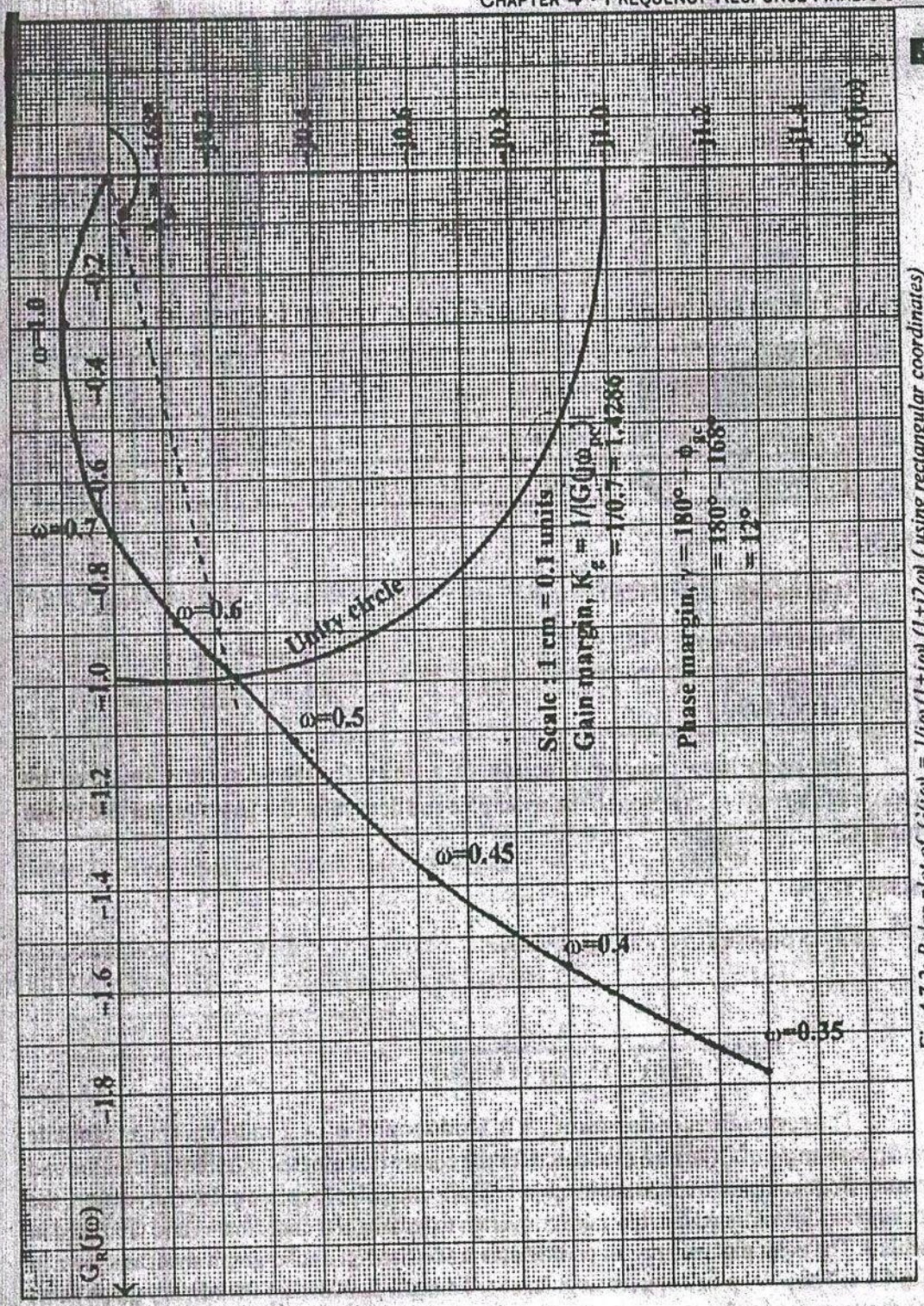
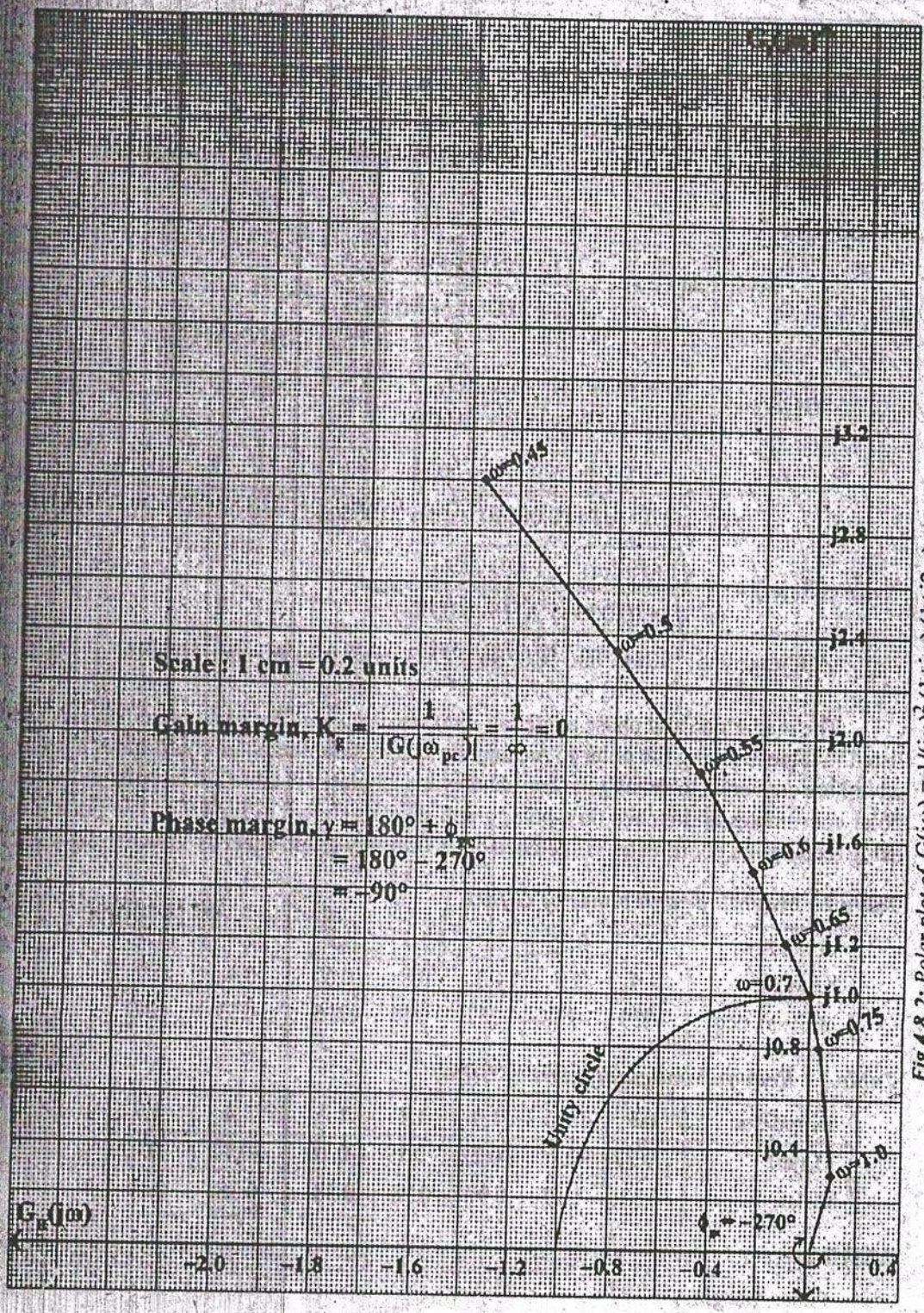


Fig 4.7.2: Polar plot of $G(j\omega) = 1/j\omega(1+j\omega)(1+j2\omega)$ (using rectangular coordinates)

$$G(s) = \frac{1}{s^2(1+s)(1+2s)}$$



Scale : 1 cm = 0.2 units

Gain margin, $K_g = \frac{1}{|G(j\omega_{pc})|} = \frac{1}{\infty} = 0$

Phase margin, $\gamma = 180° + \phi_{pc}$
$= 180° - 270°$
$= -90°$

Fig 4.8.2: Polar plot of $G(j\omega) = 1/(j\omega)^2 (1+j\omega)(1+j2\omega)$, (using rectangular coordinates)

$$= \frac{1}{\omega^2\sqrt{1+\omega^2+4\omega^2+4\omega^4}}$$

$$= \frac{1}{\omega^2\sqrt{1+5\omega^2+4\omega^4}}$$

$$\angle G(j\omega) = -180^\circ - \tan^{-1}\omega - \tan^{-1}2\omega$$

Magnitude and phase plot of $(G(j\omega))$ at various frequencies.

ω	0.45	0.5	0.55	0.6	0.65	0.7	0.75	1.0
$\lvert G(j\omega)\rvert$	8.3	2.5	1.9	1.5	1.2	0.97	0.8	0.3
$\angle G(j\omega)$	-246	-251	-256	-261	-265	-269	-273	-288

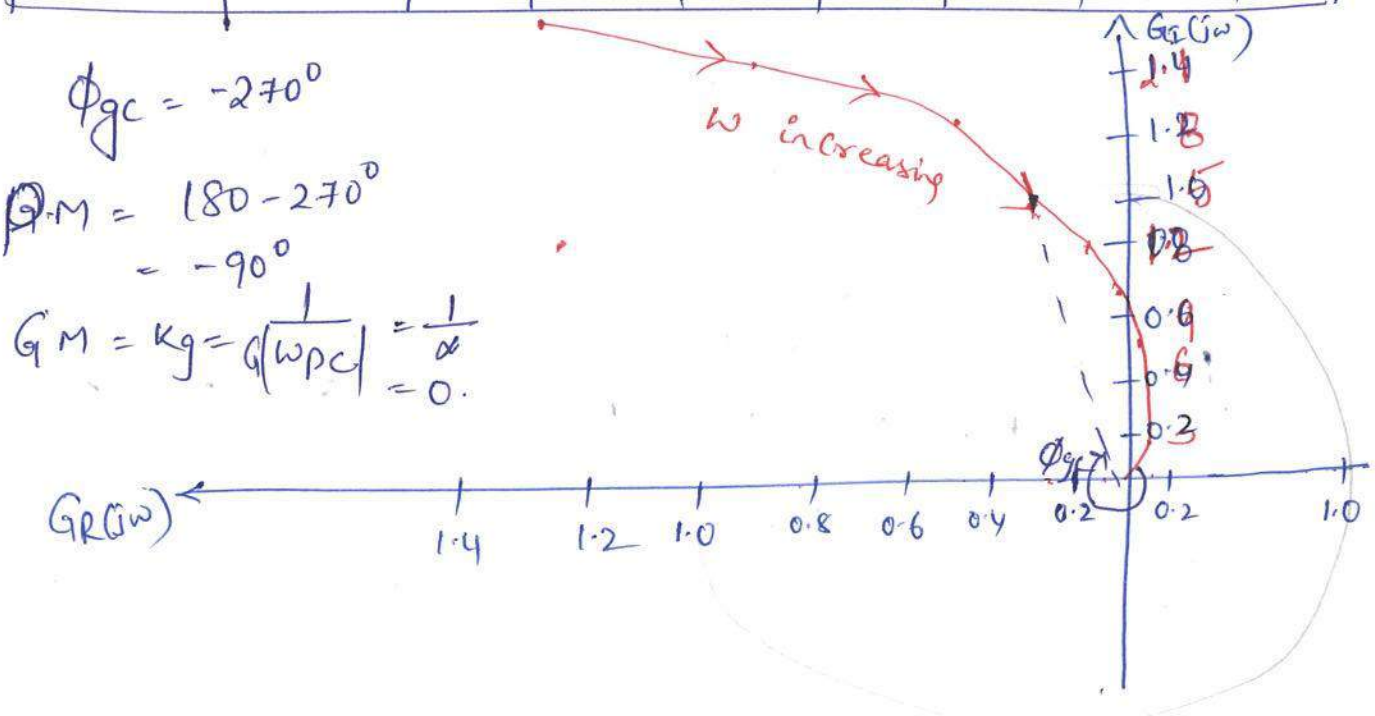
Real and imaginary part of $G(j\omega)$

ω	0.45	0.5	0.55	0.6	0.65	0.7	0.75	1.0
$G_R(j\omega)$	-1.34	-0.81	-0.46	-0.23	-0.1	-0.02	0.04	0.09
$G_I(j\omega)$	3.01	2.36	1.84	1.48	1.2	1.0	0.8	0.29

$\phi_{gc} = -270^\circ$

P.M $= 180 - 270^\circ$
$= -90^\circ$

$GM = K_g = G\left(\frac{1}{\omega_{pc}}\right) = \frac{1}{\alpha}$
$= 0.$



Problem3:

The open loop transfer function of unity feedback system is given by $G(s) = \frac{1}{s(1+s)^2}$. sketch the polar plot and determine the gain and phase margin.

Sol:

$$G(s) = \frac{1}{s(1+s)^2}$$

put $s = j\omega$

$$G(j\omega) = \frac{1}{j\omega(1+j\omega)^2} \Rightarrow \frac{1}{j\omega(1+j\omega)(1+j\omega)}$$

$$|G(j\omega)| = \frac{1}{\sqrt{\omega^2}\sqrt{1+\omega^2}\sqrt{1+\omega^2}} \Rightarrow \frac{1}{\omega\sqrt{(1+\omega^2)^2}}$$

$$= \frac{1}{\omega\sqrt{1+\omega^4+2\omega^2}} \Rightarrow \frac{1}{\omega(1+\omega^2)}$$

$$\angle G(j\omega) = -\tan^{-1}\left(\frac{\omega}{0}\right) - \tan^{-1}\left(\frac{\omega}{1}\right) - \tan^{-1}\left(\frac{\omega}{1}\right)$$

$$= -90^\circ - 2\tan^{-1}\omega.$$

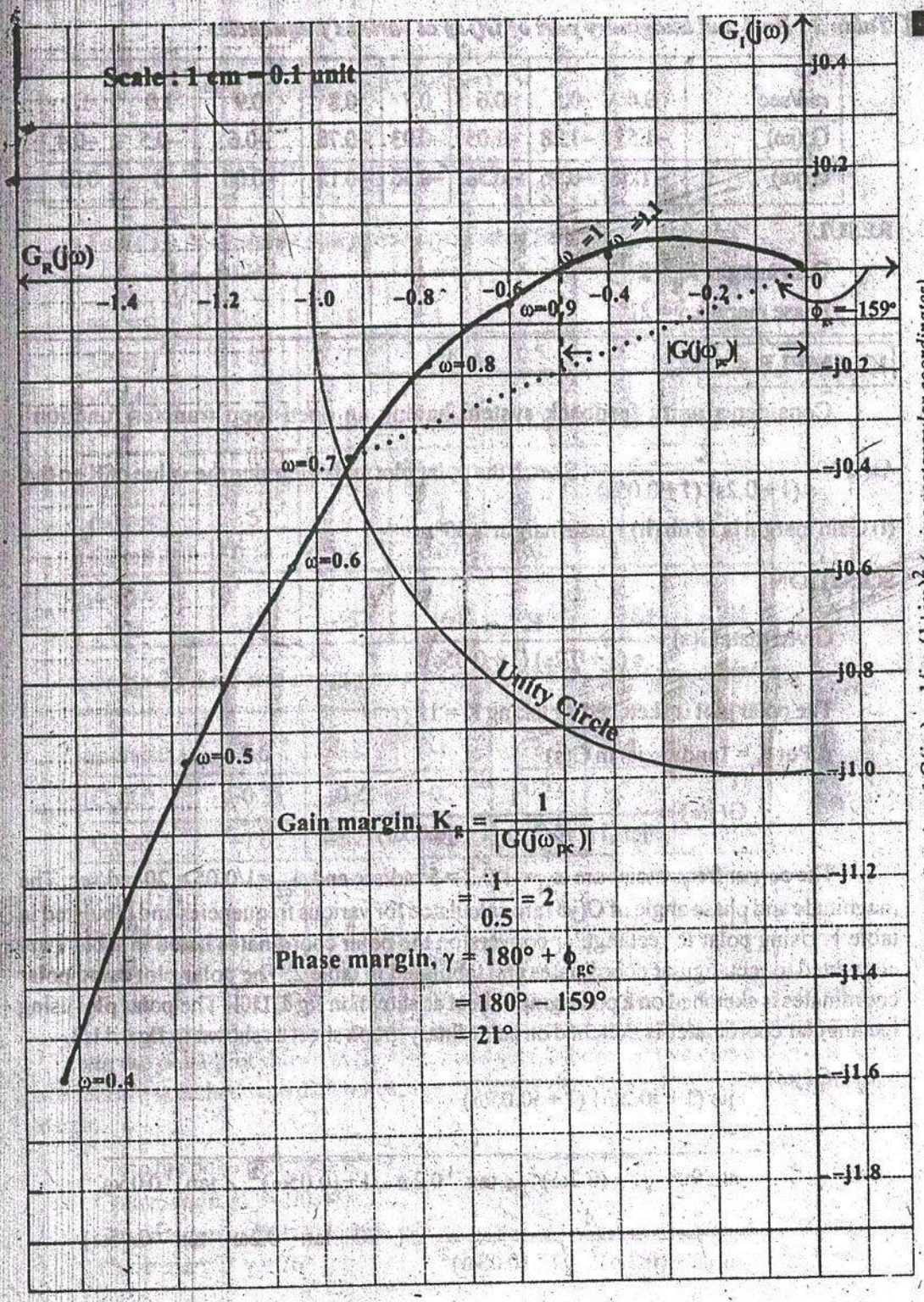
Magnitude and phase of $G(j\omega)$ at various frequencies

ω (rad/sec)	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1
$\lvert G(j\omega)\rvert$	2.2	1.6	1.2	1	0.8	0.6	0.5	0.4
$\angle G(j\omega)$	-134	-143	-151	-159	-167	-174	-180	-185

Real and imaginary part of $G(j\omega)$ at various frequencies

ω	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1
$G_R(j\omega)$	-1.53	-1.28	-1.05	-0.93	-0.78	-0.6	-0.5	-0.4
$G_I(j\omega)$	-1.58	-0.96	-0.58	-0.36	-0.18	-0.06	0	0.03

$G(s) = \frac{1}{s(1+s)^2}$



Gain margin, $K_g = \frac{1}{|G(j\omega_{gc})|}$

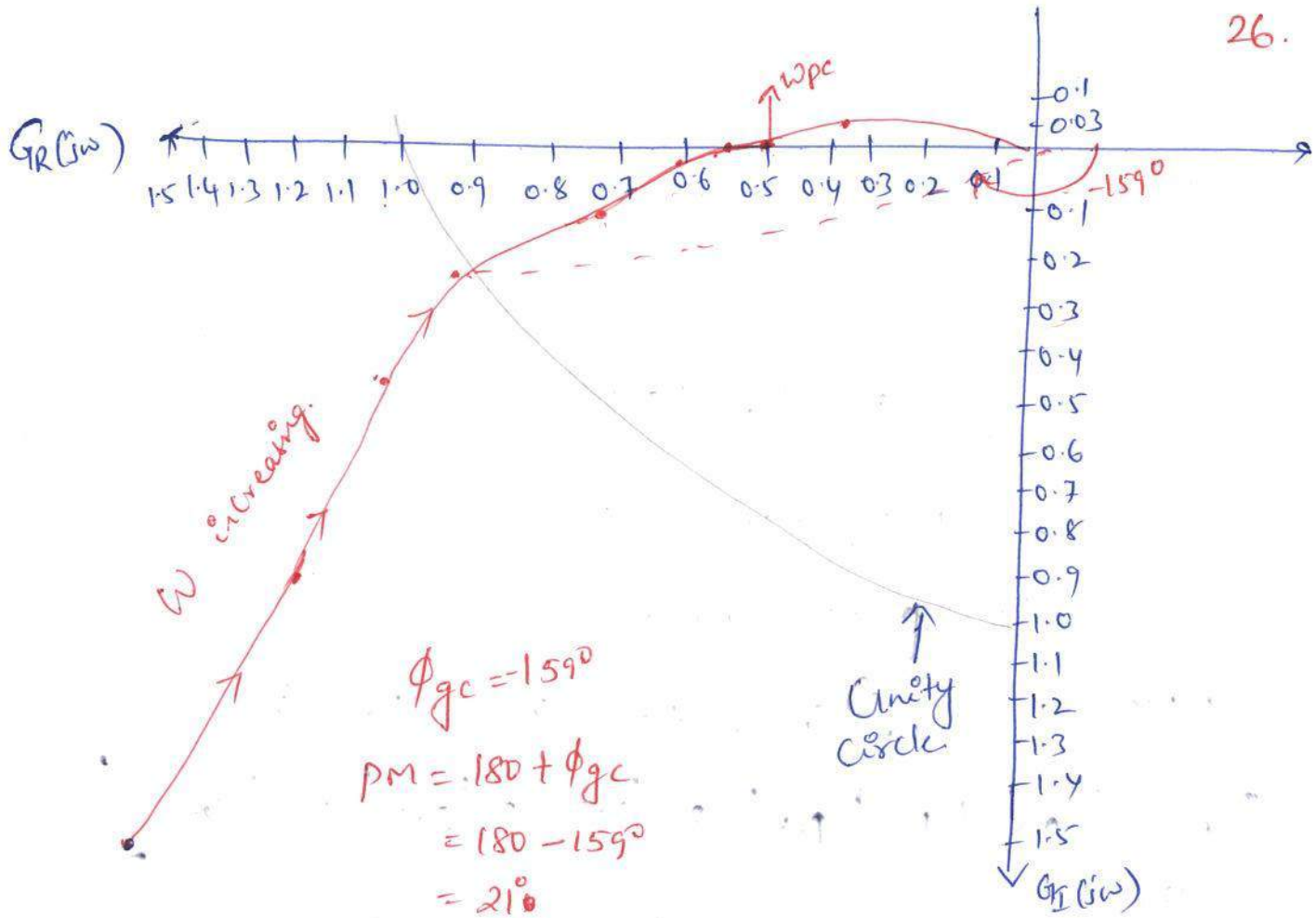
$= \frac{1}{0.5} = 2$

Phase margin, $\gamma = 180° + \phi_{gc}$

$= 180° - 159°$

$= 21°$

Fig 4.10.2: Polar plot of $G(j\omega) = 1/j\omega\,(1+j\omega)^2$ *(using rectangular coordinates)*



$$\phi_{gc} = -159^\circ$$

$$PM = 180 + \phi_{gc}$$
$$= 180 - 159^\circ$$
$$= 21^\circ$$

$$GM = K_g = \frac{1}{G(\omega_{pc})}$$
$$= \frac{1}{0.5} = 2$$

***

Problem 4:

Sketch the polar plot of the following transfer function

$$G(s) = \frac{10(1+s)}{(2+s)(4+s)}$$

Sol:

$$G(s) = \frac{10(1+s)}{(2+s)(4+s)}$$ → Convert into time constant form.

$$= \frac{10(1+s)}{2\left(1+\frac{s}{2}\right)4\left(1+\frac{s}{4}\right)} \Rightarrow \frac{10(1+s)}{8(1+0.5s)(1+0.25s)}$$

$$\Rightarrow \frac{1.25(1+s)}{(1+0.5s)(1+0.25s)}$$

$\omega_{c_2} = \frac{1}{0.5} = 2$; $\omega_{c_3} = \frac{1}{0.25} = 4$ $\omega_{c_1} = \frac{1}{1} = 1$

$$|G(j\omega)| = \frac{1.25\,(1+j\omega)}{(1+0.5j\omega)\,(1+0.25j\omega)}$$

$$= \frac{1.25\sqrt{1+\omega^2}}{\sqrt{1+0.25\omega^2}\sqrt{1+0.0625\omega^2}} \Rightarrow \frac{1.25\sqrt{1+\omega^2}}{1+0.25\omega^2+0.0625\omega^2+0.0156\omega^4}$$

$$\Rightarrow \frac{1.25\sqrt{1+\omega^2}}{1+0.3125\omega^2+0.0156\omega^4}$$

$$\angle G(j\omega) = \tan^{-1}(\omega) - \tan^{-1}(0.5\omega) - \tan^{-1}(0.25\omega).$$

ω	0.5	1	1.5	2	2.5	3	3.5	4	4.5
$G(j\omega)$									
$\angle G(j\omega)$									

ω	0.5	1	1.5	2	2.5	3	3.5	4	4.5
$G_R(j\omega)$									
$G_I(j\omega)$									

Problem 5:

Consider a unity feedback system having an open loop transfer function $G(s) = \frac{K}{s(1+0.2s)(1+0.05s)}$. Sketch the polar plot and determine the value of K so that i) Gain margin is 18 db. ii) phase margin is 60°.

Sol:

$$G(s) = \frac{K}{s(1+0.2s)(1+0.05s)}.$$

The polar plot sketched by taking $K=1$

$\therefore$ put $K=1$ and $s=j\omega$ in $G(s)$

$$G(j\omega) = \frac{K}{j\omega(1+j0.2\omega)(1+j0.05\omega)}$$

Corner frequency $\omega_{c_1} = 1/0.2 = 5$ rad/sec

$\omega_{c_2} = 1/0.05 = 20$ rad/sec.

$$G(j\omega) = \frac{1}{j\omega(1+j0.2\omega)(1+j0.05\omega)}$$

$$|G(j\omega)| = \frac{1}{\sqrt{\omega^2}\sqrt{(1+(0.2\omega)^2}\sqrt{1+(0.05\omega)^2}}$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1}(0.2\omega) - \tan^{-1}(0.05\omega),$$

ω	0.6	0.8	1	2	3	4	5	6	7	8	9	10	11	44
$\lvert G(j\omega)\rvert$	1.65	1.23	1.0	0.5	0.3	0.2	0.14	0.1	0.07	0.05	0.05	0.04	0.03	0.02
$\angle G(j\omega)$	-98	-101	-104	-117.5	-129.4	-140	-149	-157	-164	-175	-185	-180	-184	-195

ω (rad/sec)	0.6	0.8	1	2	3	4	5	6	7	9	10	11	14
$G_R(j\omega)$	-023	-023	-0.24	-0.23	-0.19	-0.15	-0.120	-0.092	-0.067	-0.050	-0.04	-0.030	-0.019
$G_I(j\omega)$	-1.63	-1.31	-0.97	-0.44	-0.23	-0.13	-0.072	-0.089	-0.019	-0.0044	0	0.002	0.005

From the polar plot, with $K=1$

Gain margin $k_g = \frac{1}{0.04} = 25$

Gain margin in db $= 20 \log 25 = 28$ db.

Phase margin $\gamma = 76^\circ$

Case i) With $K=1$, Let $G(j\omega)$ Cut the -180° axis at

gain at the point is $G_B = 0.04$.

The gain margin of 28 db with $K=1$ has to be reduced to 18db and so K has to increased to a value greater than one.

Let G_A be the gain at -180° for a gain margin of 18 db.

Now $20 \log \frac{1}{G_A} = 18$

$$\log \frac{1}{G_A} = \frac{18}{20}$$

$$\frac{1}{G_A} = 10^{18/20}$$

$$\therefore G_A = \frac{1}{10^{18/20}} = 0.125$$

The value of K is given by $K = \frac{G_A}{G_B} = \frac{0.125}{0.04}$

$$= 3.125$$

$$\boxed{K = 3.125}$$

$G(s) = \frac{K}{s(1+0.2s)(1+0.05s)}$

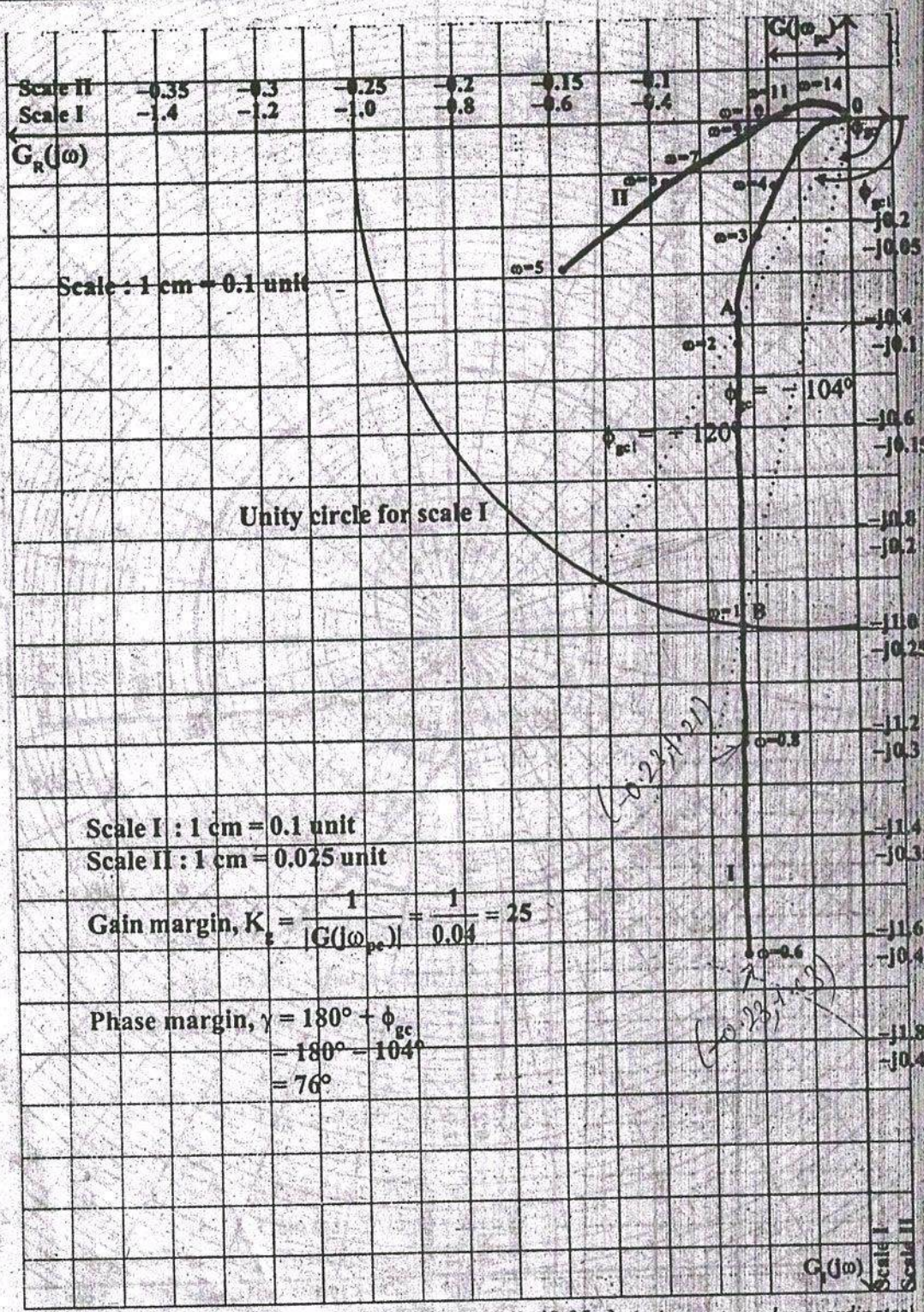


Fig 4.11.2: Polar plot of $G(j\omega) = 1/j\omega\,(1+j0.2\omega)\,(1+j0.05\omega)$, *(using rectangular coordinates)*

Case ii) With $K=1$, the phase margin is 76°. This has to be reduced to 60°. Hence gain has to be increased

Let ϕ_{gc_2} be the phase of $G(j\omega)$ for a phase margin of 60°

$$\therefore 60^\circ = 180 + \phi_{gc_2}$$

$$\phi_{gc_2} = 60^\circ - 180^\circ = -120^\circ$$

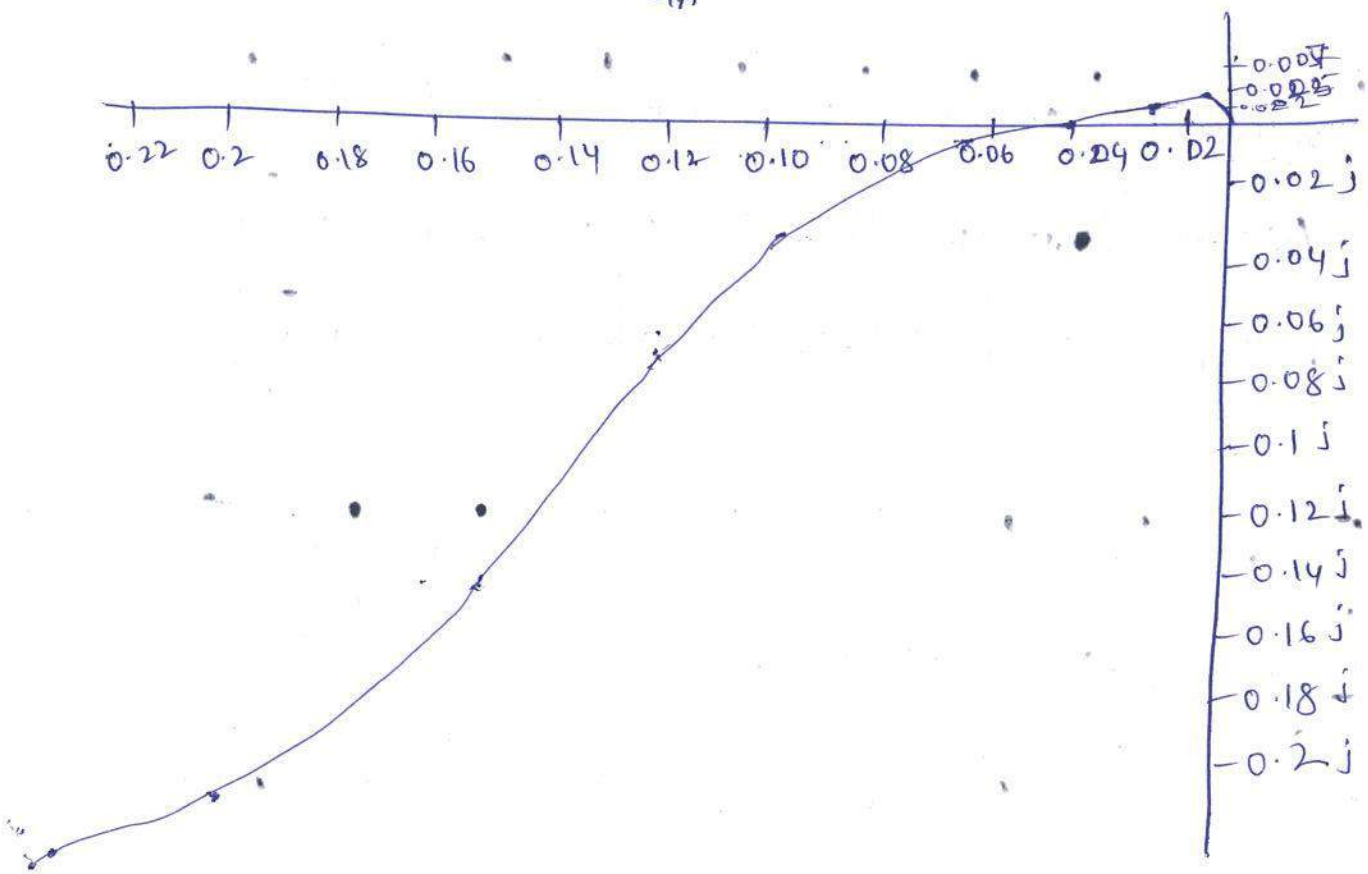
In the polar plot the -120° line cut the locus of $G(j\omega)$ at point A and cut the unity circle at point B.

Let G_A = Magnitude of $G(j\omega)$ at point A

G_B = " " " " " B.

From the polar plot $G_A = 0.425$ and $G_B = 1$

$$\text{Now } k = \frac{G_B}{G_A} = \frac{1}{0.425} = 2.353.$$



Problem 6:

Consider a unity feedback system having an open loop transfer function $G(s) = \frac{K}{s(1+0.5s)(1+4s)}$. Sketch the polar plot and determine the value of K so that i) Gain margin is 20 db ii) phase margin 30°.

Sol:

$$G(s) = \frac{K}{s(1+0.5s)(1+4s)}$$

put $K=1$, $s=j\omega$

$$G(j\omega) = \frac{1}{j\omega(1+j0.5\omega)(1+j4\omega)}$$

$\omega_{c_1} = 0.25$ rad/sec $\quad \omega_{c_2} = 2$ rad/sec.

$$|G(j\omega)| = \frac{1}{\sqrt{\omega^2}\sqrt{1+(0.5\omega)^2}\sqrt{1+(4\omega)^2}}$$

$$= \frac{1}{\omega\sqrt{1+0.25\omega^2}\sqrt{1+16\omega^2}}$$

$$\angle G(j\omega) = -90 - \tan^{-1}0.5\omega - \tan^{-1}4\omega.$$

ω (rad/sec)	0.3	0.4	0.5	0.6	0.8	1.0	1.2
$\lvert G(j\omega)\rvert$	2.11	1.3	0.87	0.61	0.35	0.22	0.15
$\angle G(j\omega)$	-149	-159	-167	-174	-184	-193	-199

ω	0.3	0.4	0.5	0.6	0.8	1.0	1.2
$G_R(j\omega)$	-1.8	-1.21	-0.85	-0.61	-0.35	-0.21	-0.14
$G_I(j\omega)$	-1.09	-0.47	-0.2	-0.06	0.02	0.05	0.05

from the polar plot with $k=1$

Gain margin $k_g = 1/0.44 = 2.27$

Gain margin in db $= 20 \log 2.27 = 7.12$ db

phase margin $\gamma = 15^\circ$

Case I: with $k=1$, Let $G(j\omega)$ cut -180° axis at point B and gain corresponding to that point be G_B. From the polar plot $G_B = 0.44$, The gain margin of 7.12db with $K=1$ has to be increased to 20db and so k has to be decreased to a value less than one.

Let G_A be the gain at -180° for gain margin of 20 db

Now $20 \log \frac{1}{G_A} = 20$

$$\log \frac{1}{G_A} = \frac{20}{20} = 1$$

$$\frac{1}{G_A} = 10^1 = 10$$

$$G_A = \frac{1}{10} = 0.1$$

The value of k is given by $= K = \frac{G_A}{G_B} = \frac{0.1}{0.44} = 0.227$

Case ii) with $k=1$ the phase margin is 15°. This has to be increased to 30°. Hence the gain has to be decreased.

Let ϕ_{gc_2} be the phase of $G(j\omega)$ for a phase margin of 30°

$\therefore 30^\circ = 180 + \phi_{gc_2}$

$\phi_{gc_2} = 30 - 180^\circ$

$= -150^\circ$.

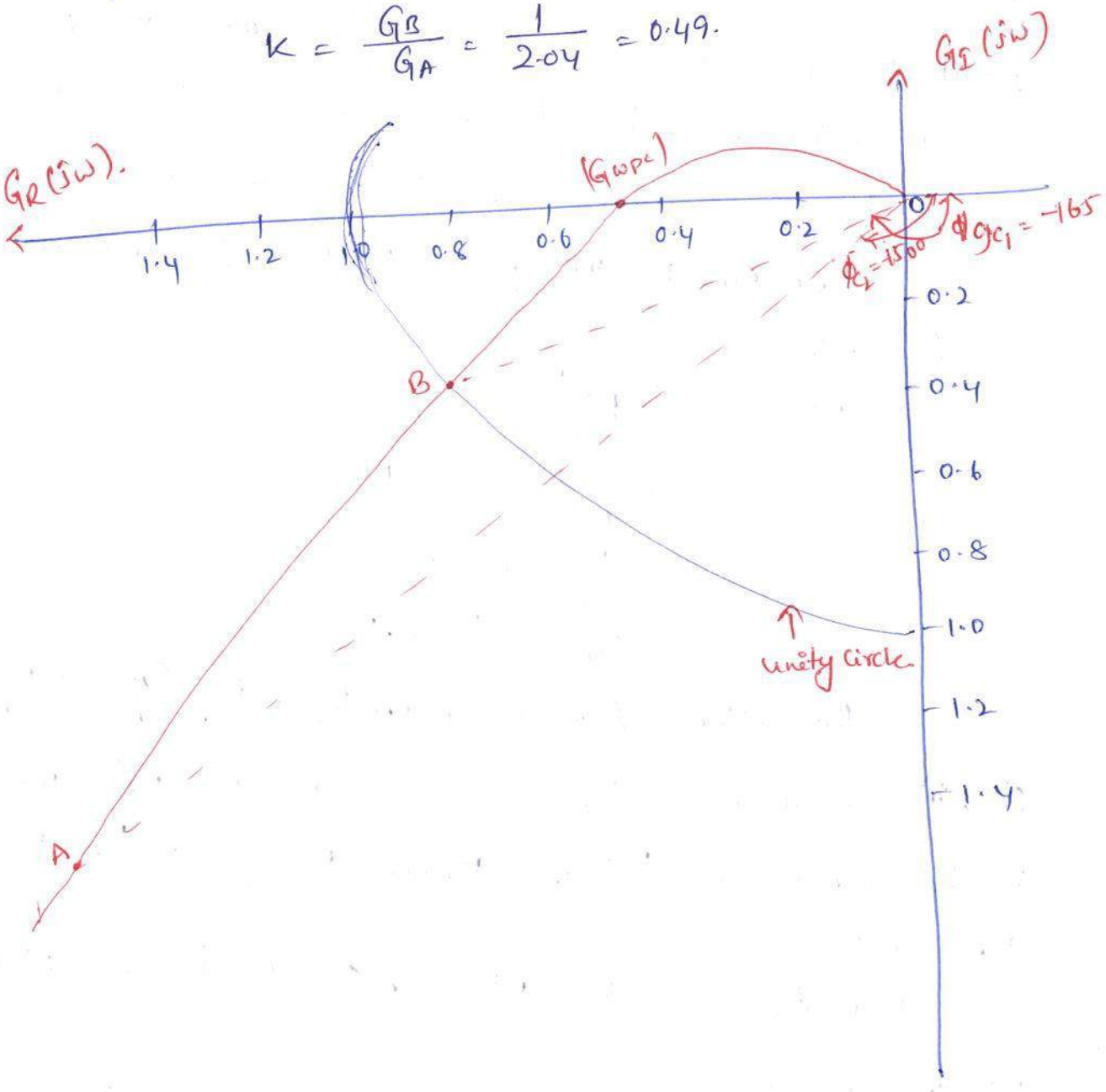
in the polar plot the -150° line cuts the locus of $G(j\omega)$ at point A and cut the unity circle at point B.

Let G_A = Magnitude of $G(j\omega)$ at point A

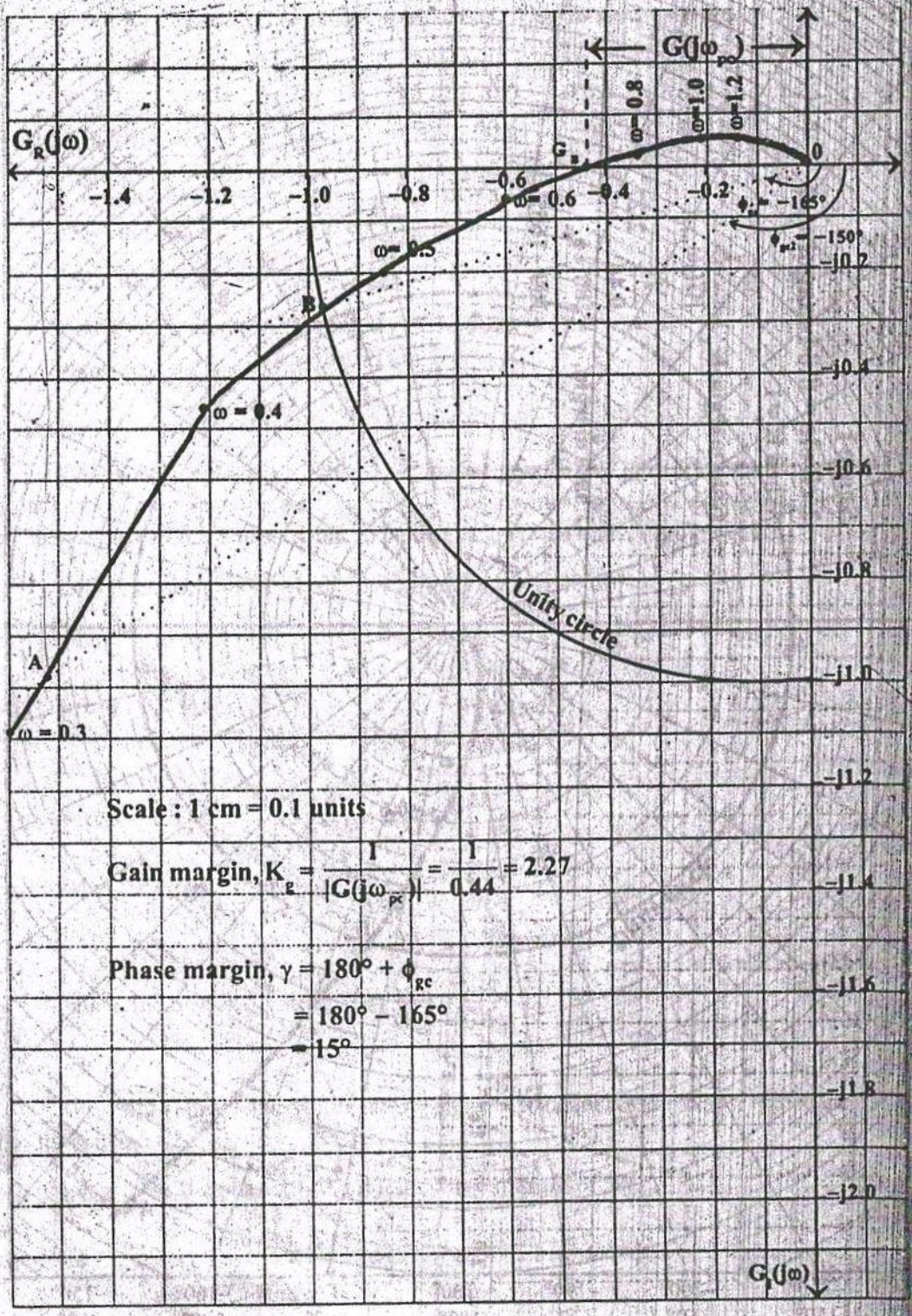
G_B = " " " " B

from the polar plot $G_A = 2.04$ & $G_B = 1$

$$K = \frac{G_B}{G_A} = \frac{1}{2.04} = 0.49.$$



$G(s) = \frac{K}{s(1+0.5s)(1+4s)}$



Scale : 1 cm = 0.1 units

Gain margin, $K_g = \frac{1}{|G(j\omega_{pc})|} = \frac{1}{0.44} = 2.27$

Phase margin, $\gamma = 180° + \phi_{gc}$

$= 180° - 165°$

$= 15°$

Nyquist Stability Criterion :

Consider the closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

The characteristic equation of the system is given by $1+G(s)H(s)=0$.

Let $F(s) = 1+G(s)H(s)$

T-f of G can be expressed as

$$G(s)H(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

where $m \le n$

$$\therefore F(s) = 1+G(s)H(s) = 1 + \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$$= \frac{(s+p_1)(s+p_2)\cdots(s+p_n) + K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$$= \frac{(s+z_1')(s+z_2')+\cdots(s+z_n')}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

Where $z_1', z_2' \cdots z_n'$ are zeros of $F(s)$ which are obtained by combining the numerator and denominator polynomial of $G(s)H(s)$

For the condition $F(s)=0$, the numerator of $F(s)$ should be equal to zero.

$$\therefore (s+z_1')(s+z_2')\cdots(s+z_n')=0 \quad \text{---} (1)$$

Eq (1) represents the characteristic equation of the system. for the stability of the system of the characteristic equation should not lie on the right half of the S-plane. The roots of Characteristic equation are zeros of F(s) and also they are poles of closed loop transfer function

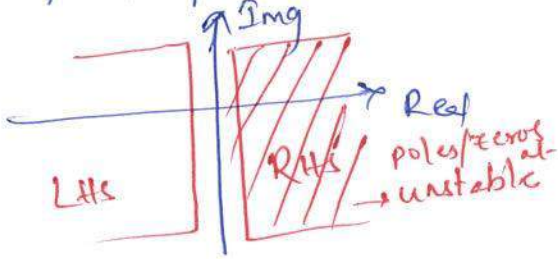
Hence we can conclude that for the stability of closed loop system the zeros of F(s) should not lies on the right half of s-plane.

Stability using polar plot

→ polar plot stability analysis is only applicable to minimum phase system.

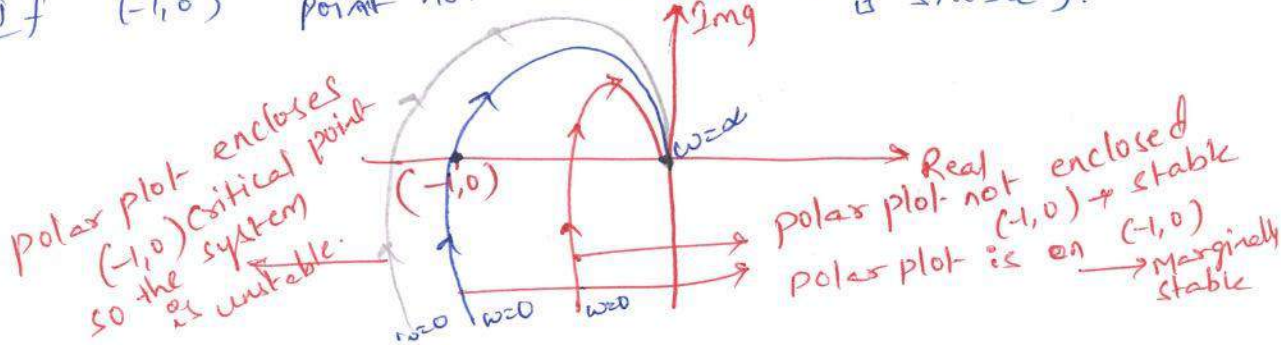
→ for minimum phase system, all the poles and zeros are there in Left half of the plane.

→ To check stability, see the Critical point (-1,0), check enclosement around (-1,0) in polar plot.



→ If (-1,0) point enclosed → [closed loop system is unstable]

→ If (-1,0) point not enclosed → [closed loop system is stable).



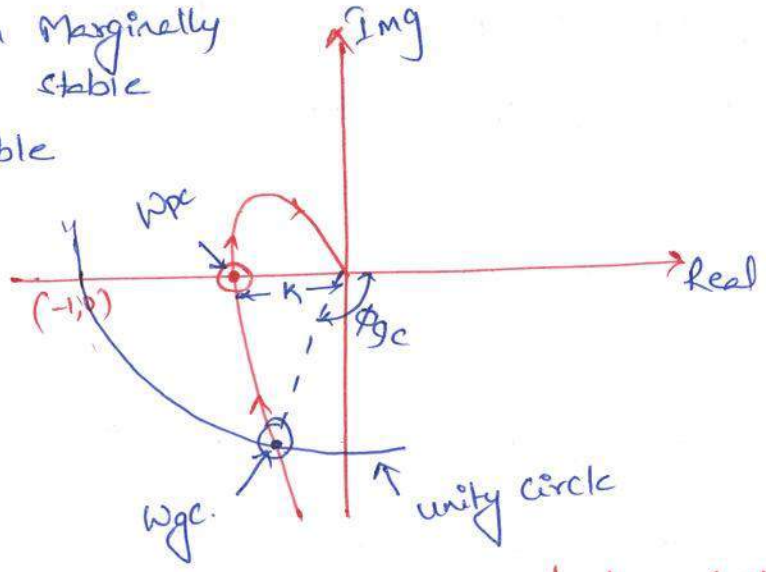
$\omega_{pc} > \omega_{gc}$ → System is stable

$\omega_{pc} = \omega_{gc}$ → " " Marginally stable

$\omega_{pc} < \omega_{gc}$ → unstable

Gain Margin

$= \frac{1}{k} = 20 \log\left(\frac{1}{k}\right)$



Gain Margin and phase Margin of a closed loop system:

Gain Margin is a factor by which the system gain can be increased to drive the system to the verge of stability. The magnitude of $G(j\omega)H(j\omega)$ when it crosses real axis and the phase corresponding to that point is $-180°$. The frequency corresponding to that point be ω_{pc}. If the gain of the system is increased by a factor $1/A$ then the magnitude at the frequency ω_{pc} will be $A \cdot \frac{1}{A} = 1$, Now the $G(j\omega)H(j\omega)$ locus will pass through 1, K_g of the system be defined as the reciprocal of the gain at which the phase angle is $-180°$. The frequency at which the phase angle is $-180°$ is called phase cross over frequency.

$$\text{Gain Margin, } K_g = \left| \frac{1}{G(j\omega)H(j\omega)} \right|_{\omega = pc}$$

$$\text{Gain margin in db} = 20 \log \frac{1}{|G(j\omega)H(j\omega)|}$$

The phase Margin is defined as the amount of additional phase lag at gain Crossover frequency required to bring the system to verge of instability. The frequency at which the magnitudes of $G(j\omega)H(j\omega)$ equals unity is called gain crossover frequency, ω_{gc}.

The phase angle Correeponding to the meeting point of unity circle and $G(j\omega)H(j\omega)$ locus is $-180^\circ + \gamma$. Now with magnitude remaining unity if an additional phase lag equal to γ is introduced then the net phase angle becomes -180° and $G(j\omega)H(j\omega)$ locus will pass through $-1+j0$ point driving the system to the verge of instability. This additional phase lag γ is known as phase Margin.

Let $\phi_{gc} = \angle G(j\omega)H(j\omega)\big|_{\omega=\omega_{pc}}$.

$$= -180^\circ + \gamma = \phi_{gc}$$

$\therefore$ phase Margin $\gamma = 180^\circ + \phi_{gc}$.

for stability of closed loop system the gain margin of open loop system should be greater than 1 or if it is expressed in db it should be positive and phase Margin of open loop system should be positive.

Stability:

A System is said to be stable, if it's output is under control. Otherwise, it is said to be unstable. A stable system produces a bounded output for a given bounded input.

Absolutely stable system:

If the system is stable for all the range of system component values, then it is known as "absolutely stable system". The open loop control system is absolutely stable if all the poles of the open loop transfer function present in left half of s-plane. Similarly, the closed loop control system is absolutely stable if all the poles of the closed loop transfer function present in the left half of the s-plane.

Conditionally stable system:

If the system is stable for a certain range of system component values, then it is known as Conditionally stable system.

Marginally stable system:

If the system is stable by producing an output signal with constant amplitude

And constant frequency of oscillations for bounded input, then it is known as "marginally stable system".

The open loop Control system is marginally stable if any two poles of the open loop transfer function is present on the imaginary axis. Similarly, the Closed loop Control system is marginally stable if any two poles of the closed loop transfer function is present on the imaginary axis.

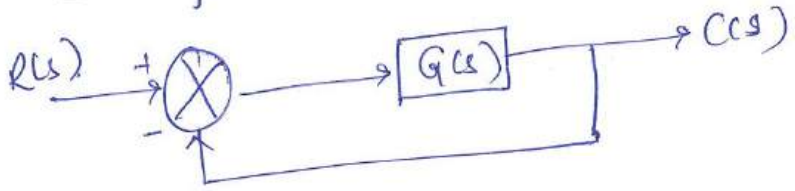
Steady state accuracy:

The deviation of output of Control system from desired response during steady state is known as steady state error from which we get steady state accuracy. It is represented as e_{ss}. We can find steady state error using the final value theorem as follows.

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} s\,E(s)$$

$E(s)$ is the Laplace transform of the error signal $e(t)$.

Steady state errors for unity feedback systems.



By the transfer function of unity negative feedback closed loop ~~tran~~ control system as

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$C(s) = \frac{R(s)\,G(s)}{1+G(s)} \quad \text{—— (1)}$$

The output of the summing point is

$$E(s) = R(s) - C(s) \quad \text{—— (2)}$$

Substitute (1) in (2)

$$E(s) = R(s) - \frac{R(s)\,G(s)}{1+G(s)}$$

$$= \frac{R(s)}{1+G(s)}$$

Substitute (2) in steady state error formula.

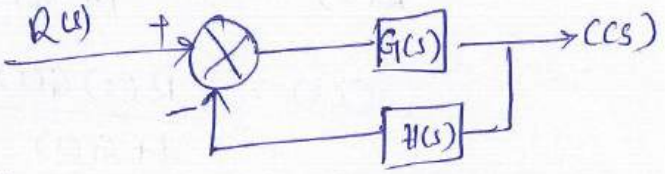
$$e_{ss} = \lim_{s\to 0} \frac{s\,R(s)}{1+G(s)}$$

Input signal	Steady state error	Error Constant
Unit step signal	$\frac{1}{1+k_p}$	$k_p = \lim_{s\to 0} G(s)$
unit ramp signal	$\frac{1}{k_v}$	$k_v = \lim_{s\to 0} s\,G(s)$
Unit parabolic signal	$\frac{1}{k_a}$	$k_a = \lim_{s\to 0} s^2 G(s)$

Where k_p, k_v and k_a are position error Constant, velocity error constant and acceleration error Constant respectively.

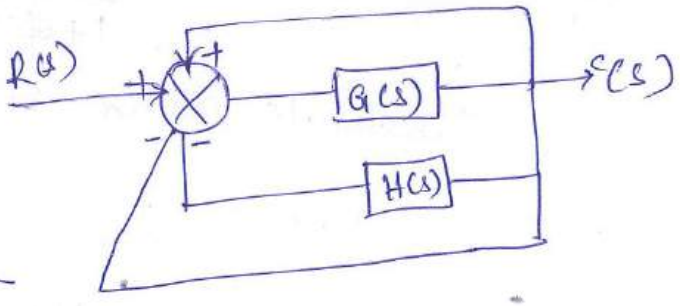
Steady state error for Non-unity feedback System:

Consider the following block diagram of closed loop Control system, which is having non unity negative feedback.

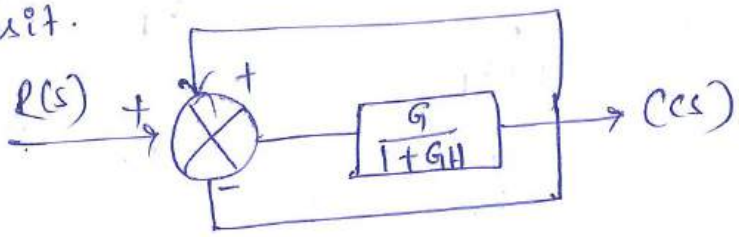


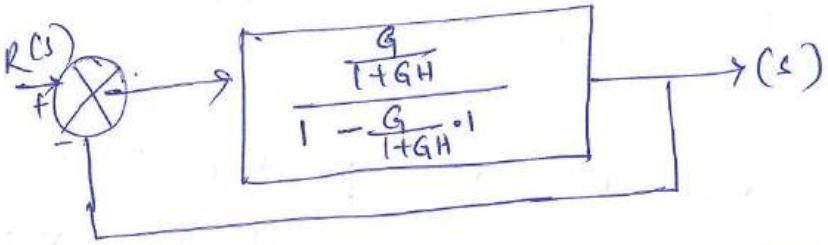
We can find the steady state errors only for the unity feedback system. So we have to Convert the non-unity feedback system into unity feedback system.

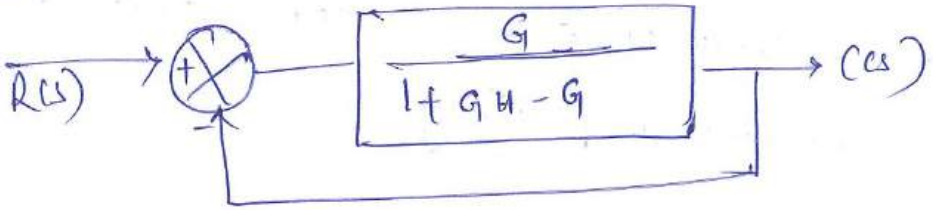
For the above include one unity negative feedback and unity positive feedback path in the block diagram.



Simplify the block diagram by keeping the unity negative feedback ckt.







$\therefore$ Consider $G(s)$ as $\dfrac{G(s)}{1+G(s)H(s)-G(s)}$

for the Calculation of steady state error

$$\frac{C(s)}{R(s)} = \frac{\dfrac{G(s)}{1+G(s)H(s)-G(s)}}{1+\dfrac{G(s)}{1+G(s)H(s)-G(s)}\cdot 1}$$

$$= \frac{\dfrac{G(s)}{1+G(s)H(s)-\cancel{G(s)}}}{\dfrac{1+G(s)H(s)-\cancel{G(s)}+\cancel{G(s)}}{1+G(s)H(s)-\cancel{G(s)}}}$$

$$= \frac{G(s)}{1+G(s)H(s)}$$

[$\because$ same as unity -ve feedback system].

$$C(s) = \frac{G(s)R(s)}{1+G(s)H(s)}$$

$\therefore E(s) = R(s) - C(s)$

$$E(s) = R(s) - \frac{G(s)R(s)}{1+G(s)H(s)}$$

$$= R(s) - \frac{G(s)R(s)}{1+G(s)} \quad [\because H(s)=1.$$

$$= \frac{R(s)}{1+G(s)}$$

$$e_{ss} = \lim_{s\to 0} s\,\frac{R(s)}{1+G(s)}.$$

$=$

Transient accuracy:

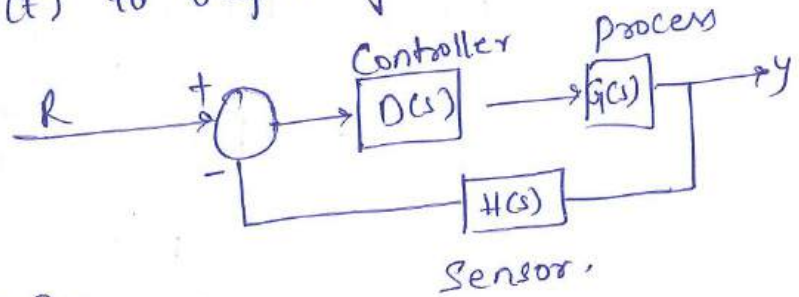
Transient response of Control system means Changing, so, this occurs mainly after two conditions and these two conditions are written as follows.

1. Just after switching 'ON' the system that means at the time of application an input signal to the system.
2. Just after any abnormal conditions. Abnormal conditions may include sudden change in the load short circuiting ...etc.

Disturbance rejection:

The most important effect of feedback in a Control system is the Control and partial elimination of the effects of disturbance signals. A feedback Control system with a disturbance input is shown in fig.

N(s) is plant transfer function from the disturbance input w(t) to output y(t).



$$Y(s) = \frac{D(s)G(s)}{1+D(s)G(s)H(s)} + \frac{N(s)}{1+D(s)G(s)H(s)} W(s)$$

with $R(s) = 0$, the system output becomes.

$$Y(s) = \frac{N(s)}{1+D(s)G(s)H(s)} W(s)$$

For the openloop system.

$$Y(s) = N(s)\, W(s)$$

The feedback reduces the effect of feedback on Control system. Thus to reject disturbances, the loop gain must be increased in manner such that disturbance i/p from the system o/p is not increased.

Sensitivity and Robustness:

The Robustness refers to the ability of a Control system to with-stand parameter variations in the plant transfer function, and still maintain the stability and performance goal.

Robustness is characterized interms of the sensitivity of the closed loop transfer function T(s) to variation in one or more of the plant parameters.

System sensitivity function:

The system sensitivity function is defined as the ratio of percentage change in the Closed loop transfer function to percentage change in the plant transfer function i.e assuming unity feedback $(H(s) = 1)$.

$$S_G^T = \frac{dT/T}{dG/G} = \frac{dT}{dG} \cdot \frac{G}{T} = \frac{1}{1+kG(s)}$$

Hence, increasing loop gain in a feedback Control system sensitivity function, denoted as S(s) is the same as the transfer function from the error, e(s), to output y(s). The sensitivity function is, thus, a function of frequency.

$$S(s) + T(s) = 1$$

where T(s) is closed loop transfer function

Parameter sensitivity:

The sensitivity of the closed loop transfer function, T(s) to percentage change in a plant parameter, α is given as

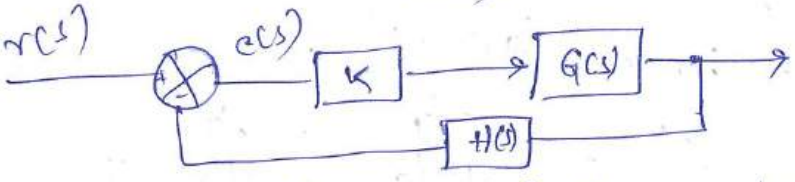
$$S_\alpha^T = S_G^T S_\alpha^G.$$

Root loci Method of feedback Controller Design:

The Root locus Constitutes a graph of the closed loop root locations, with variation in static feedback Controller gain k. Inorder to develop the Rootlocus Concepts, Consider a typical feedback Control system where k represents a Controller.

G(s) is the transfer function

H(s) is the sensor transfer functi



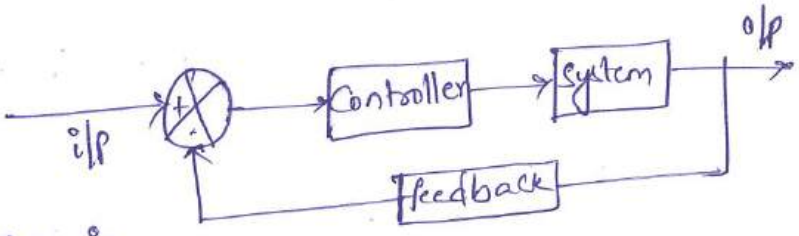
The loop gain for the feedback loop includes

the Controller, the plant, and the sensor transfer function.

Controllers :-

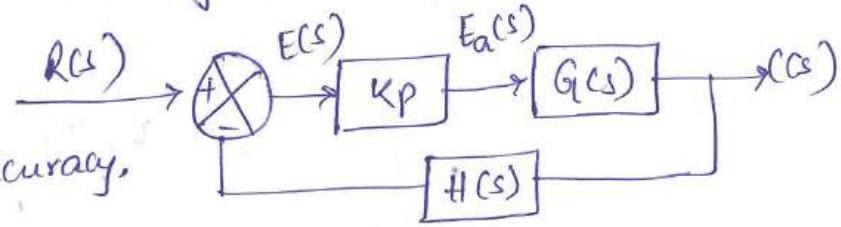
1. Controller is a device, when it is introduced in feedback of forward path of the system, controls the steady state and transient response as per the requirement
2. Controller Converts the input to some other form of error due to which steady state and transient response gets improved.
3. i.e, A Controller is a device introduced in the system to modify the error signal and to produce the Control signal.

Proportional Control:



In proportional Control, the actuating signal $E_a(s)$ is proportional to the error signal $E(s)$. Hence the name proportional Control system.

The block diagram is given as



without sacrifying the steady state accuracy, the maximum overshoot can be reduced to some extent by modifying the actual signal.

$$E_a(s) = K_p \cdot E(s)$$

↳ proportional Gain.

~~Proportional~~

* Derivative Control~~ler~~ : (P-D Controller)

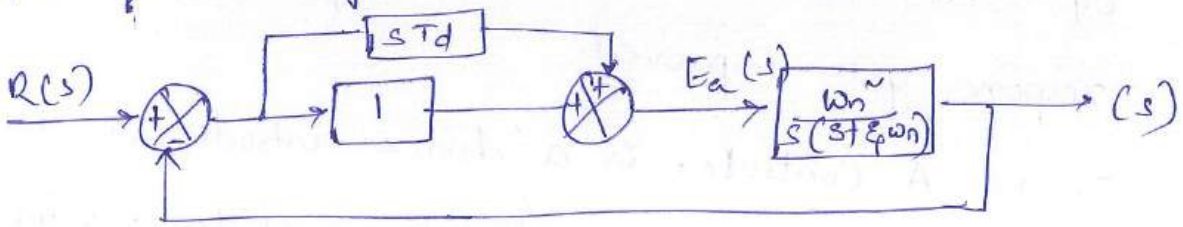
In ~~proportional~~ derivative Controller, the actuating signal. Consists of proportional error signal and derivative of error signal

$$e_a(t) = e(t) + T_d \cdot \frac{d}{dt}[e(t)]$$

$$E_a(s) = E(s) + T_d s \cdot E(s)$$

$$E_a(s) = E(s)[1 + T_d s]$$

The block diagram is



$$\frac{C(s)}{R(s)} = \frac{(1 + sT_d)\dfrac{\omega_n^2}{s(s+2\xi\omega_n)}}{1 + (1+sT_d)\cdot \dfrac{\omega_n^2}{s(s+2\xi\omega_n)}}$$

$$s^2 + (2\xi\omega_n + \omega_n^2 T_d)s + \omega_n^2 = 0 \quad \text{---- (1)}$$

Compare eq(1) with $s^2 + 2\xi'\omega_n s + \omega_n^2$

$$2\xi'\omega_n = 2\xi\omega_n + \omega_n^2 T_d$$

$$\xi' = \frac{2\xi\omega_n + \omega_n^2 T_d}{2\omega_n}$$

$$\boxed{\xi' = \xi + \frac{\omega_n T_d}{2}}$$

Damping ratio increased using derivative Control, Mp decreases.

Error function

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)}$$

$$= \frac{1}{1+\frac{\omega_n^2(1+sT_d)}{s(s+2\xi\omega_n)}}$$

$$= \frac{s(s+2\xi\omega_n)}{s(s+2\xi\omega_n)+\omega_n^2(1+sT_d)}$$

$$\frac{E(s)}{R(s)} = \frac{s(s+2\xi\omega_n)}{s^2+(2\xi\omega_n+\omega_n^2T_d)s+\omega_n^2}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2T_d\left(s+\frac{1}{T_d}\right)}{s^2+2\xi\omega_n s+\omega_n^2}$$

$r(t) = t$

$R(s) = \frac{1}{s^2}$

$$E(s) = \frac{(s+2\xi\omega_n)}{s\left(s^2+(2\xi\omega_n+\omega_n^2T_d)s+\omega_n^2\right.}$$

$$e_{ss}.(t) = \lim_{s\to 0} s.E(s)$$

$$= \lim_{s\to 0} \frac{s+2\xi\omega_n}{s^2+(2\xi\omega_n+\omega_n^2T_d)s+\omega_n^2}$$

$$= \frac{2\xi\omega_n}{\omega_n^2}$$

$$= \frac{2\xi}{\omega_n}$$

Comparing the steady state error of 2nd order system with and without derivative control, both values are same. Thus e_{ss} not effected by derivative control action.

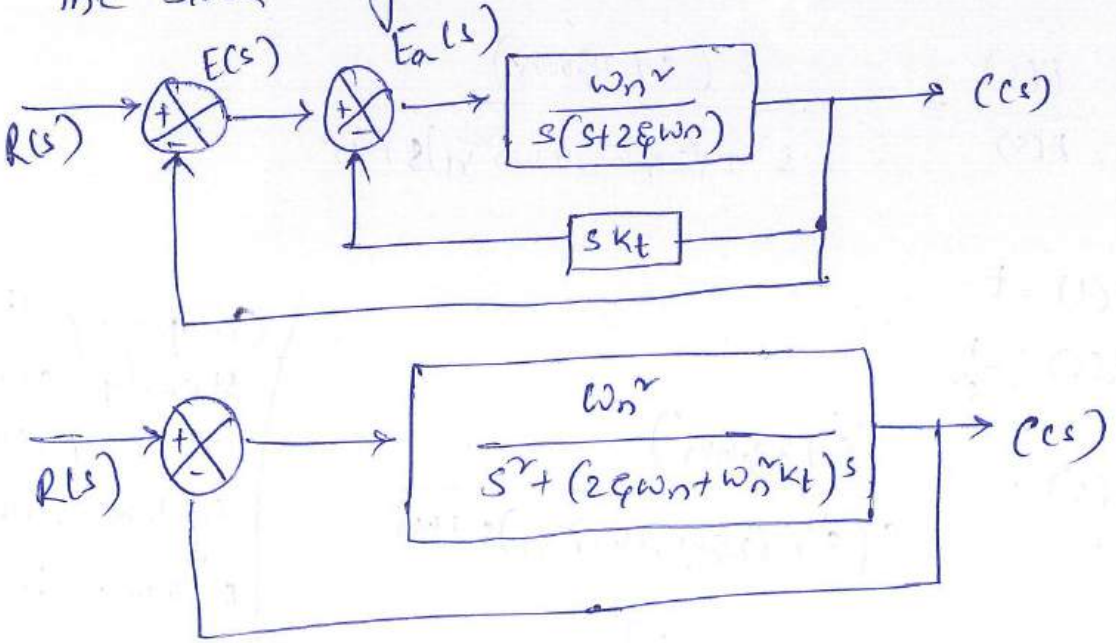
Derivative feedback Control :-

Derivative feedback Control or rate feedback Control is also known as tachometer feedback Control. In this Actuating signal is the difference of proportional error and derivative of output signal.

$$E_a(t) = e(t) - k_t \cdot \frac{d}{dt} c(t)$$

$$E_a(s) = E(s) - k_t \cdot s\, C(s)$$

The block diagram is



$$\frac{C(s)}{R(s)} = \frac{\dfrac{\omega_n^2}{s^2 + (2\xi\omega_n + \omega_n^2 k_t)s}}{1 + \dfrac{\omega_n^2}{s^2 + (2\xi\omega_n + \omega_n^2 k_t)s}}$$

$$= \frac{\omega_n^2}{s^2 + (2\xi\omega_n + \omega_n^2 k_t)s + \omega_n^2}$$

$$s^2 + (2\xi\omega_n + \omega_n^2 k_t)s + \omega_n^2 = 0$$

$$\Rightarrow s^2 + 2\xi' \omega_n s + \omega_n^2 = 0.$$

Standard equation $s^2 + 2\xi\omega_n s + \omega_n^2 = 0$

By Comparing $2\xi'\omega_n = 2\xi\omega_n + \omega_n^2 k_t$

$$\xi' = \xi + \frac{\omega_n k_t}{2}$$

Using derivative feedback Control, damping ratio is increased; M_p will be reduced

$$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)\cdot H(s)}$$

$$= \frac{1}{1 + \frac{\omega_n^2}{s^2 + (2\xi\omega_n + \omega_n^2 k_t)s}}$$

$$= \frac{s^2 + (2\xi\omega_n + \omega_n^2 k_t)s}{s^2 + (2\xi\omega_n + \omega_n^2 k_t)s + \omega_n^2}$$

$r(t) = t$

$R(s) = \frac{1}{s^2}$

$$E(s) = \frac{1}{s^2} \cdot \frac{s^2 + (2\xi\omega_n + \omega_n^2 k_t)s}{s^2 + (2\xi\omega_n + \omega_n^2 k_t)s + \omega_n^2}$$

$$e_{ss}(t) = \lim_{s\to 0} sE(s)$$

$$= \lim_{s\to 0} \cancel{s}\cdot\frac{1}{\cancel{s^2}} \cdot \frac{\cancel{s}(s + 2\xi\omega_n + \omega_n^2 k_t)}{s^2 + (2\xi\omega_n + \omega_n^2 k_t)s + \omega_n^2}$$

$$= \frac{2\xi\omega_n + \omega_n^2 k_t}{\omega_n^2}$$

$$= \frac{2\xi}{\omega_n} + k_t$$

$\Downarrow$

Steady Error increased.

~~Proportional +~~

Integral Controller : (P-I controller)

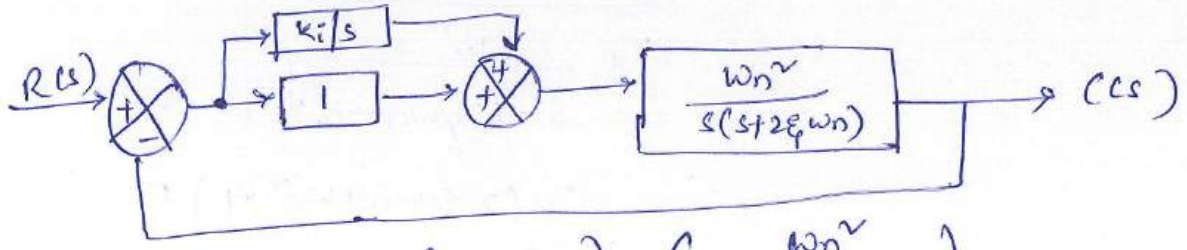
for ~~proportional~~ integral action, the actuating signal consists of proportional error signal added with integral error signal.

$$e_a(t) = e(t) + k_i \int e(t)\,dt$$

$$E_a(s) = E(s) + \frac{k_i}{s} E(s)$$

$$E_a(s) = E(s)\left[1 + \frac{k_i}{s}\right]$$

The block diagram is



$$\frac{C(s)}{R(s)} = \frac{\left(1 + \frac{k_i}{s}\right)\left(\frac{\omega_n^2}{s^2+2\xi\omega_n s}\right)}{1 + \left(1 + \frac{k_i}{s}\right)\left(\frac{\omega_n^2}{s^2+2\xi\omega_n s}\right)}$$

$$= \frac{(s+k_i)\omega_n^2}{s^3 + 2\xi\omega_n s^2 + \omega_n^2 s + k_i\omega_n^2}$$

Characteristic equation is $s^3 + 2\xi\omega_n s^2 + \omega_n^2 s + k_i\omega_n^2 = 0$

The second order system has became as a third order system. due to the inclusion of integral Control.

$$G(s) = \frac{(s+k_i)\omega_n^2}{s^2(s+2\xi\omega_n)} \qquad H(s) = 1$$

$$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)H(s)} = \frac{1}{1 + \frac{(s+k_i)\omega_n^2}{s^2(s+2\xi\omega_n)}}$$

$$= \frac{s^2(s+2\xi\omega_n)}{s^3+2\xi\omega_n s^2+\omega_n^2 s+k_i\omega_n^2}$$

$$\Rightarrow E(s) = \frac{R(s)}{1+G(s)\cdot H(s)} = \frac{1}{s^2}\cdot\frac{s^2(s+2\xi\omega_n)}{s^3+2\xi\omega_n^2 s+k_i\omega_n^2}$$

But $R(s) = \frac{1}{s^2}$ for Ramp input.

$$= \frac{(s+2\xi\omega_n)}{s^3+2\xi\omega_n s^2+\omega_n^2 s+k_i\omega_n^2}$$

steady state error

$$e_{ss} = \lim_{t\to\infty} e(t)$$

$$= \lim_{s\to 0} s\cdot E(s)$$

$$= \lim_{s\to 0} s\cdot\frac{s+2\xi\omega_n}{s^3+2\xi\omega_n s^2+\omega_n^2 s+k_i\omega_n^2}$$

$$= 0.$$

for parabolic i/p.

$$r(t) = t^2/2$$

$$R(s) = 1/s^3$$

$$e_{ss} = \lim_{s\to 0} s\cdot E(s)$$

$$= \lim_{s\to 0} s\cdot\frac{1}{s^3}\cdot\frac{s^2(s+2\xi\omega_n)}{s^3+2\xi\omega_n s^2+\omega_n^2 s+k_i\omega_n^2}$$

$$= \frac{2\xi\omega_n}{k_i\omega_n^2}$$

$$= \frac{2\xi}{k_i\omega_n}.$$

$=$

Proportional-plus-integral-plus-Derivative Control (PID Control)

In PID Control, The actuating signal consists of proportional error signal added with integral and derivative error signal.

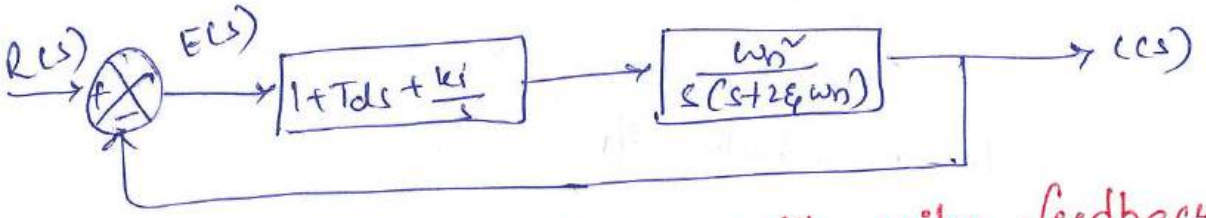
$$e_a(t) = e(t) + T_d \frac{d}{dt} e(t) + k_i \int e(t) \cdot dt$$

Apply Laplace Transform on both sides.

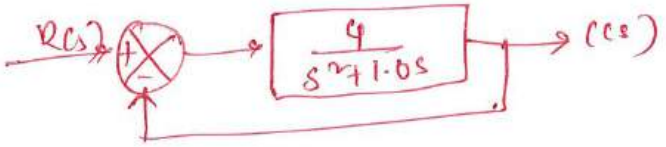
$$E_a(s) = E(s) + T_d s E(s) + k_i \frac{1}{s} E(s)$$

$$E_a(s) = E(s) \left[1 + T_d s + \frac{k_i}{s} \right]$$

The block diagram is



pb: A closed loop control system with unity feedback given below. By using derivative control, is to be made 0.75. Determine the value of T_C. Also determine t_r, t_p, M_p with out and with derive control, when input is unit step.



Given $G(s) = \frac{4}{s^2+s}$ (from given fig).

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\frac{4}{s^2+s}}{1+\frac{4}{s^2+s}}$$

$$\frac{C(s)}{R(s)} = \frac{4}{s^2+s+4}$$

The characteristic eq is $s^2+s+4=0$.

$\omega_n^2 = 4$ $\qquad 2\xi\omega_n = 1$

$\omega_n = 2$ $\qquad \xi = \frac{1}{2\omega_n}$

$= \frac{1}{2\times 2} = 0.25$.

The damping ratio without derivative control

$\xi = 0.25$

$$t_r = \frac{\pi - \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)}{\omega_n\sqrt{1-\xi^2}}$$

$$= \frac{\pi - \tan^{-1}\left(\frac{\sqrt{1-(0.25)^2}}{0.25}\right)}{2\sqrt{1-(0.25)^2}}$$

$= 0.942$ sec.

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

$$= \frac{\pi}{2\sqrt{1-(0.25)^2}}$$

$= 1.622$ sec

$$M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100$$

$$= e^{-\pi\times 0.25/\sqrt{1-(0.25)^2}} \times 100$$

$= 0.44 \times 100$

$\%M_p = 44.43\%$.

→ With Derivative Control.

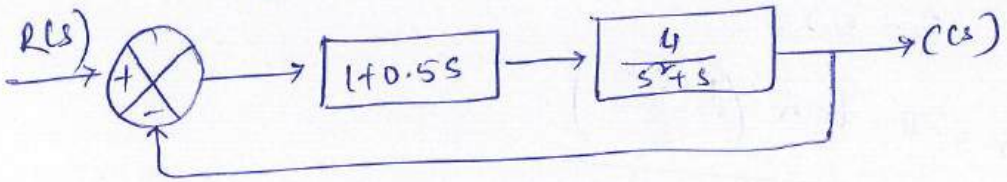
Damping ratio with Derivative control

$$\xi' = \xi + \frac{\omega_n T_d}{2}$$

$$= 0.75 = 0.25 + \frac{2 \times T_d}{2}$$

$$\Rightarrow T_d = 0.5.$$

Block diagram is



$$G(s) = \frac{4(1+0.5s)}{s^2+s}$$

$$= \frac{4+2s}{s^2+s}$$

$$H(s) = 1$$

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$= \frac{\frac{4+2s}{s^2+s}}{1+\frac{4+2s}{s^2+s}\cdot 1}$$

$$= \frac{4+2s}{s^2+s+4+2s} = \frac{4+2s}{s^2+3s+4}$$

$r(t) = 1$

$R(s) = 1/s$

$\Rightarrow C(s) = R(s) \cdot T(s)$

$$= \frac{1}{s} \cdot \frac{4+2s}{s^2+3s+4}$$

$$= \frac{4+2s}{s(s^2+3s+4)}$$

Using partial expansion Method.

$$C(s) = \frac{4+2s}{s(s^2+3s+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+3s+4}$$

$$4+2s = A(s^2+3s+4) + Bs^2+Cs$$

Put $s \Rightarrow 0$

$$4 = 4A$$

$$A = 1$$

Compare s^2 terms

$0 = A + B$

$B = -A$

$B = -1, \quad A = 1$

Compare s terms

$3A + C = 2$

$3 + C = 2$

$C = -1$

$$\therefore C(s) = \frac{1}{s} - \frac{(s+1)}{s^2+3s+4}$$

$$= \frac{1}{s} - \frac{(s+1)}{(s+1.5)^2+1.75}$$

$$= \frac{1}{s} - \frac{(s+1.5)}{(s+1.5)^2+1.75} - \frac{0.5}{(s+1.5)^2+1.75}$$

$$= \frac{1}{s} - \frac{(s+1.5)}{(s+1.5)^2+1.75} + \frac{0.5}{\sqrt{1.75}} \cdot \frac{\sqrt{1.75}}{(s+1.5)^2+1.75}$$

$$c(t) = L^{-1}(C(s)) = L^{-1}\left[\frac{1}{s} - \frac{(s+1.5)}{(s+1.5)^2+(\sqrt{1.75})^2} + \frac{0.378 \times \sqrt{1.75}}{(s+1.5)^2+(\sqrt{1.75})^2}\right]$$

$$= 1 - e^{-1.5t}\cos\sqrt{1.75}\,t + 0.378\, e^{-1.5t}\sin\sqrt{1.75}\,t$$

$$= 1 - e^{-1.5t}\left[\cos 1.32t + 0.378 \sin 1.32t\right].$$

t_r :-

At $t = t_r \Rightarrow c(t) = 1$

$$\therefore 1 - e^{-1.5t_r}\left[\cos 1.32 t_r - 0.378 \sin 1.32 t_r\right] = 1$$

$$\Rightarrow e^{-1.5t_r}\left[\cos 1.32 t_r - 0.378 \sin 1.32 t_r\right] = 0$$

$e^{-1.5t_r} \neq 0 \Rightarrow \cos 1.32 t_r - 0.378 \sin 1.32 t_r = 0$

$$0.378 \sin 1.32 t_r = \cos 1.32 t_r$$

$$\tan 1.32 t_r = \frac{1}{0.378}$$

$$t_r = \tan^{-1}\left(\frac{1}{0.378}\right) / 1.32$$

$$\boxed{t_r = 0.916 \text{ sec}}$$

t_p :-

At $t=t_p$: $\frac{d}{dt}[c(t)]=0$

$$= \frac{d}{dt}\left[1 - e^{-1.5t}(\cos 1.32t - 0.378 \sin 1.32t)\right]_{t=t_p}$$

$$= 0$$

$$e^{-1.5t_p}\left[-(\sin 1.32t_p)(1.32) - 0.378 \times 1.32 \times \cos 1.32t_p\right] - (\cos 1.32t_p - 0.378 \sin 1.32t_p) \times e^{-1.5t_p} \times (-1.5) = 0$$

$$0.753 \sin 1.32t_p = -1.999 \cos 1.32t_p$$

$$\tan 1.32t_p = \frac{-1.999}{0.753}$$

$$\tan 1.32t_p = -2.655$$

$$t_p = \frac{\tan^{-1}(-2.655)}{1.32}$$

$$= \frac{\pi - \tan^{-1}(2.655)}{1.32}$$

$$= 1.463 \text{ sec}$$

$$\boxed{t_p = 1.463 \text{ sec}}$$

Mp :-

$$c(\infty) = 1$$

$$c(t_p) = 1 - e^{-1.5 \times 1.463}\left[\cos(1.32 \times 1.463) - 0.378 \sin(1.32 \times 1.46\right.$$

$$= 1 - e^{-1.5 \times 1.463}[\cos(1.93) - 0.378 \sin(1.93)]$$

$$= 1 - e^{-1.5 \times 1.463}[\cos(110.58) - 0.378 \sin(110.58)]$$

$$= 1.079$$

$$= 1.08$$

$$M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} = \frac{1.08 - 1}{1} = 0.08.$$

$$\boxed{\% M_p = 8\%}$$

Response with P, PI, PD and PID Controllers.

In feedback Control systems a Controller is used to modify error signal and to achieve better Control action. The Controllers will modify the transient response and the steady state error of the system.

Proportional Controller (P-Controller):

The proportional Controller produces an output signal which is proportional to error signal. The Transfer function of proportional Controller is K_p. The term K_p is called the gain of the Controller. Hence the proportional Controller amplifies the error signal and increases the loop gain of the system.

The following aspects of system behaviour are improved by increasing loop gain of the system.

1. Steady state tracking accuracy.
2. Disturbance signal Rejection
3. Relative stability.

Advantage:

Increased in loop gain, it decreases the sensitivity of the system to parameter variations

Disadvantage:

It produces Constant steady state error.

The proportional Controller is a device that produces a Control signal u(t) proportional to the input error signal e(t).

$$u(t) \propto e(t)$$

$$\therefore u(t) = k_p e(t)$$

where k_p = proportion gain (or) Constant.

On taking Laplace Transform on both side

$$U(s) = k_p E(s)$$

$$\therefore \frac{U(s)}{E(s)} = k_p$$

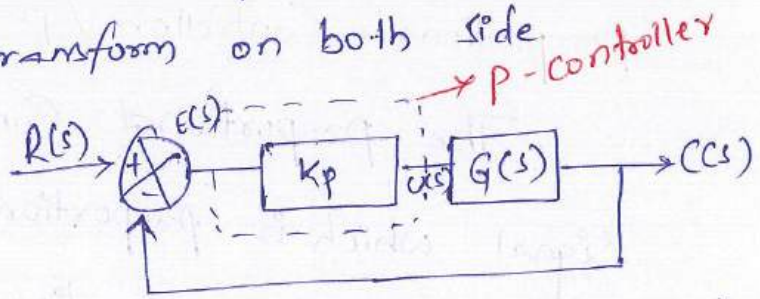


fig: Block diagram of proportional Controller.

The proportional Controller can be realized by an amplifier with adjustable gain, either the Non-inverting op-amp or the inverting op-amp followed by Sign changer will work as a proportional Controller.

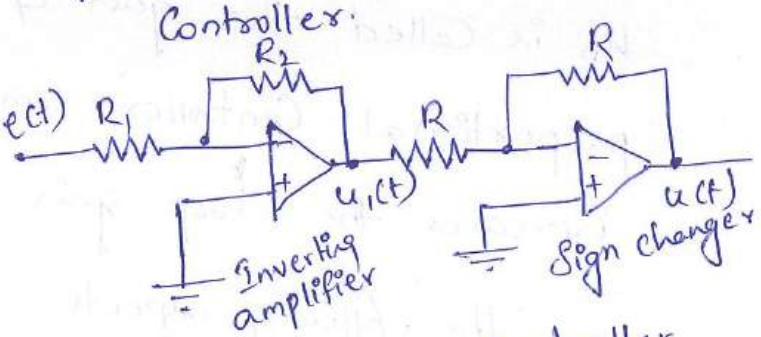


fig: op-amp - P- Controller using inverting amplifier.

Integral Controller (I- Controller):

The Integral Controller is a device that produces a Control signal u(t) which is proportional to integral of the input error signal. e(t).

$$u(t) \propto \int e(t)\, dt.$$

$\therefore u(t) = K_I \int e(t)\,dt$

where K_I is integral gain or Constant.

On taking Laplace transform on both sides we get.

$$U(s) = K_I \frac{E(s)}{s}$$

$$\therefore \frac{U(s)}{E(s)} = \frac{K_I}{s}$$

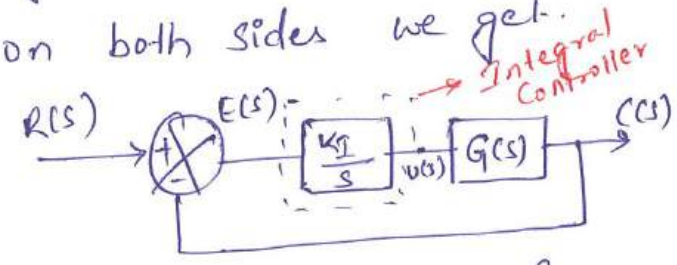
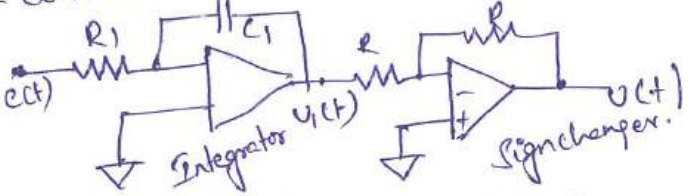


fig: Block diagram of integral Controller.

Advantage:-

The integral Controller removes or reduces the steady state error without the need for manual reset. Hence I-Controller is also called as "Automatic Reset".



Disadvantage:-

May lead to oscillatory response of increasing or decreasing amplitude, so this system may become unstable.

Derivative Controller:-

The Derivative Controller is a device that produces a control signal u(t) which is proportional to the Derivative of the input error signal e(t)

$$u(t) \propto \frac{d}{dt} e(t)$$

$$\text{i.e.} \quad u(t) = K_D \frac{de(t)}{dt}$$

where K_D is Derivative gain (or) Constant.

On taking Laplace transform on both sides

$$U(s) = K_D s E(s).$$

$$\therefore \frac{U(s)}{E(s)} = K_D s$$

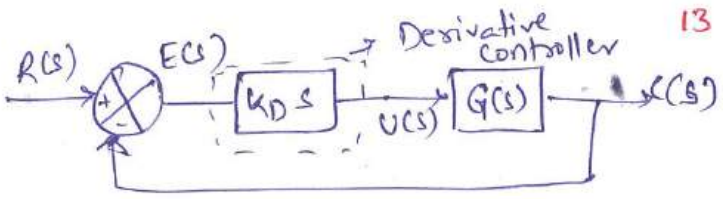


fig: Block diagram of Derivative Controller.

Advantage:-

Increases stability.

Disadvantage:-

Decreases steady state error.

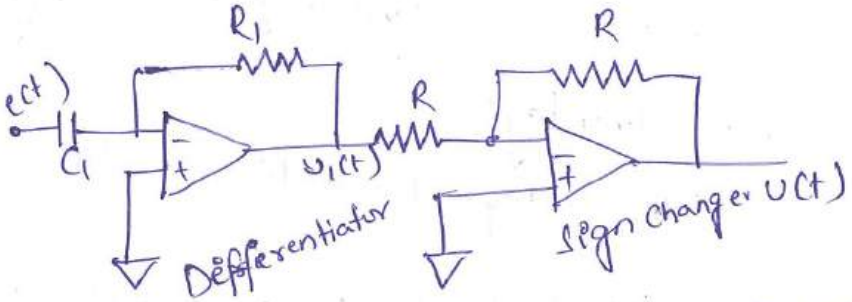


fig: D-Controller using op-amp.

PI Controller (Proportional - Integral Controller):-

The PI Controller produces an output signal consisting of two terms: 1) proportional to the error signal
2) proportional to the integral of error signal.

$$u(t) \propto \left[e(t) + \int e(t)\right]$$

$$u(t) = K_p e(t) + \frac{K_I}{s} \int e(t)$$

where K_p is proportional gain (or) Constant
K_I is Integral gain (or) Constant.

on taking Laplace transform on both sides

$$U(s) = K_p E(s) + \frac{K_I}{s} E(s)$$

$$= E(s)\left[K_p + \frac{K_I}{s}\right]$$

$$\therefore \frac{U(s)}{E(s)} = K_p + \frac{K_I}{s}$$

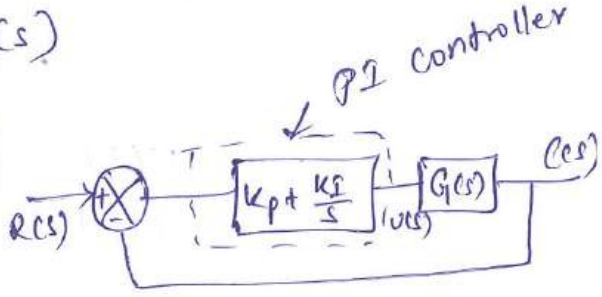


fig: Block diagram of PI Controller.

Advantage:

PI Controller increases the loop gain and makes the system less sensitive to variations of system parameters.

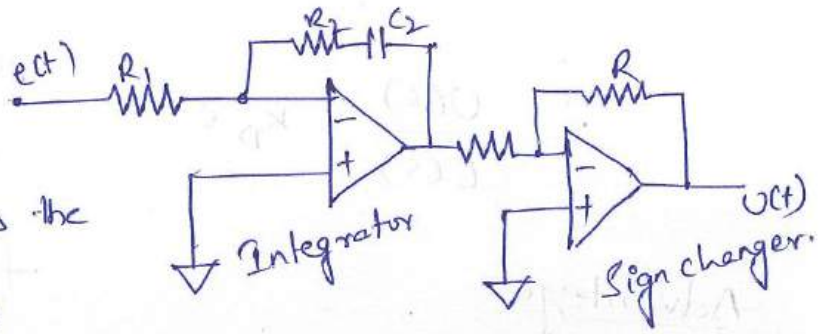


fig: PI Controller using op-amp.

Disadvantage

The integral action reduces the steady state error.

PD Controller: (Proportional - Derivative Controller):-

The proportional plus derivative Controller produces an output signal Consisting of two terms:

1. proportional to the error signal.
2. proportional to the Derivative of error signal.

$$u(t) \propto \left[e(t) + \frac{d}{dt} e(t)\right]$$

$$u(t) = \left[k_p e(t) + k_D \frac{d}{dt} e(t)\right]$$

where k_p is proportional Controller gain (or) Constant

k_D is Derivative Controller gain (or) Constant.

On taking Laplace Transform on both sides.

$$U(s) = k_p E(s) + k_D s \cdot E(s)$$

$$= E(s)\left[k_p + k_D s\right]$$

$$\therefore \frac{U(s)}{E(s)} = \left[k_p + k_D s\right]$$

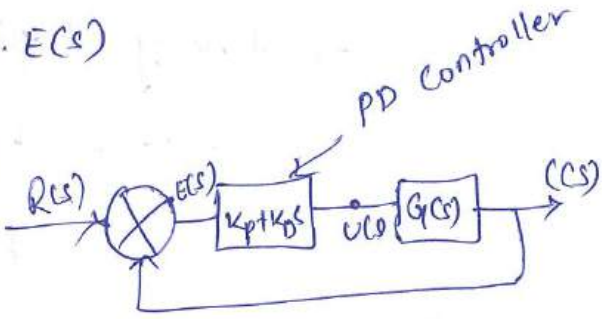


fig: Block diagram of PD Controller.

Advantage:

Increases the Stability of the System

Disadvantage

It amplifies noise signals and may cause a saturation effect in the actuator.

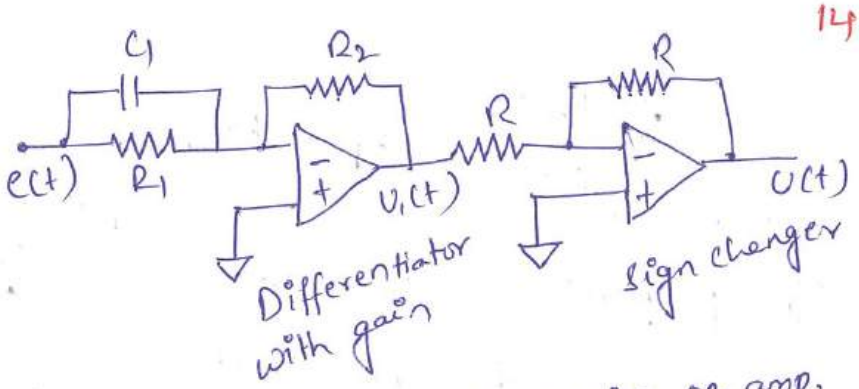


fig: PD Controller using op-amp.

PID Controller (Proportional - Integral - Derivative Controller)

The PID Controller produces an output signal consisting of three terms:

1. proportional to error signal
2. proportional to the Derivative of the error signal
3. " " Integral " " " ".

$$u(t) \propto \left[e(t) + \int e(t) + \frac{d}{dt} e(t) \right]$$

$$u(t) = \left[K_p e(t) + K_I \int e(t) + K_D \frac{d}{dt} e(t) \right]$$

Where K_p is proportional Constant (or) gain

K_I " Integral " " "

K_D " Derivative " " "

On taking Laplace Transform on both sides

$$U(s) = K_p E(s) + \frac{K_I}{s} E(s) + K_D s E(s)$$

$$= E(s) \left[K_p + \frac{K_I}{s} + K_D s \right]$$

$$\therefore \frac{U(s)}{E(s)} = K_p + \frac{K_I}{s} + K_D s$$

Adv & Disadv

The proportional Controller stabilizes the gain but produces a steady state error. The integral Controller reduces or eliminates the steady state error. The derivative Controller reduces the rate of change of error.

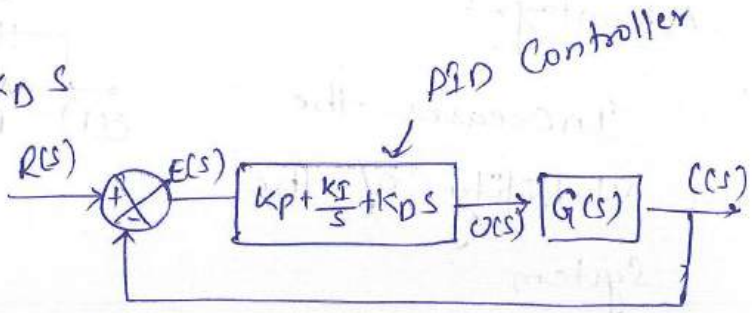


fig: Block diagram of PID Controller.

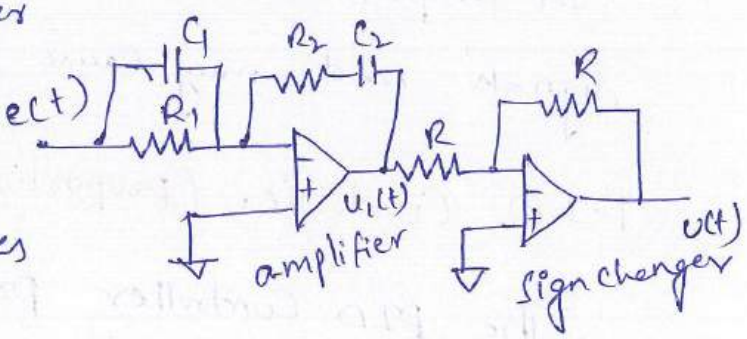


fig: PID Controller using op-amp.

Effect of PI Controller :

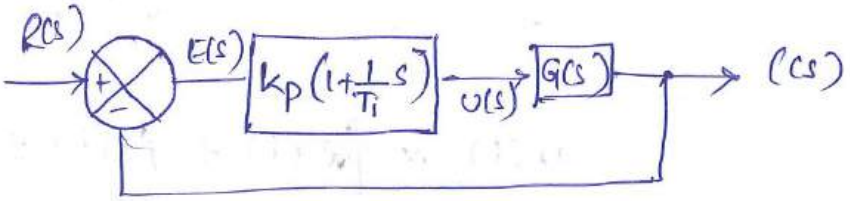
PI Controller Transfer function is $G_C(s)$

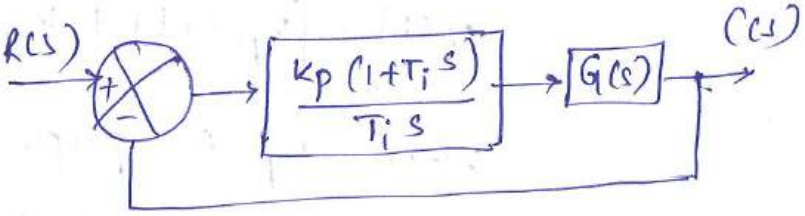
$$G_C(s) = K_p + K_p \frac{1}{T_i s}$$

$$\left[\because \frac{K_p}{T_i} = K_I\right]$$

$$G_C(s) = K_p \left(1 + \frac{1}{T_i s}\right)$$

$$= \frac{K_p (1 + T_i s)}{T_i s}$$





Let $G(s)$ be a second order system Transfer function.

Open loop Transfer function $G(s) = \frac{\omega_n^2}{s(s+2\xi\omega_n)}$

Total open loop TF $\rightarrow$ $G_C(s)\, G(s)\, H(s)$

$$= K_p \left(\frac{1+(T_i s)}{T_i s}\right) \times \frac{\omega_n^2}{s(s+2\xi\omega_n)} \cdot 1$$

$$= \frac{K_p \omega_n^2 (1+T_i s)}{s^2 T_i (s+2\xi\omega_n)}$$

Closed loop Transfer function is given by

$$\frac{C(s)}{R(s)} = \frac{G_c(s) G(s)}{1+G_c(s) G(s)}$$

$$= \frac{\dfrac{K_p \omega_n^2 (1+T_i s)}{s^2 T_i (s+2\xi\omega_n)}}{1+\dfrac{K_p \omega_n^2 (1+T_i s)}{s^2 T_i (s+2\xi\omega_n)}}$$

$$= \frac{K_p \omega_n^2 (1+T_i s)}{s^2 T_i (s+2\xi\omega_n) + K_p \omega_n^2 (1+T_i s)}$$

$$= \frac{K_p \omega_n^2 (1+T_i s)}{s^3 T_i + 2\xi\omega_n T_i s^2 + K_p \omega_n^2 T_i s + K_p \omega_n^2}$$

$$= \frac{\left(\dfrac{K_p}{T_i}\right) \omega_n^2 (1+T_i s)}{s^3 + 2\xi\omega_n s^2 + K_p \omega_n^2 s + \dfrac{K_p}{T_i} \omega_n^2} \qquad \left[\because K_i = \frac{K_p}{T_i}\right]$$

$$= \frac{K_i \omega_n^2 (1+T_i s)}{s^3 + 2\xi\omega_n s^2 + K_p \omega_n^2 s + K_i \omega_n^2}$$

Effect of PD Controller:

The Transfer function of PD Controller $G_c(s) = K_p + K_D s$

$[\because K_D = K_p \cdot T_d]$

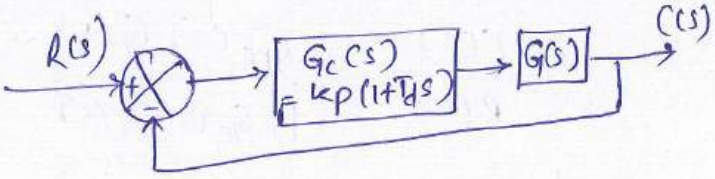
$$G_c(s) = K_p + K_p T_d s$$

$$= K_p (1 + T_d s)$$

where K_p is proportional gain

T_d is derivative time.

Open Loop Transfer function



$$G(s) = \frac{\omega_n^2}{s(s+2\xi\omega_n)}$$

Loop Transfer function $G_c(s)\, G(s)\, H(s)$

$$= K_p (1 + T_d s) \cdot \frac{\omega_n^2}{s(s+2\xi\omega_n)} \cdot 1$$

$$= \frac{K_p \omega_n^2 (1 + T_d s)}{s(s+2\xi\omega_n)}$$

Closed Loop Transfer function

$$\frac{C(s)}{R(s)} = \frac{G_c(s)\, G(s)}{1 + G(s)\, G_c(s)} = \frac{\dfrac{K_p (1+T_d s)\omega_n^2}{s(s+2\xi\omega_n)} \cdot \dfrac{\omega_n^2}{s(s+2\xi\omega_n)}}{1 + \dfrac{K_p (1+T_d s)\omega_n^2}{s(s+2\xi\omega_n)}}$$

$$= \frac{K_p (1+T_d s)\omega_n^2}{s(s+2\xi\omega_n) + K_p (1+T_d s)\omega_n^2}$$

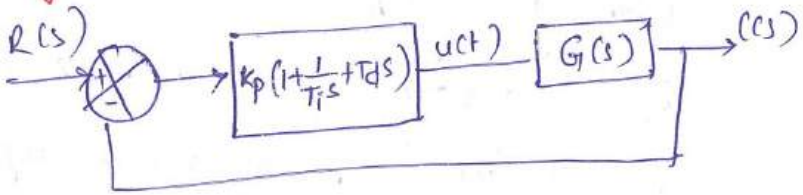
$$= \frac{K_p \omega_n^2 (1+T_d s)}{s^2 + 2\xi\omega_n s + K_p\omega_n^2 + K_p\omega_n^2 T_d s}$$

$$= \frac{K_p \omega_n^2 (1+T_d s)}{s^2 + (2\xi\omega_n + K_p \omega_n^2 T_d) s + K_p \omega_n^2}$$

$$= \frac{\omega_n^2 (K_p + K_d s)}{s^2 + (2\xi\omega_n + K_d \omega_n^2) s + K_p \omega_n^2}$$

Effect of proportional · Integral Derivative (PID) Controller

$$G(s) = \frac{\omega_n^2}{s(s+2\xi\omega_n)}$$



Open loop T.f = $G(s)\, G_c(s)\, H(s)$

$$= \frac{\omega_n^2}{s(s+2\xi\omega_n)} \cdot K_p \left(1 + \frac{1}{T_i s} + T_d s\right) \cdot 1$$

Closed loop T.f = $\dfrac{G(s)\, G_c(s)}{1 + G(s)\, G_c(s)}$

$$= \frac{\omega_n^2 \left(K_p T_i s + K_p T_i T_d s^2 + K_p\right)}{T_i s^2 (s + 2\xi\omega_n)}$$

$$\frac{}{1 + \dfrac{\omega_n^2 \left(K_p T_i s + K_p T_i T_d s^2 + K_p\right)}{T_i s^2 (s+2\xi\omega_n)}}$$

$$= \frac{\omega_n^2 \left(K_p (T_i s + T_i T_d s^2 + 1)\right)}{T_i s^2 (s+2\xi\omega_n) + \omega_n^2 \left(K_p (T_i s + T_i T_d s^2 + 1)\right)}$$

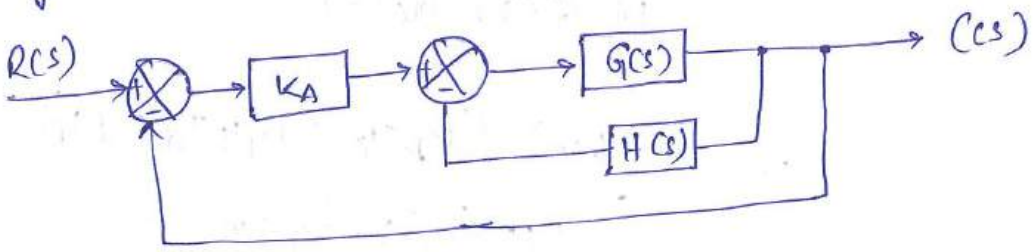
$$= \frac{K_p \left(1 + T_i s + T_D T_i s^2\right) \omega_n^2}{T_i s^3 + 2\xi\omega_n s^2 T_i + K_p \omega_n^2 + K_p \omega_n^2 T_i s + K_p \omega_n^2 T_D T_i s^2}$$

$$= \frac{K_I \left(1 + T_i s + T_d T_i s^2\right) \omega_n^2}{s^3 + (2\xi\omega_n + K_p \omega_n^2 T_d) s^2 + K_p \omega_n^2 s + K_I \omega_n^2}$$

Problem 1 :-

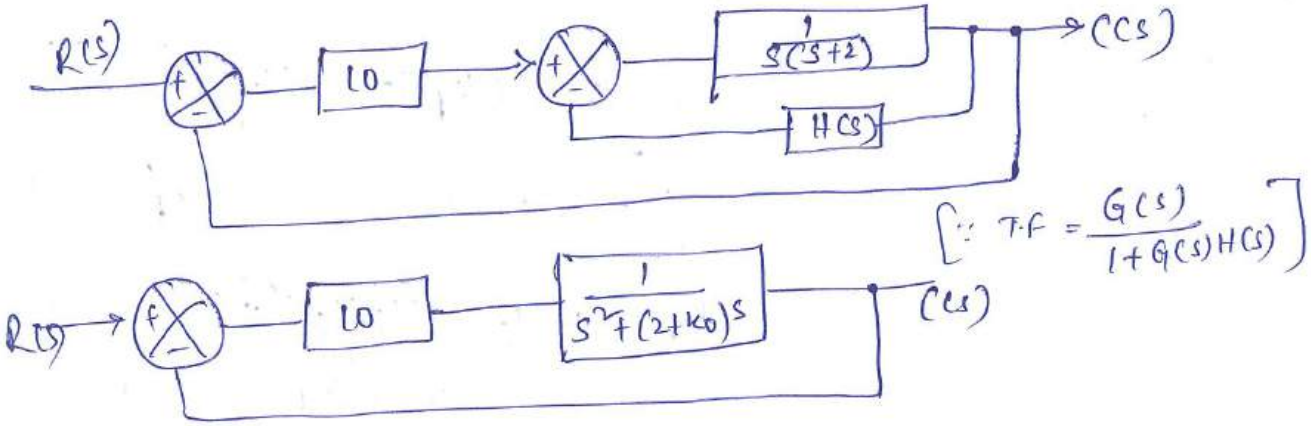
A unity feedback Control System has an Amplifier with gain $K_A = 10$ and gain ratio $G(s) = \frac{1}{s(s+2)}$ in feed farward path. A derivative feedback, $H(s) = s K_0$ is introduced as a minor loop around $G(s)$, Determine the the derivative feedback Constant, K_0 so that the System damping factor is 0.6.

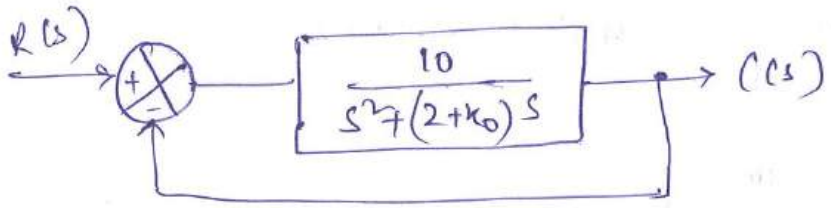
Sol: The given System Can be represented by the blocke-diagram as shown in fig.



Here $K_A = 10$

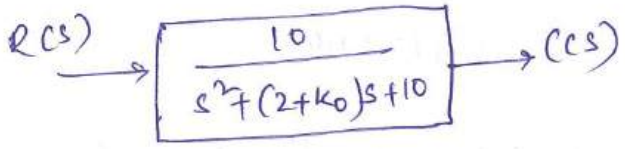
$G(s) = \frac{1}{s(s+2)}$, $H(s) = s K_0$





$$\frac{C(s)}{R(s)} = \frac{\frac{10}{s^2+(2+k_0)s}}{1+\frac{10}{s^2+(2+k_0)s}\cdot 1}$$

$$\frac{C(s)}{R(s)} = \frac{10}{s^2+(2+k_0)s+10} \quad \text{——— (1)}$$



Compare eq (1) with standard form of second order Transfer function i.e., $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$ ——— (2)

from (1) & (2)

$\omega_n^2 = 10$

$\omega_n = 3.162$ rad/sec

$2+k_0 = 2\xi\omega_n$

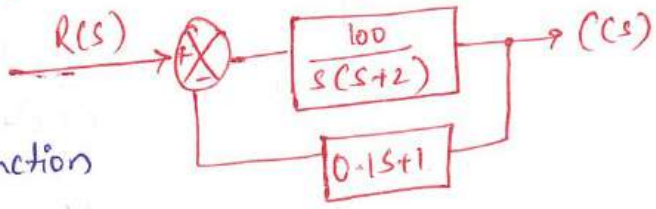
$k_0 = 2\xi\omega_n - 2$

$= (2\times 0.6\times 3.162) - 2$

$\boxed{k_0 = 1.794}$

Problem 2:

A positional Control system with velocity feedback is shown in fig. What is the response of the system for the unit step input.



The closed loop Transfer function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$G(s) = \frac{100}{s(s+2)}, \quad H(s) = 0.1s + 1$$

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{100}{s(s+2)}}{1 + \frac{100}{s(s+2)}(0.1s+1)}$$

$$= \frac{\frac{100}{s(s+2)}}{\frac{s(s+2) + 100(0.1s+1)}{s(s+2)}}$$

$$= \frac{100}{s^2 + 2s + 10s + 100} \Rightarrow \frac{100}{s^2 + 12s + 100}$$

The roots of the characteristic equation is

$$s_1, s_2 = \frac{-12 \pm \sqrt{144 - 400}}{2} = -6 \pm j8.$$

$$s_1 = -6 + j8$$

$$s_2 = -6 - j8.$$

The Response in s domain $C(s) = R(s) \cdot \frac{100}{s^2 + 12s + 100}$

input is unit step $R(s) = 1/s$

$$C(s) = \frac{1}{s} \cdot \frac{100}{s^2 + 12s + 100}$$

By partial fraction expansion

$$C(s) = \frac{100}{s(s^2 + 12s + 100)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 12s + 100}$$

$$100 = A(s^2 + 12s + 100) + Bs^2 + Cs$$

Compare $s \Rightarrow 0 \Rightarrow 100 = A(100)$

$$A = 1$$

Compare s^2 terms $\quad A+B=0$

$$B=-A$$
$$B=-1$$

Compare s terms $\quad 12A+C=0$

$$12+C=0$$
$$C=-12$$

$\therefore \quad A=1, \quad B=-1, \quad C=-12$

$$C(s) = \frac{1}{s} + \frac{-1(s)-12}{s^2+12s+100}$$

$$= \frac{1}{s} - \frac{s+12}{s^2+12s+100}$$

$$= \frac{1}{s} - \frac{s+6+6}{(s+6)^2+s^2+12s+36+64}$$

$$= \frac{1}{s} - \frac{s+6+6}{(s+6)^2+64}$$

$$= \frac{1}{s} - \frac{s+6}{(s+6)^2+64} - \frac{6}{(s+6)^2+64}$$

$$= \frac{1}{s} - \frac{s+6}{(s+6)^2+64} - \frac{6}{8}\cdot\frac{8}{(s+6)^2+64}$$

By taking inverse Laplace Transform of $C(s)$

$$C(t) = L^{-1}\left[\frac{1}{s} - \frac{s+6}{(s+6)^2+8^2} - \frac{6}{8}\cdot\frac{8}{(s+6)^2+8^2}\right]$$

$$= 1 - e^{-6t}\cos 8t - \frac{6}{8}\, e^{-6t}\sin 8t$$

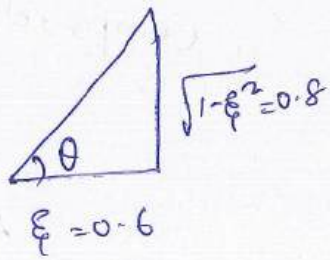
$$= 1 - e^{-6t}\left[\frac{6}{8}\sin 8t + \cos 8t\right].$$

$$\boxed{C(t) = 1 - e^{-6t}\left(\frac{6}{8}\sin 8t + \cos 8t\right)}$$ Time Response

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{100}{s^2 + 12s + 100}$$

$\omega_n^2 = 100$ | $2\xi\omega_n = 12$

$\omega_n = 10$ | $\xi = \frac{12}{2\omega_n} = 0.6$



$\sin\theta = 0.8$, $\cos\theta = 0.6$, $\tan\theta = \frac{0.8}{0.6}$

$\theta = \tan^{-1}\frac{0.8}{0.6} = 53^0$.

$= 0.925$ rad.

The Response in time domain

$$C(t) = 1 - e^{-6t}\left[\frac{6}{8}\sin 8t + \cos 8t\right]$$

$$= 1 - e^{-6t}\frac{10}{8}\left[\frac{6}{10}\sin 8t + \frac{8}{10}\cos 8t\right]$$

$$= 1 - \frac{10}{8}e^{-6t}\left[\sin 8t \times 0.6 + \cos 8t \times 0.8\right]$$

$$= 1 - 1.25\, e^{-6t}\left[\sin 8t\cos\theta + \cos 8t \sin\theta\right]$$

$$= 1 - 1.25\, e^{-6t}\left[\sin(8t+\theta)\right]$$

$[\because \theta = 0.925 \text{ rad}]$.

$$= 1 - 1.25\, e^{-6t}\sin(8t + 0.925)$$

Compensators:

Compensator is an extra device which is used to alter the behaviour of the system so as to achieve given specifications is called compensator.

Types of Compensators:

An external device, compensator can be introduced in a system anywhere as per the convenience and the requirement. The various types of compensation are:

1. Series Compensation
2. parallel Compensation.
3. series-parallel Compensation.

1. Series Compensation:

The compensator is a physical device, whose transfer function is denoted as $G_c(s)$. If the Compensator is placed in series with the forward path transfer function of the plant is called "series Compensation".

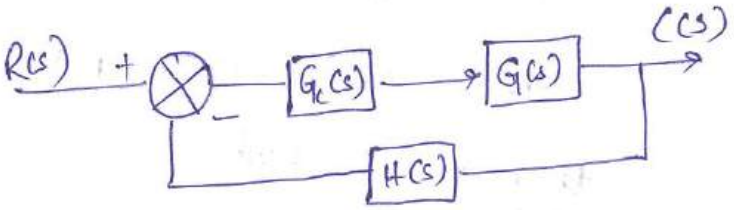


fig: Series Compensation.

This scheme is also called as "Cascade Compensation". The flow of signal in such a series scheme is from lower energy level towards higher energy level. This requires additional amplifiers to increase the gain and also to provide

isolation. The number of Components required in series scheme is more than in parallel scheme.

Parallel Compensation:

A Compensator is required and introduced in feedback path to provide an additional internal feedback loop. Such Compensation is called "feedback Compensation" or "parallel Compensation".

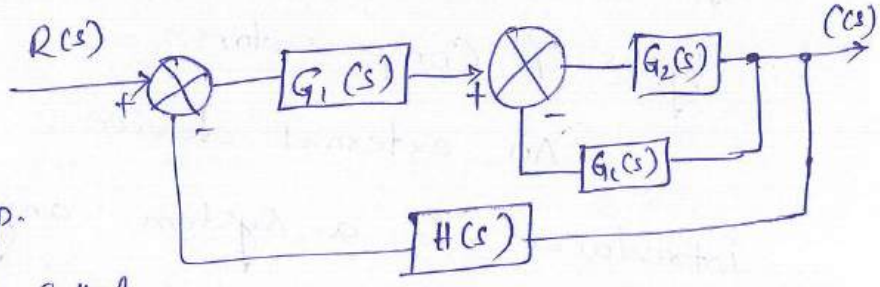


fig: parallel Compensation

The energy transfer in such parallel scheme from higher energy level towards lower energy level. Hence in such parallel scheme the additional amplifiers are not required. Thus the number of Components required less than required in the series scheme.

Series-parallel Compensation:

It is necessary to provide both types of Compensations in some cases called "series-parallel Compensation"

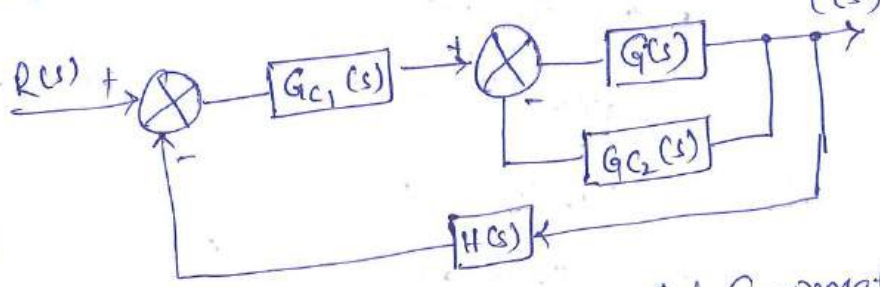


fig: series-parallel Compensati

The selection of the proper Compensation scheme depends on the nature of the signals available in the system.

Compensating Networks:

The Compensator is a physical device, it may be an electrical network, Mechanical unit, pneumatic, hydraulic or combinations of various types of devices.

The commonly used electrical Compensating Networks are:

1. Lead Network or lead Compensator
2. Lag Network or Lag compensator
3. Lag-Lead Network or Lag-lead Compensator.

When a sinusoidal input is applied to a network and it produces a sinusoidal steady state output having a phase lead with respect to input then the network is called "lead Network". If the steady state output has phase Lag then the network is called "Lag Network". In the Lag-Lead Network both phase Lag and lead occur but in the differential frequency regions.

The phase Lag occurs in the low frequency region. while the phase lead occurs in the high frequency region.

Lag Compensator :-

A Compensator having the characteristics of a lag Network is called a lag Compensator. If a sinusoidal signal is applied to lag network, then in steady state the output will have a phase lag with respect to input.

Lag Compensation results in a large improvement in steady state performance but results in slower response due to reduced bandwidth.

S-plane Representation of Lag Compensator

The Lag Compensator is an electrical network which produces sinusoidal output having phase lag when sinusoidal input is applied.

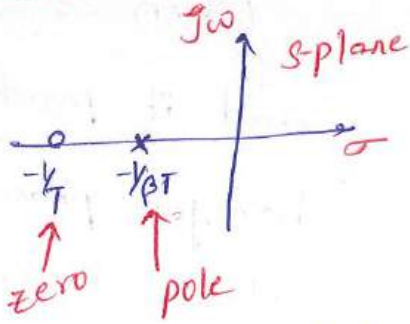
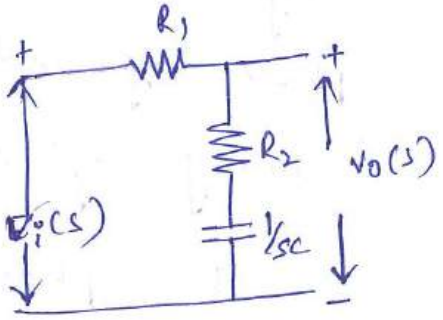


fig: pole zero plot of Lag Compensator

$$G_c(s) = \frac{s+z}{s+p}$$

(General T.F for any Compensator)



Consider $p = \frac{1}{\beta z} = \frac{z}{\beta}$ [∵ Consider $z = 1/z$]

$$\beta = \frac{z}{p}$$

$$G_c(s) = \frac{s+z}{s+\frac{z}{\beta}}$$

$$= \frac{s+1/z}{s+\frac{1}{\beta z}} \quad \text{---(1)}$$

[∵ Consider $\phi = \tan^{-1}\left(\frac{1}{z}\right) - \tan^{-1}\left(\frac{1}{\beta z}\right)$ = -ve]

$\angle G_c(s)$ is '-ve' because we are considering Lag Compensator (phase is lag indicated as '-ve').

so, $\frac{p}{z} < 1 \Rightarrow \frac{z}{p} > 1$

$\Rightarrow \beta > 1 \quad [\because \beta = \frac{z}{p}]$

"To get Bode plot we need to find Magnitude and phase at Maximum Lag Condition

Let $s = j\omega$

$\frac{d\phi}{d\omega}\Big|_{\omega=\omega_m} = 0$

from (1) $G_c(j\omega) = \frac{j\omega + \frac{1}{z}}{j\omega + \frac{1}{\beta z}}$

$= \frac{\beta(1 + j\omega z)}{(1 + j\omega\beta z)}$ ——— (2) $\quad [\because T_1 = z, \; T_2 = \beta z]$

Maximum Lag condition is $\omega_m = \sqrt{\omega_{c_1} \cdot \omega_{c_2}}$

$\frac{d\phi}{d\omega}\Big|_{\omega=\omega_m} = 0$

$= \sqrt{\frac{1}{z} \cdot \frac{1}{\beta z}}$

$= \frac{1}{z\sqrt{\beta}}$

To get ϕ

from (2) $\phi = \tan^{-1}(\omega z) - \tan^{-1}(\omega\beta z)$

$\tan\phi = \frac{\omega z - \omega\beta z}{1 + \omega^2 z^2 \beta}\Big|_{\omega=\omega_m = \frac{1}{z\sqrt{\beta}}}$ $\quad [\because \tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}]$

$= \frac{\frac{1}{z\sqrt{\beta}} z - \frac{1}{z\sqrt{\beta}} \beta z}{1 + \frac{1}{z^2\beta} z^2 \beta}$

$= \frac{\frac{1}{\sqrt{\beta}} - \sqrt{\beta}}{1 + 1} \Rightarrow \frac{1 - \sqrt{\beta}\sqrt{\beta}}{\frac{\sqrt{\beta}}{2}}$

$\Rightarrow \frac{1 - \sqrt{\beta}\sqrt{\beta}}{2\sqrt{\beta}} \Rightarrow \frac{1 - \beta}{2\sqrt{\beta}}$

$$\phi_m = \tan^{-1}\left(\frac{1-\beta}{2\sqrt{\beta}}\right)$$

$$\left[\because \sin\phi = \frac{1-\beta}{1+\beta} \quad \beta = \frac{1-\sin\phi}{1+\sin\phi}\right]$$

To get Maximum magnitude.

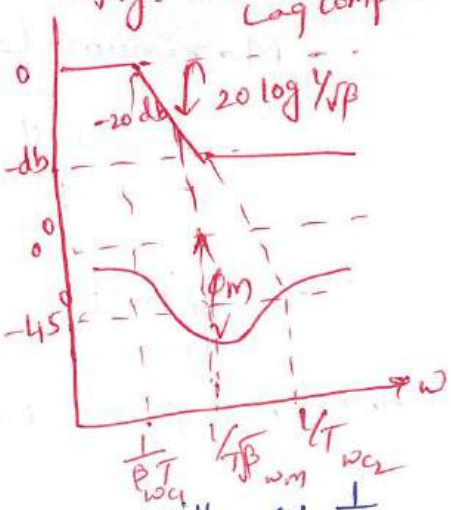
from (2) $$|G_c(j\omega)| = \frac{\sqrt{1+\omega^2\tau^2}}{\sqrt{1+\omega^2\beta^2\tau^2}}\Bigg|_{\omega=\omega_m=\frac{1}{\tau\sqrt{\beta}}}$$

$$= \frac{\sqrt{1+\frac{1}{\tau^2\beta}\tau^2}}{\sqrt{1+\frac{1}{\tau^2\beta}\beta^2\tau^2}}$$

$$= \frac{\sqrt{1+\frac{1}{\beta}}}{\sqrt{1+\beta}} \Rightarrow \frac{\sqrt{\beta+1}}{\sqrt{\beta}\sqrt{1+\beta}} \Rightarrow \frac{1}{\sqrt{\beta}}$$

$$|G_c(j\omega)|_{\omega=\omega_m} = \frac{1}{\sqrt{\beta}}$$

fig: Bode plot of Lag compensator



Compare with the Electrical network

$$T.F = \frac{V_o}{V_i} = \frac{R_2+\frac{1}{sC}}{R_1+R_2+\frac{1}{sC}}$$

$$= \frac{1+R_2sC}{s(R_1+R_2)C+1}$$

$R_2 \to \tau$, $\frac{R_2}{R} \to \beta$

$$\tau = R_2C$$

$$\beta = \frac{R+R_2}{R_2}$$

$$=$$

Compare with $\frac{s+\frac{1}{\tau}}{s+\frac{1}{\beta\tau}}$

(or)

$$\frac{1+j\omega\tau}{1+j\omega\beta\tau}$$

Effects of Lag Compensator:

1. Stability increases
2. Bandwidth decreases
3. Rise time increases
4. Transient response become poor
5. Steady state Response increase.

Problem:

A unity feedback system has an open loop transfer function $G(s) = \frac{K}{s(1+2s)}$. Design a suitable Lag compensator so that phase margin is 40° and the steady state error for ramp input is less than or equal to 0.2.

Sol: proceducer for the design of Lag Compensator using Bodeplot.

Step1: Choose the value of K in uncompensated system to meet steady state error requirement.

Given that $e_{ss} \leq 0.2$ for ramp input

Let $e_{ss} = 0.2$

we know that $e_{ss} = \frac{1}{K_v}$ for ramp input

$\therefore$ velocity error Constant $K_v = \frac{1}{e_{ss}}$

$= \frac{1}{0.2}$

$= 5$

By the defination of velocity error Constant

$K_v = \lim_{s \to 0} s\, G(s)\, H(s)$ $[\because H(s) = 1]$

$\therefore K_v = \lim_{s \to 0} s \cdot G(s)$

$$K_v = \lim_{s\to 0} s \cdot \frac{K}{s(1+2s)} = K$$

$$\boxed{K_v = K = 5}$$

Step2: Sketch the bode plot for uncompensated system.

put $s = j\omega$

$$G(s) = \frac{5}{s(1+2s)}$$

$$G(j\omega) = \frac{5}{j\omega(1+2j\omega)}$$

Magnitude plot

The Corner frequency is $\omega_c = \frac{1}{2} = 0.5$ rad/sec

Term	Cornerfrequency rad/sec	slope db/dec	Change in slope db/dec
$\frac{5}{j\omega}$	–	-20	–
$\frac{1}{1+j2\omega}$	$\omega_c = 0.5$	-20	-20 -20 = -40

Choose a low frequency ω_l such that $\omega_l < \omega_c$ and

" " high " ω_h " " $\omega_h > \omega_c$

Let $\omega_l = 0.1$ rad/sec & $\omega_h = 10$ rad/sec.

At $\omega = \omega_l \Rightarrow A|_{\omega=\omega_l} = 20 \log \left|\frac{5}{j\omega}\right|$

$= 20 \log \frac{5}{0.1}$

$= 34$ db

At $\omega = \omega_c \Rightarrow A|_{\omega=\omega_c} = 20 \log \left|\frac{5}{j\omega}\right|$

$= 20 \log \frac{5}{0.5}$

$= 20$db

At $\omega=\omega_h$, $A\big|_{\omega=\omega_h} = [\text{slope from } \omega_c \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_c}] + A\big|_{\omega=\omega_c}$

$= -40 \times \log\frac{10}{0.5} + 20$

$= -32\,db.$

Phase plot:

The phase angle of $G(j\omega)$ as a function of ω is given by $\phi = \angle G(j\omega) = -90^\circ - \tan^{-1}(2\omega)$

ω rad/sec	0.1	0.5	1.0	5	10
ϕ degrees	-101	-135	-153	-174	-177.

Step3: Determine the phase margin of the uncompensated system from the bode plot. If the phase margin does not satisfy the requirement then lag compensation is required.

Let ϕ_{gc} = phase of $G(j\omega)$ at gain cross over frequency (ω_{gc}).

γ = phase margin of uncompensated system.

from the bode plot of uncompensated system we get

$\phi_{gc} = -162^\circ$

$\gamma = 180^\circ + \phi_{gc}$

$= 180^\circ - 162^\circ$

$= 18^\circ$

The system requires a phase margin of 40°, but the available phase margin is 18° and so Lag Compensation should be improve the phase margin.

Step 4: Choose a suitable value for the phase margin of Compensated System.

The desired phase margin $\gamma_d = 40^\circ$

∴ phase margin of Compensated System $\gamma_n = \gamma_d + \epsilon$

Where ϵ is Additional phase lag to Compensate for shift in gain cross over frequency.

Choose initial value of $\epsilon = 5^\circ$. (Given in procedure)

$$\therefore \gamma_n = \gamma_d + \epsilon$$
$$= 40^\circ + 5^\circ$$
$$= 45^\circ.$$

Step 5: Determine the new gain Cross over frequency, ω_{gcn}

The new ω_{gcn} is the frequency Corresponding to a phase margin of γ_n on the bode plot of uncompensated system.

Let ϕ_{gcn} = phase of $G(j\omega)$ at new gain Cross over frequency, ω_{gcn}

Now $\gamma_n = 180^\circ + \phi_{gcn}$

$$\phi_{gcn} = \gamma_n - 180^\circ$$

The new gain Cross over frequency ω_{gcn} is given by the frequency at which the phase of $G(j\omega)$ is ϕ_{gcn}.

Let ω_{gcn} = New gain Crossover frequency and

ϕ_{gcn} = phase of $G(j\omega)$ at ω_{gcn}.

Now $\gamma_n = 180^\circ + \phi_{gcn}$

$\therefore \phi_{gcn} = \gamma_n - 180^\circ$

$= 45^\circ - 180^\circ$

$= -135^\circ$.

From the bodeplot we found that the frequency corresponding to a phase of -135° is 0.5 rad/sec.

$\therefore$ New gain Crossover frequency $\omega_{gcn} = 0.5$ rad/sec.

Step 6:

Determine the parameter, β of the Compensator. The Value of β is given by the magnitude of $G(j\omega)$ at new gain Crossover frequency, ω_{gcn}, Find the db gain (A_{gcn}) at new gain Crossover frequency, ω_{gcn}.

Now, $A_{gcn} = 20 \log_{10} \beta$

$\log_{10} \beta = A_{gcn}/20$

$\beta = 10^{A_{gcn}/20}$.

From the bode plot we found that the db magnitude at ω_{gcn} is 20db.

$|G(j\omega)|$ in db at $(\omega = \omega_{gcn})$

$A|_{\omega_{gcn}} = 20$ db

$A_{gcn} = 20 \log \beta \Rightarrow \beta = 10^{20/20} = 10$.

$\beta = 10$.

Step 7: Determine the transfer function of Lag Compensator

place the zero of the Compensator arbitarly at 1/10th of the new gain Cross over frequency, ω_{gcn}.

$\therefore$ Zero of the Lag Compensator $Z_c = \frac{1}{T} = \frac{\omega_{gcn}}{10}$

Now $T = \frac{10}{\omega_{gcn}}$

pole of the Lag Compensator $P_c = \frac{1}{\beta T}$

Transfer function of the Lag Compensator $G_c(s) = \frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}$

$= \beta\left(\frac{1+sT}{1+s\beta T}\right)$.

Zero of the Lag Compensator $Z_c = \frac{1}{T} = \frac{\omega_{gcn}}{10}$

$T = \frac{10}{\omega_{gcn}} = \frac{10}{0.5} = 20$.

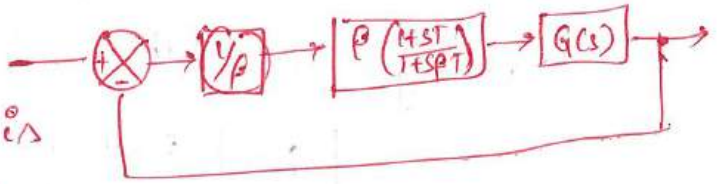
pole of the Lag Compensator $P_c = \frac{1}{\beta T} = \frac{1}{10\times 20} = \frac{1}{200}$

$= 0.005$.

Transfer function of the Lag Compensator $G_c(s) = \beta\left(\frac{1+sT}{1+s\beta T}\right)$

$= 10\left(\frac{1+20s}{1+200s}\right)$

Step 8: Determine the open loop transfer function of Compensated System. The Lag Compensator is Connected in series with plant as shown in fig.



When the Lag Compensator is inserted in series with plant, the open loop gain of the system is amplified by the factor β ($\because \beta > 1$).

If the gain produced is not required then attenuator with gain $\frac{1}{\beta}$ can be introduced in series with the lag Compensator to nullify the gain produced by lag Compensator.

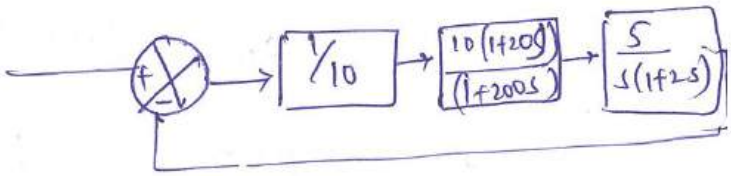
The openloop transfer function of the Compensated system

$$G_0(s) = \frac{1}{\beta} G_c(s) \cdot G(s)$$

$$= \frac{1}{\beta} \cdot \beta \left(\frac{1+sT}{1+s\beta T}\right) \cdot G(s)$$

$$G_0(s) = \left(\frac{1+sT}{1+s\beta T}\right) G(s)$$

openloop Transfer function of Compensated system



$$G_0(s) = \frac{1}{10} \times \frac{10(1+20s)}{(1+200s)} \times \frac{5}{s(1+2s)}$$

$$= \frac{5(1+20s)}{s(1+200s)(1+2s)}$$

Step 9: Determine the actual phase margin of Compensated System. Calculate the actual phase angle of the Compensated System using the Compensated transfer function at new gain Crossover frequency ω_{gen}.

Let ϕ_{gen} = phase of $G_0(j\omega)$ at $\omega = \omega_{gen}$

Actual phase margin of the Compensated System

$$\gamma_0 = 180^\circ + \phi_{gen}.$$

If the actual phase margin satisfy the given specification then the design is accepted. otherwise repeat the procedure from step 4 to step 9 by taken ϵ as 5° more

than previous design.

To determine the actual phase margin of Compensated system.

Let $s = j\omega$ in $G_0(s)$

$$G_0(j\omega) = \frac{5(1+j20\omega)}{j\omega(1+j200\omega)(1+j2\omega)}$$

Let ϕ_0 = phase of $G_0(j\omega)$

ϕ_{gco} = phase of $G_0(j\omega)$ at $\omega = \omega_{gcn}$

$$\phi_0 = \tan^{-1}20\omega - 90^\circ - \tan^{-1}200\omega - \tan^{-1}2\omega$$

At $\omega = \omega_{gcn}$, $\phi_0 = \phi_{gco} = \tan^{-1}20\omega_{gcn} - 90^\circ - \tan^{-1}200\omega_{gcn} - \tan^{-1}2\omega_{gcn}$

$$\therefore \phi_{gco} = \tan^{-1}(20\times0.5) - 90^\circ - \tan^{-1}(200\times0.5) - \tan^{-1}(2\times0.5)$$

$$= -140^\circ.$$

Actual phase margin of Compensated system $\gamma_0 = 180^\circ + \phi_{gco}$

$$\gamma_0 = 180^\circ - 140^\circ$$

$$= 40^\circ.$$

H/W The open loop transfer function of certain unity feedback control system is given by $G(s) = \frac{k}{s(s+4)(s+80)}$. It is desired to have the phase margin to be atleast 38° and the velocity error constant $K_v = 30\,sec^{-1}$. Design a phase lag series compensator.

Lead Compensators :

A Compensator having the characteristics of a lead network is called a lead Compensator. If a sinusoidal signal is applied to the lead network, then in steady state the output will have a phase lead with respect to the input.

The lead Compensation increases the bandwidth, which improves the speed of response and also reduces the amount of overshoot. Lead compensation improves the transient response, where as there is a small change in steady state accuracy. Generally, Lead Compensation provided to make an unstable to a stable system.

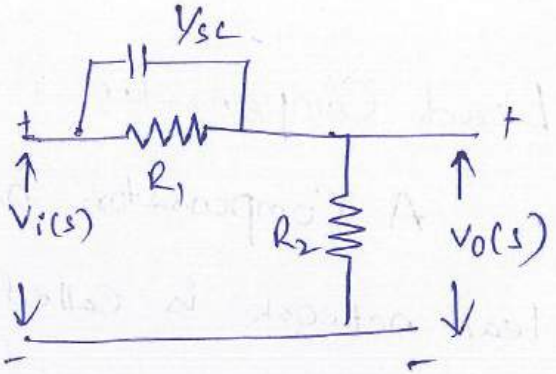
A lead Compensator is a basically a high pass filter and so it amplifies high frequency noise signals. If the pole introduced by the Compensator is not Cancelled by a zero in the system, then lead Compensation increases the order of the systems by one.

S-plane representation of Lead Compensator:-

Lead Compensator is a electrical N/w which produces sinusoidal o/p having phase lead when a sinusoidal i/p is applied.

$$G_c(s) = \frac{s+z}{s+p}$$

(General T.F of any Compensator)



$$\alpha = \frac{z}{p} \quad , \quad p = \frac{z}{\alpha}$$

$$G_c(s) = \frac{s+z}{s+\frac{z}{\alpha}} \qquad \left[\because z = 1/z\right]$$

$$= \frac{s+1/z}{s+\frac{1}{\alpha z}} \Rightarrow \frac{\alpha(zs+1)}{(\alpha s z+1)} \quad \text{---(1)}$$

put $s = j\omega$

$$G_c(j\omega) = \frac{\alpha(j\omega z+1)}{(j\omega\alpha z+1)} \quad \text{---(2)}$$

$$\angle G_c(j\omega) = \tan^{-1}(\omega z) - \tan^{-1}(\omega\alpha z) = \phi$$

$$\tan\phi = \frac{\omega z - \omega\alpha z}{1+\omega^2 z^2\alpha} \qquad \left[\because \tan\phi = \frac{\tan a - \tan b}{1+\tan a\tan b}\right] \quad \text{---(3)}$$

Maximum compensation Condition

$$\left.\frac{d\phi}{d\omega}\right|_{\omega=\omega_m} = 0.$$

$$\omega_m = \sqrt{\omega_{c_1}\cdot\omega_{c_2}}$$

from (1)

$$\omega_{c_1} = \frac{1}{z} \quad , \quad \omega_{c_2} = \frac{1}{\alpha z}$$

$$\omega_m = \sqrt{\frac{1}{z}\cdot\frac{1}{\alpha z}}$$

$$= \frac{1}{z\sqrt{\alpha}}$$

from (3)

$$\tan\phi_{\omega=\omega_m} = \frac{\frac{1}{z\sqrt{\alpha}}z - \frac{1}{z\sqrt{\alpha}}\alpha z}{1+\omega\frac{1}{z^2\alpha}z^2\cdot\alpha}$$

$$= \frac{\frac{1}{\sqrt{\alpha}} - \sqrt{\alpha}}{2}$$

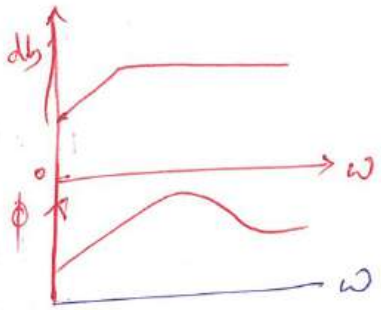
$\therefore \sin\phi = \left(\frac{1-\alpha}{1+\alpha}\right)$

$\alpha = \frac{1-\sin\phi}{1+\sin\phi}$

$$\boxed{\tan\phi\big|_{w=w_m} = \frac{1-\alpha}{2\sqrt{\alpha}}} \Rightarrow \boxed{\phi = \tan^{-1}\left(\frac{1-\alpha}{2\sqrt{\alpha}}\right)}$$

Magnitude :-

(2) $\Rightarrow$ $$|G_c(jw)|_{w=w_m} = \frac{\sqrt{1+w^2T^2}}{\sqrt{1+w^2\alpha^2T^2}}$$



$$= \frac{\sqrt{1+\frac{1}{T^2\alpha}\cdot T^2}}{\sqrt{1+\frac{1}{T^2\alpha}\alpha^2T^2}}$$

$$= \frac{\sqrt{1+\frac{1}{\alpha}}}{\sqrt{1+\alpha}} \Rightarrow \frac{\sqrt{\frac{\alpha+1}{\alpha}}}{\sqrt{1+\alpha}}$$

$$\Rightarrow \sqrt{\frac{\alpha+1}{\alpha}} \times \frac{1}{\sqrt{1+\alpha}} \Rightarrow \frac{1}{\sqrt{\alpha}}$$

$$\boxed{|G_c(jw)|_{w=w_m} = \frac{1}{\sqrt{\alpha}}}$$

Comparing with Electrical Network

~~$R_1 + \frac{1}{sC}$ is parallel. So, $\frac{R_1 \cdot \frac{1}{sC}}{R_1 + \frac{1}{sC}}$~~

~~and R_2 is in series~~

$$\frac{\frac{R_1}{sC}}{\frac{sR_1C+1}{sC}} + R_2 \Rightarrow \frac{R_1}{1+sR_1C} + R_2$$

$$\Rightarrow \frac{R_1 + R_2 + sR_1R_2C}{1+sR_1C}$$

$$\frac{V_o}{V_i} = \frac{R_2}{R_2 + \frac{R_1\frac{1}{sC}}{R_1+\frac{1}{sC}}} \Rightarrow \frac{R_2}{R_2 + \frac{R_1}{R_1sC+1}}$$

$$= \frac{R_2(R_1sC+1)}{R_1R_2sC+R_2+R_1}$$

$$= \frac{R_2R_1sC+R_2}{R(1+R_2)+R_2R_1sC}$$

$$= \frac{R_2(1+R_1sC)}{R_1+R_2(1+R_1R_2sC)}$$

$$= \frac{R_2}{R_1+R_2}\left[\frac{1+R_1sC}{1+\frac{R_1R_2}{R_1+R_2}sC}\right] \quad \text{---} (4)$$

by Comparing (2) & (4)

$$\alpha = \frac{R_2}{R_1+R_2} \; ; \quad \tau = R_1C.$$

[∵ by substituting α, τ & P we can get T.F for Lead Compensator]

Effects of Lead Compensator:

1. Stability increases
2. Bandwidth increases
3. Rise time decreases
4. Transient Response is improved
5. Noise enters into the system at high frequencies.

Lag - Lead Compensator:

Lag Lead Compensator is an electrical network which produces phase lag at one frequency region and phase lead at another frequency region.

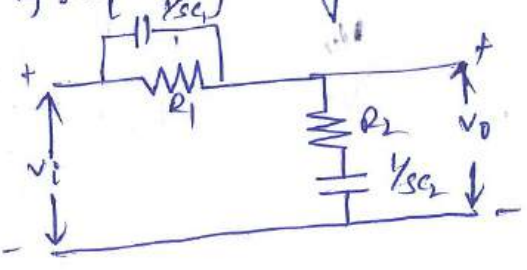


fig: Lag-Lead Network.

Consider $G_c(s) = \left(\frac{s+\frac{1}{T_1}}{s+\frac{1}{\alpha T_1}}\right)\left(\frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}}\right)$

Lead section $\alpha < 1$ Lag section. $\beta > 1$

$$= \frac{\alpha(1+sT_1)}{(1+s\alpha T_1)} \quad \frac{\beta(1+sT_2)}{1+s\beta T_2}$$

$$G_c(s) = \left(\frac{1+sT_1}{1+s\alpha T_1}\right)\left(\frac{1+sT_2}{1+s\beta T_2}\right) \quad \text{--- (1)} \quad [\because \text{we can neglect } \alpha \text{ \& } \beta$$

from the Lead-Lag Network

$$\frac{V_o}{V_i} = \frac{R_2+\frac{1}{sC_2}}{R_2+\frac{1}{sC_2}+\frac{R_1}{1+R_1sC_1}}$$

$$= \frac{R_2sC_2+1}{R_2sC_2+1+\frac{R_1sC_2}{1+R_1sC_1}}$$

$$= \frac{(R_2sC_2+1)(1+R_1sC_1)}{R_1R_2s^2C_1C_2+1+R_1sC_1+R_2sC_2+R_1sC_2}$$

$$= \frac{(R_2sC_2+1)(1+R_1sC_1)}{R_1R_2C_1C_2\left[s^2+s\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_2C_1}\right)+\frac{1}{R_1R_2C_1C_2}\right.}$$

$$= \frac{\left(s+\frac{1}{R_2C_2}\right)\left(s+\frac{1}{R_1C_1}\right)}{s^2+s\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_2C_1}\right)+\frac{1}{R_1R_2C_1C_2}} \quad \text{--- (2)}$$

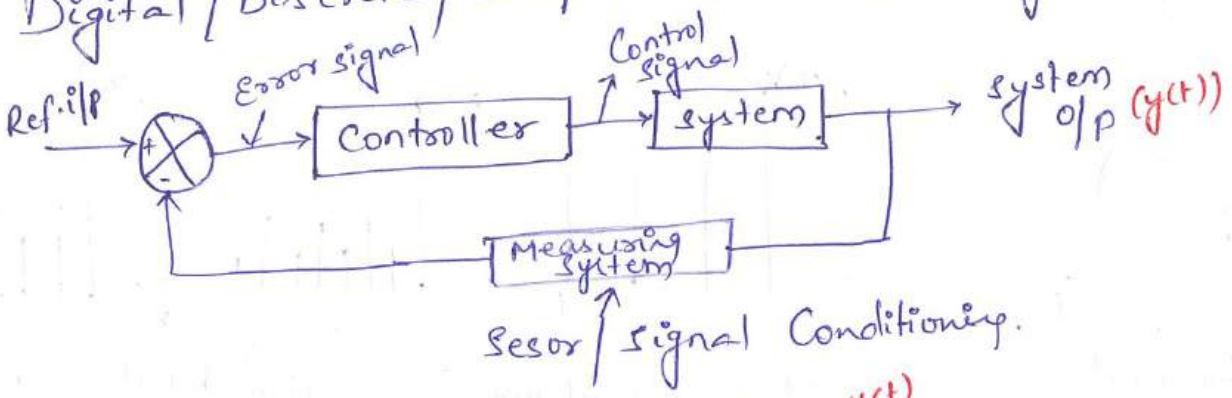
from (1) & (2)

$\therefore T_1 = R_1C_1$, $T_2 = R_2C_2$ $\quad \alpha\beta = 1$

By using this Lag-Lead N/w or Compensator we are going to improve Transient and steady state Response"

Analog & Digital implementation of Controller:

Digital / Discrete / sampled data Control system:



By observing Tracking & Regulation fig's. In case of Regulation the system response is already settled at some point and some sudden disturbance may come, this disturbance must be nullified (filtered) with the desired dynamics (Controllers / compensators). This can be accomplished by proper designing of a Controller.

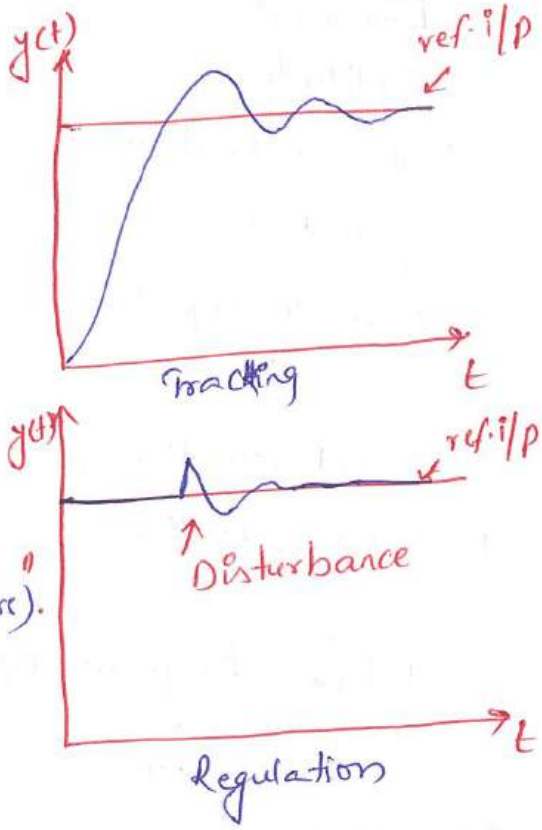


fig: Tracking & Regulation of Control system response.

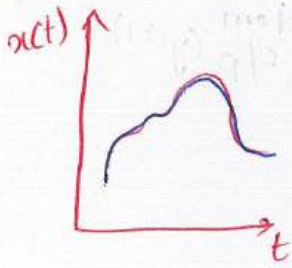
Equation (Analog) → Software (Digital)

To reduce the Noise / disturbances so many Analog devices are available (Amplifies, Micro processors controlers, filters) but by using these Performance degradation is came in to the picture.

With the use of Analog devices & Software (Digital) we can greatly reduce disturbances with out degration of performance of the system.

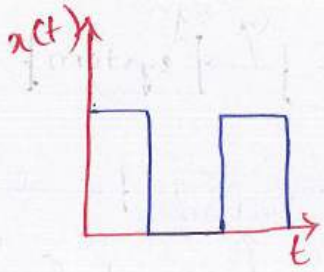
→ Analog V/s digital

(Continuous time) (Discrete time)



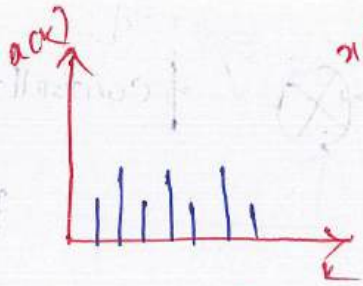
Time and amplitude both are Continues in nature.

CT. analog signal



Time is Continuous but amplitude (2 cases) H/L is discrete. So it is also a Continuous time also "digital signal"

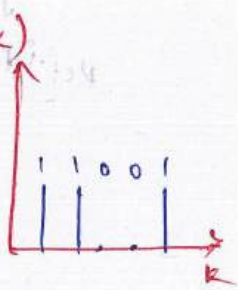
C.T digital signal



Time is discrete, Amplitude is Continuous

Called "discrete time analog signal"

D.T analog signal



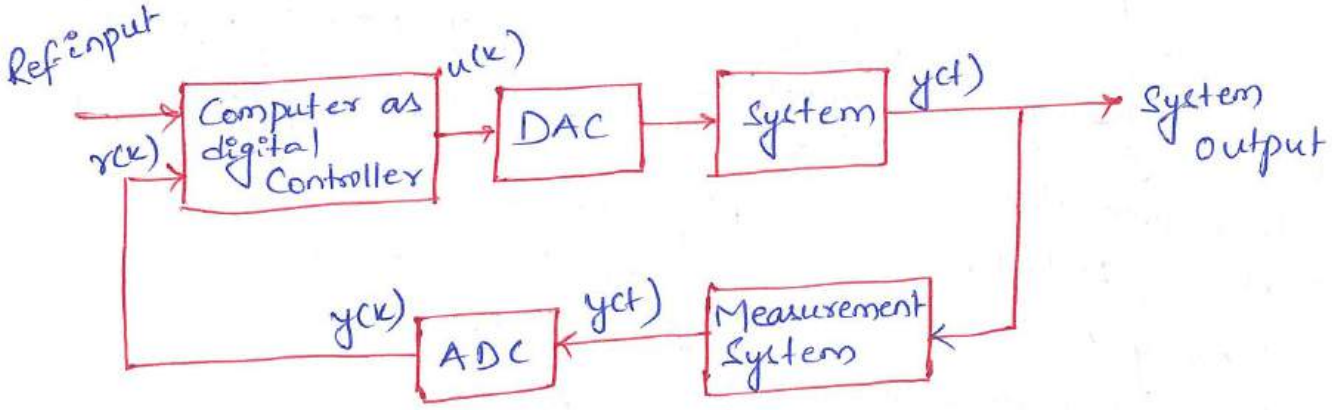
Time discrete amplitude have 2 levels (H/L) so the signal Called "digital signal"

DT digital signal

CT - Continuous time

DT - Discrete time.

→ Block diagram of digital / sampled data / discrete time Control System.



The Reference input need to apply to the Computer with 0 to 5V powersupply, Computer Contains the algorithm of a Controller, Controller (algorithms) generates u(k).

Steps

1. Determine error $e(k) = r(k) - y(k)$
2. Apply e(k) to control algorithm to generate u(k) (digital)
3. U(k) is applied to DAC (Digital to Analog Converter, it converts into continuous time analog signal u(t) which can serve the control objective to the system.
4. The o/p of system y(t) is measured and fed back to input through Analog digital Converter (ADC). ADC converts into digital y(k) and applied to the input.

Factors affecting accuracy (performance) of digital control system:

1. Accuracy of measurement system.
2. Resolution of ADC & DAC
3. Speed of Computor (gives the accuracy of the system).
4. Controller algorithm (Decides the performance of the system)

Advantages

1. Accuracy is high
2. Speed (High computation speed)
3. Flexibility (we can change the controller algorithm (program) no need to change the hardware).
4. Noise and other environmental factors (influence) ↓ (Temp. change, power changes)
5. Reliability ↑ (failure → one of the component is fails we can change the u but in the software chang fails is less).

6. Ability to implement Complex algorithm

Limitations:

1. finite word length
2. Stability
3. Cost may be higher
4. Limitation on sampling rate.

PROCEDURE FOR THE DESIGN OF LAG COMPENSATOR USING BODE PLOT

The following steps may be followed to design a lag compensator using bode plot and to be connected in series with transfer function of uncompensated system, G(s).

Step-1 : Choose the value of K in uncompensated system to meet the steady state error requirement.

Step-2 : Sketch the bode plot of uncompensated system. *[Refer Chapter-4 for the procedure to sketch bode plot].*

Step-3 : Determine the phase margin of the uncompensated system from the bode plot. If the phase margin does not satisfy the requirement then lag compensation is required.

Step-4 : Choose a suitable value for the phase margin of the compensated system.

Let γ_d = Desired phase margin as given in specifications.

γ_n = Phase margin of compensated system.

Now, $\gamma_n = \gamma_d + \epsilon$

where ϵ = Additional phase lag to compensate for shift in gain crossover frequency

Choose an initial value of $\epsilon = 5°$.

Step-5 : Determine the new gain crossover frequency, ω_{gcn}. The new ω_{gcn} is the frequency corresponding to a phase margin of γ_n on the bode plot of uncompensated system.

Let, ϕ_{gcn} = Phase of $G(j\omega)$ at new gain crossover frequency, ω_{gcn}

Now, $\gamma_n = 180° + \phi_{gcn}$ (or) $\phi_{gcn} = \gamma_n - 180°$

The new gain crossover frequency, ω_{gcn} is given by the frequency at which the phase of $G(j\omega)$ is ϕ_{gcn}.

Step-6 : Determine the parameter, β of the compensator. The value of β is given by the magnitude of $G(j\omega)$ at new gain crossover frequency, ω_{gcn}. Find the db gain (A_{gcn}) at new gain crossover frequency, ω_{gcn}

Now, $A_{gcn} = 20\log\beta$ (or) $\frac{A_{gcn}}{20} = \log\beta$, $\therefore \beta = 10^{A_{gcn}/20}$

Step-7 : Determine the transfer function of lag compensator.

Place the zero of the compensator arbitrarily at $1/10^{th}$ of the new gain crossover frequency

$\therefore$ Zero of the lag compensator, $z_c = \frac{1}{T} = \frac{\omega_{gcn}}{10}$

Now, $T = \frac{10}{\omega_{gcn}}$.

Pole of the lag compensator, $p_c = 1/\beta T$

Transfer function of lag compensator $G_c(s) = \frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}} = \beta\left(\frac{1+sT}{1+s\beta T}\right)$

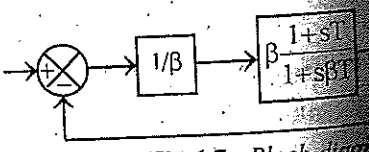


Fig 6.7 : Block diagram of lag compensated system

Step-8 : Determine the open loop transfer function of compensated system. The lag compensator connected in series with plant as shown in fig 6.7.

...n the lag compensator is inserted in series with plant, the open loop gain of the system is ...dified by the factor $\beta(\because \beta>1)$. If the gain produced is not required then attenuator with gain ...can be introduced in series with the lag compensator to nullify the gain produced by lag ...pensator.

...e open loop transfer function of the compensated system,

$$G_o(s) = \frac{1}{\beta}.G_c(s).G(s) = \frac{1}{\beta}.\beta\frac{(1+sT)}{(1+s\beta T)}.G(s) = \frac{(1+sT)}{(1+s\beta T)}\ .\ G(s)$$

...etermine the actual phase margin of compensated system. Calculate the actual phase angle of ...e compensated system using the compensated transfer function at new gain crossover ...requency, ω_{gcn}.

...Let, ϕ_{gco} = Phase of $G_o(j\omega)$ at $\omega = \omega_{gcn}$

...Actual phase margin of the compensated system, $\gamma_o = 180° + \phi_{gco}$

...If the actual phase margin satisfies the given specification then the design is accepted. Otherwise ...repeat the procedure from step 4 to 9 by taking $\in$ as 5° more than previous design.

...DURE FOR DESIGN OF LAG COMPENSATOR USING ROOT LOCUS

...The following steps may be followed to design a lag compensator using root locus and to be ...connected in series with the transfer function of uncompensated system.

...: Draw the root locus of uncompensated system. *[Refer chapter-5 for the proeedure to construct ...root locus].*

...: Determine the dominant pole, s_d. Draw a straight line through the origin with an angle $\cos^{-1}\zeta$ with ...respect to negative real axis. The intersection point of the straight line with root locus gives the ...dominant pole, s_d.

...3 : Determine the open loop gain of the uncompensated system at $s = s_d$. Let this gain be K. The open ...loop gain K at $s = s_d$ on root locus is given by,

$$K = \frac{\text{Product of vector lengths from } s_d \text{ to open loop poles}}{\text{Product of vector lengths from } s_d \text{ to open loop zeros}}$$ (vector length measured to scale)

...-4 : Calculate the parameter, β of the compensator.

Let, K_{vu} = Velocity error constant of uncompensated system.

K_{vd} = Desired velocity error constant.

$$K_{vu} = \lim_{s\to 0} sG(s)$$

Let A be the factor by which the velocity error constant of the system has to be increased,

where, $A = K_{vd}/K_{vu}$

Choose β such that it is 10 to 20% greater than A.

$\therefore \beta = (1.1 \text{ to } 1.2) \times A$.

Step-5 : Determine the transfer function of lag compensator. The zero of the lag compensator (1/T) is chosen to be 10% of the second pole of uncompensated system.

$\therefore$ Zero of the compensator, $z_c = (-1/T) = 0.1 \times$ (second pole of G(s))

Now, $T = 1/[-0.1 \times$ (second pole of G(s))]

Pole of the lag compensator, $p_c = -1/\beta T$.

$$\left.\begin{matrix}\text{Transfer function}\\ \text{of lag compensator}\end{matrix}\right\} G_c(s) = \frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}} = \beta\frac{(1+sT)}{(1+s\beta T)}$$

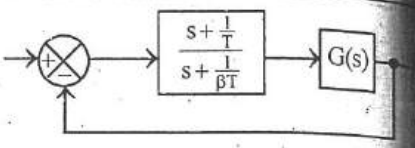


Fig 6.8 : Block diagram of lag compensated system.

Step-6 : Determine the open loop transfer function of the compensated system. The lag compensator is connected in series with the plant as shown in fig 6.8

$$\left.\begin{matrix}\text{Open loop transfer fuction}\\ \text{of compensated system}\end{matrix}\right\} G_o(s) = G_c(s).G(s) = \frac{(s+\frac{1}{T})}{(s+\frac{1}{\beta T})}.G(s)$$

Step-7 : Check whether the compensated system satisfies the steady state error requirement. If it is satisfied, then the design is accepted otherwise repeat the design by modifying the locations of poles and zeros of the compensator.

EXAMPLE 6.1

A unity feedback system has an open loop transfer function, G(s) = K/s(1+2s). Design a suitable lag compensator so that phase margin is 40° and the steady state error for ramp input is less than or equal to 0.2.

SOLUTION

Step-1: Calculation of gain, K.

Given that, $e_{ss} \le 0.2$ for ramp input. Let $e_{ss} = 0.2$

We know that, $e_{ss} = 1/K_v$ for ramp input.

$$\therefore \text{Velocity error constant, } K_v = \frac{1}{e_{ss}} = \frac{1}{0.2} = 5.$$

By definition of velocity error constant, $K_v = \underset{s\to 0}{Lt}\ s\,G(s)H(s)$.

Since the system is unity feedback system, H(s) = 1.

$$\therefore K_v = \underset{s\to 0}{Lt}\ s\,G(s) = \underset{s\to 0}{Lt}\ s\frac{K}{s(1+2s)} = K \qquad \therefore K = 5$$

Step-2: Bode plot of uncompensated system.

Given that, G(s) = 5/s(1+2s)

Let s = jω, ∴ G(jω) = 5/jω(1+j2ω).

MAGNITUDE PLOT

The corner frequency is, $\omega_c = 1/2 = 0.5$ rad/sec

The various terms of G(jω) are listed in table-1. Also the table shows the slope contributed by each term and the change in slope at the corner frequency.

TABLE-1

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\frac{5}{j\omega}$	–	–20	–
$\frac{1}{1+j2\omega}$	$\omega_c = \frac{1}{2} = 0.5$	–20	–20–20 = –40

Choose a low frequency ω_l such that $\omega_l < \omega_c$ and choose a high frequency ω_h such that $\omega_h > \omega_c$.

Let $\omega_l = 0.1$ rad/sec and $\omega_h = 10$ rad/sec

Let $A = |G(j\omega)|$ in db

$$\text{At } \omega = \omega_l, \quad A = 20 \log\left|\frac{5}{j\omega}\right| = 20 \log \frac{5}{0.1} = 34 \text{ db}$$

$$\text{At } \omega = \omega_c, \quad A = 20 \log\left|\frac{5}{j\omega}\right| = 20 \log \frac{5}{0.5} = 20 \text{ db}$$

$$\text{At } \omega = \omega_h, \quad A = \left[\text{slope from } \omega_c \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_c}\right] + A_{(\text{at } \omega = \omega_c)} = -40 \times \log\frac{10}{0.5} + 20 = -32 \text{ db}$$

Let the points a, b and c be the points corresponding to frequencies ω_l, ω_c and ω_h respectively on the magnitude plot. In a semilog graph sheet choose appropriate scales and fix the points a, b and c. Join the points by straight lines and mark the slope on respective region. Magnitude plot is shown in fig 6.1.1.

PHASE PLOT

The phase angle of $G(j\omega)$ as a function of ω is given by, $\phi = \angle G(j\omega) = -90° - \tan^{-1}2\omega$

The phase angle of $G(j\omega)$ are calculated for various values of ω and listed in table-2.

TABLE-2

ω rad/sec	0.1	0.5	1.0	5	10
ϕ deg	–101	–135	–153	–174	–177

On the same semilog sheet take another y-axis, choose appropriate scale and draw phase plot as shown in fig 6.1.1.

Step-3 : Determination of phase margin of uncompensated system.

Let, ϕ_{gc} = Phase of $G(j\omega)$ at gain crossover frequency (ω_{gc}).

and γ = Phase margin of uncompensated system.

From the bode plot of uncompensated system we get, $\phi_{gc} = -162°$.

Now, $\gamma = 180° + \phi_{gc} = 180° - 162° = 18°$

The system requires a phase margin of 40°, but the available phase margin is 18° and so lag compensation should be employed to improve the phase margin.

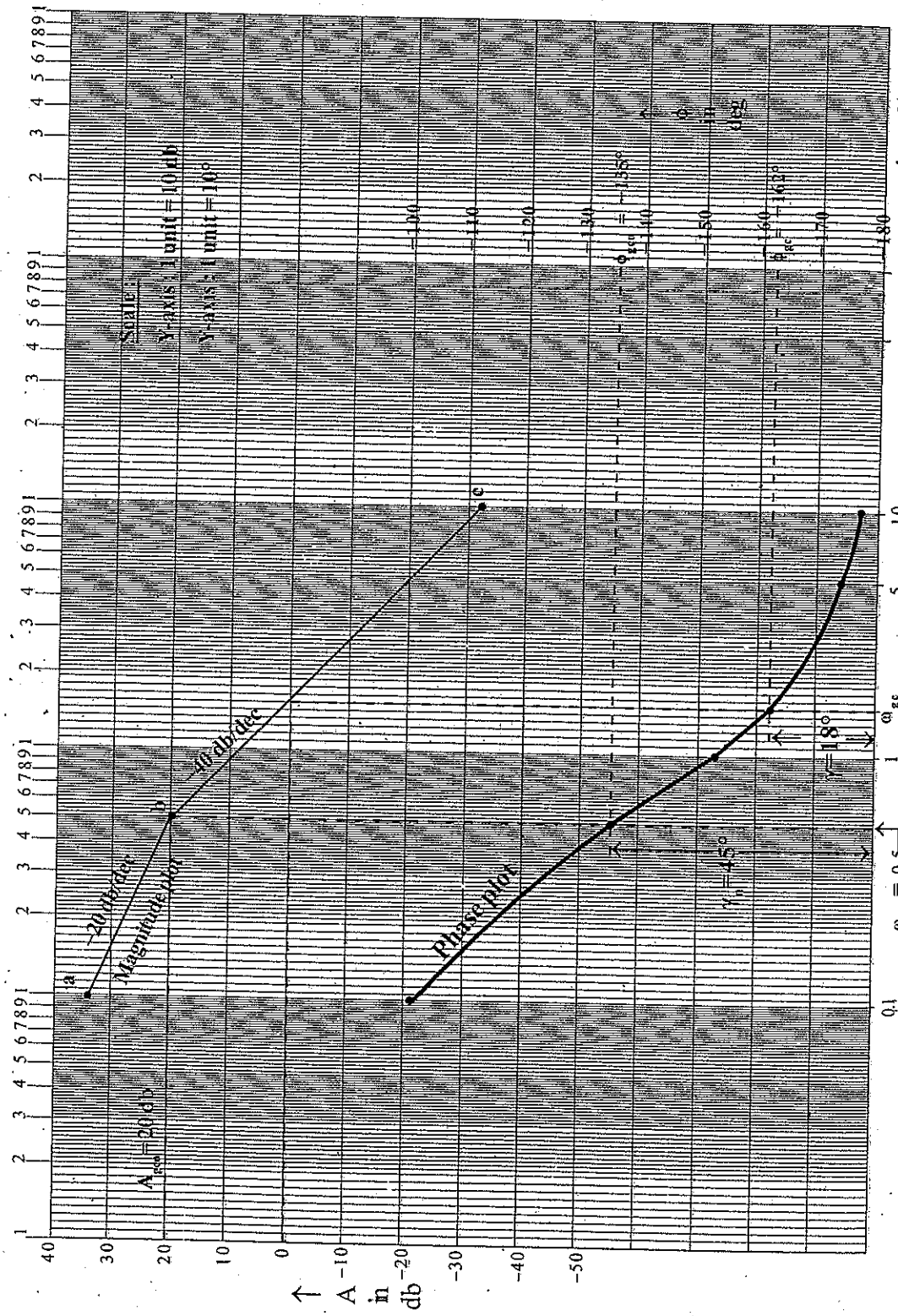


Fig 6.11 : Bode plot of $G(j\omega) = 5/j\omega(1+j2\omega)$

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Step-4 : Choose a suitable value for the phase margin of compensated system.

The desired phase margin, $\gamma_d = 40°$.

$\therefore$ Phase margin of compensated system, $\gamma_n = \gamma_d + \epsilon$

Let initial choice of $\epsilon = 5°$

$\therefore \gamma_n = \gamma_d + \epsilon = 40° + 5° = 45°$

Step 5 : Determine new gain crossover frequency.

Let ω_{gcn} = New gain crossover frequency and ϕ_{gcn} = Phase of $G(j\omega)$ at ω_{gcn}

Now, $\gamma_n = 180° + \phi_{gcn}$

$\therefore \phi_{gcn} = \gamma_n - 180° = 45° - 180° = -135°$

From the bode plot we found that, the frequency corresponding to a phase of $-135°$ is 0.5 rad/sec.

$\therefore$ New gain crossover frequency, $\omega_{gcn} = 0.5$ rad/sec.

Step-6 : Determine the parameter, β

From the bode plot we found that, the db magnitude at ω_{gcn} is 20 db.

$\therefore |G(j\omega)|$ in db at $(\omega = \omega_{gcn}) = A_{gcn} = 20$ db

Also, $A_{gcn} = 20 \log \beta$; $\therefore \beta = 10^{A_{gcn}/20} = 10^{20/20} = 10$.

Step-7 : Determine the transfer function of lag compensator.

The zero of the compensator is placed at a frequency one-tenth of ω_{gcn}.

$\therefore$ Zero of the lag compensator, $z_c = \dfrac{1}{T} = \dfrac{\omega_{gcn}}{10}$

$$\text{Now, } T = \frac{10}{\omega_{gcn}} = \frac{10}{0.5} = 20$$

Pole of the lag compensator, $p_c = \dfrac{1}{\beta T} = \dfrac{1}{10 \times 20} = \dfrac{1}{200} = 0.005$

Transfer function of lag compensator, $G_c(s) = \dfrac{s + \frac{1}{T}}{s + \frac{1}{\beta T}} = \beta \dfrac{1 + sT}{1 + s\beta T} = 10 \dfrac{(1 + 20s)}{(1 + 200s)}$

Step-8 : Determine the open loop transfer function of compensated system.

The block diagram of the compensated system is shown in fig 6.1.2. The gain of compensator is nullified by introducing an attenuator in series with compensator, as shown in fig 6.1.2.

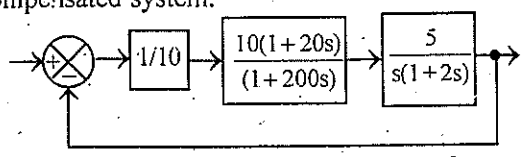

Fig 6.1.2 : Block diagram of lag compensated system.

Open loop transfer function of compensated system $\Big\}$ $G_o(s) = \dfrac{1}{10} \times \dfrac{10\,(1+20s)}{(1+200s)} \times \dfrac{5}{s(1+2s)} = \dfrac{5(1+20s)}{s(1+200s)\,(1+2s)}$

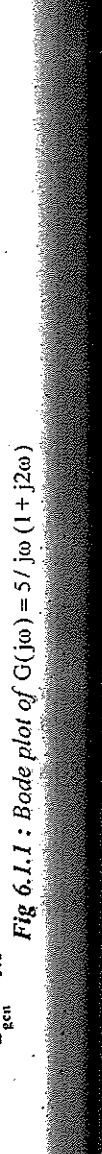

Fig 6.1.1 : Bode plot of $G(j\omega) = 5 / j\omega\,(1 + j2\omega)$

Step-9 : Determine the actual phase margin of compensated system.

On substituting $s = j\omega$ in $G_o(s)$ we get, $G_o(j\omega) = \dfrac{5(1+j20\omega)}{j\omega(1+j200\omega)(1+j2\omega)}$

Let, ϕ_o = Phase of $G_o(j\omega)$

ϕ_{gco} = Phase of $G_o(j\omega)$ at $\omega = \omega_{gcn}$

$\phi_o = \tan^{-1} 20\omega - 90° - \tan^{-1} 200\omega - \tan^{-1} 2\omega$

At $\omega = \omega_{gcn}$, $\phi_o = \phi_{gco} = \tan^{-1} 20\,\omega_{gcn} - 90° - \tan^{-1} 200\omega_{gcn} - \tan^{-1} 2\,\omega_{gcn}$

$\therefore \phi_{gco} = \tan^{-1}(20 \times 0.5) - 90° - \tan^{-1}(200 \times 0.5) - \tan^{-1}(2 \times 0.5) = -140°.$

Actual phase margin of compensated system, $\gamma_o = 180° + \phi_{gco} = 180° - 140° = 40°$

CONCLUSION

The actual phase margin of the compensated system satisfies the requirement. Hence the desig is acceptable.

RESULT

Transfer function of lag compensator, $G_c(s) = \dfrac{10(1+20s)}{(1+200s)} = \dfrac{(s+0.05)}{(s+0.005)}$

Open loop transfer function of compensated system, $G_o(s) = \dfrac{5(1+20s)}{s(1+200s)(1+2s)}$

EXAMPLE 6.2

The open loop transfer function of certain unity feedback control system is given by $G(s) = K/s(s+4)(s+80)$. It is desired to have the phase margin to be atleast 33° and the velocity error constant $K_v = 30\ sec^{-1}$. Design a phase lag series compensator.

SOLUTION

Step-1 : Calculation of gain, K

Given that, $K_v = 30\ sec^{-1}$

By definition of velocity error constant, $K_v = \underset{s\to 0}{Lt}\ s\,G(s)\,H(s)$

Since the system is unity feedback system, $H(s) = 1$.

$$\therefore K_v = \underset{s\to 0}{Lt}\ sG(s) = \underset{s\to 0}{Lt}\ s\,\frac{K}{s(s+4)(s+80)} = \frac{K}{4\times 80}$$

i.e., $\dfrac{K}{4\times 80} = 30, \quad \therefore K = 30\times 80\times 4 = 9600.$

Step-2 : Bode plot of uncompensated system

$$G(s) = \frac{9600}{s(s+4)(s+80)} = \frac{9600/4\times 80}{s(1+\frac{s}{4})(1+\frac{s}{80})} = \frac{30}{s(1+0.25s)(1+0.0125s)}$$

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Let $s = j\omega$, $\therefore G(j\omega) = \dfrac{30}{j\omega\,(1+j0.25\omega)\,(1+j0.0125\omega)}$

AGNITUDE PLOT

The corner frequencies are, $\omega_{c1} = 1/0.25 = 4$ rad/sec and $\omega_{c2} = 1/0.0125 = 80$ rad/sec. The various ms of G(jω) are listed in table-1. Also the table shows the slope contributed by each term and the nge in slope at the corner frequency.

BLE-1

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\dfrac{30}{j\omega}$	–	–20	–
$\dfrac{1}{1+j0.25\omega}$	$\omega_{c1} = \dfrac{1}{0.25} = 4$	–20	–20–20 = –40
$\dfrac{1}{1+j0.0125\omega}$	$\omega_{c2} = \dfrac{1}{0.0125} = 80$	–20	–40 – 20 = –60

Choose a low frequency ω_l such that $\omega_l < \omega_{c2}$ and choose a high frequency ω_h such that $\omega_h > \omega_{c2}$. Let $\omega_l = 1$ rad/sec and $\omega_h = 100$ rad/sec.

Let $A = |G(j\omega)|$ in db

$$\text{At } \omega = \omega_l, \quad A = 20\log\left|\frac{30}{j\omega}\right| = 20\log\frac{30}{1} = 29.5 \text{ db} \approx 30 \text{ db}$$

$$\text{At } \omega = \omega_{c1}, \quad A = 20\log\left|\frac{30}{j\omega}\right| = 20\log\frac{30}{4} = 17.5 \text{ db} \approx 18 \text{ db}$$

$$\text{At } \omega = \omega_{c2}, \quad A = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + A_{(\text{at } \omega = \omega_{c1})}$$

$$= -40\log\frac{80}{4} + 18 = -34 \text{ db}$$

$$\text{At } \omega = \omega_h, \quad A = \left[\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c2}}\right] + A_{(\text{at } \omega = \omega_{c2})}$$

$$= -60 \times \log\frac{100}{80} + (-34) = -40 \text{ db}$$

Let the points a, b, c and d be the points corresponding to frequencies ω_l, ω_{c1}, ω_{c2} and ω_h respectively the magnitude plot. In a semilog graph sheet choose appropriate scales and fix the points a, b, c and d. n the points by straight lines and mark the slope on the respective region. The magnitude plot is shown fig 6.2.1.

HASE PLOT

The phase angle of G(jω) as a function of ω is given by,

$$\phi = \angle G(j\omega) = -90° - \tan^{-1}0.25\omega - \tan^{-1}0.0125\omega$$

The phase angle of G(jω) are calculated for various values of ω and listed in table-2.

TABLE-2

ω rad/sec	1	4	10	50	80	100
ϕ deg	−104	−138	−165 ≈ −164	−207 ≈ −208	−222	−229 ≈ −230

On the same semilog sheet take another y-axis, choose appropriate scale and draw phase pl... shown in fig 6.2.1.

Step-3: Determination of phase margin of uncompensated system.

Let, ϕ_{gc} = Phase of $G(j\omega)$ at gain crossover frequency (ω_{gc}).

γ = Phase margin of uncompensated system.

From the bode plot of uncompensated system we found that, $\phi_{gc} = -168°$.

Now, $\gamma = 180° + \phi_{gc} = 180° - 168° = 12°$

The system requires a phase margin of atleast 33°, but the available phase margin is 12° and lag compensation should be employed to improve the phase margin.

Step-4 : Choose a suitable value for the phase margin of compensated system.

The desired phase margin, $\gamma_d = 33°$.

$\therefore$ Phase margin of compensated system, $\gamma_n = \gamma_d + \epsilon$

Let initial choice of $\epsilon = 5°$; $\therefore \gamma_n = \gamma_d + \epsilon = 33° + 5° = 38°$

Step 5: Determine new gain crossover frequency.

Let ω_{gcn} = New gain crossover frequency and ϕ_{gcn} = Phase of $G(j\omega)$ at ω_{gcn}

Now, $\gamma_n = 180° + \phi_{gcn}$

$\therefore \phi_{gcn} = \gamma_n - 180° = 38° - 180° = -142°$

From the bode plot we found that, the frequency corresponding to a phase of −142° i... 4.7 rad/sec.

$\therefore$ New gain crossover frequeney, $\omega_{gcn} = 4.7$ rad/sec.

Step-6 : Determine the parameter, β

From the bode plot we found that, the db magnitude at ω_{gcn} as 16 db.

$\therefore |G(j\omega)|$ in db at $(\omega = \omega_{gcn}) = A_{gcn} = 16$ db

Also, $A_{gcn} = 20 \log \beta$; $\therefore \beta = 10^{Agcn/20} = 10^{16/20} = 6.3$.

Step-7 : Determine the transfer function of lag compensator.

The zero of the compensator is placed at a frequency one-tenth of ω_{gcn}.

$\therefore$ Zero of the lag compensator, $z_c = \frac{1}{T} = \frac{\omega_{gcn}}{10}$

$$\text{Now, } T = \frac{10}{\omega_{gcn}} = \frac{10}{4.7} = 2.13$$

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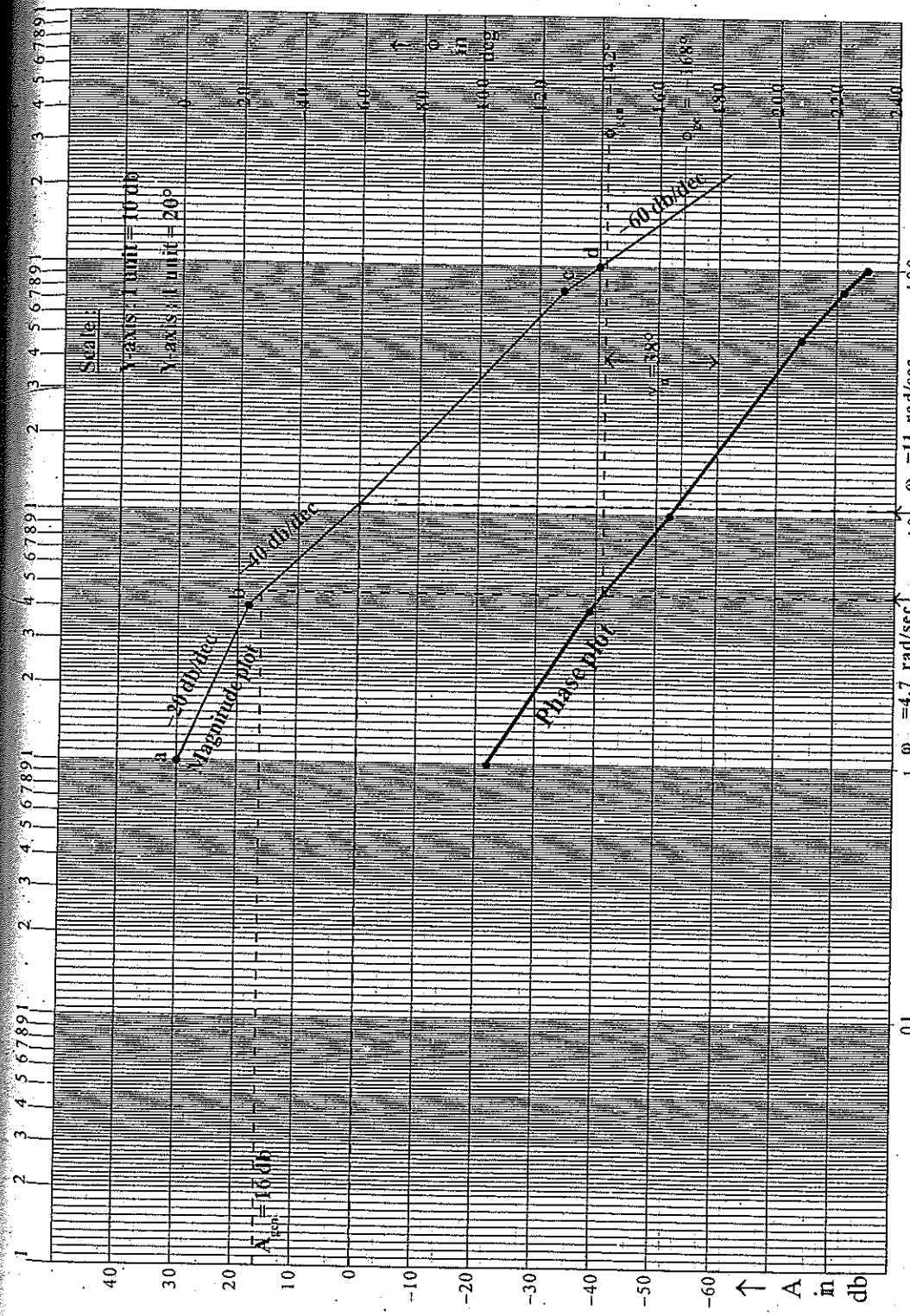

Fig 6.2.1 : Bode plot of $G(j\omega) = 30/j\omega\,(1+j0.25\omega)\,(1+j0.0125\omega)$

Pole of the lag compensator, $p_c = \frac{1}{\beta T} = \frac{1}{6.3 \times 2.13} = \frac{1}{13.419}$

Transfer function of lag compensator $G_c(s) = \frac{s + \frac{1}{T}}{s + \frac{1}{\beta T}} = \beta \frac{1 + sT}{1 + s\beta T} = 6.3 \frac{(1 + 2.13s)}{(1 + 13.419s)}$

Step-8 : Determine the open loop transfer function of the compensated system.

The block diagram of the compensated system is shown in fig 6.2.2. The gain of the compen is nullified by introducing an attenuator in series with the compensator, as shown in fig 6.

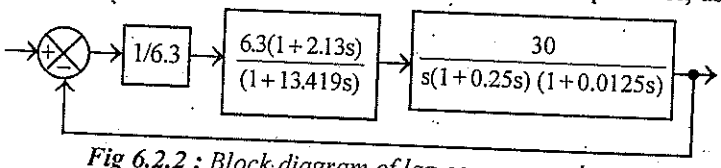

Fig 6.2.2 : Block diagram of lag compensated system.

Open loop transfer function of compensated system $G_o(s) = \frac{1}{6.3} \times \frac{6.3\,(1+2.13s)}{(1+13.419s)} \times \frac{30}{s(1+0.25s)\,(1+0.0125s)}$

$$= \frac{30\,(1+2.13s)}{s\,(1+13.419s)\,(1+0.25s)\,(1+0.0125s)}$$

Step-9 : Determine the actual phase margin of compensated system.

On substituting $s = j\omega$ in $G_o(s)$ we get,

$$G_o(j\omega) = \frac{30\,(1+j2.13\omega)}{j\omega\,(1+j13.419\omega)\,(1+j0.25\omega)\,(1+j0.0125\omega)}$$

Let ϕ_o = Phase of $G_o(j\omega)$ and ϕ_{gco} = Phase of $G_o(j\omega)$ at $\omega = \omega_{gcn}$

$\phi_o = \tan^{-1} 2.13\omega - 90° - \tan^{-1} 13.419\omega - \tan^{-1} 0.25\omega - \tan^{-1} 0.0125\omega$

At $\omega = \omega_{gcn}$, $\phi_o = \phi_{gco} = \tan^{-1} 2.13\,\omega_{gcn} - 90° - \tan^{-1} 13.419\omega_{gcn}$

$- \tan^{-1} 0.25\omega_{gcn} - \tan^{-1} 0.0125\,\omega_{gcn}$

$\therefore \phi_{gco} = \tan^{-1} (2.13 \times 4.7) - 90° - \tan^{-1} (13.419 \times 4.7)$

$- \tan^{-1} (0.25 \times 4.7) - \tan^{-1} (0.0125 \times 4.7) = -147°.$

Actual phase margin of compensated system, $\gamma_o = 180° + \phi_{gco} = 180° - 147° = 33°$

CONCLUSION

The actual phase margin of the compensated system satisfies the requirement. Hence the design is acceptable.

RESULT

Transfer function of lag compensator, $G_c(s) = \frac{6.3\,(1+2.13s)}{(1+13.419s)} = \frac{(s+0.469)}{(s+0.074)}$

Transfer function of lag compensated system $G_0(s) = \frac{30\,(1+2.13s)}{s\,(1+13.419s)\,(1+0.25s)\,(1+0.0125s)}$

EXAMPLE 6.3

The forward path transfer function of a certain unity feedback control system is given by G(s) = K/s (s+2) (s +8). Design a suitable lag compensator so that the system meets the following specifications. (i) Percentage overshoot ≤ 16% for unit step input, (ii) Steady state error ≤ 0.125 for unit ramp input.

SOLUTION

Step-1 : Sketch the root locus of uncompensated system.

To find poles of open loop system

Given that, G(s) = K / s(s+2) (s+8).

The poles of open loop transfer function are the roots of the equation, s(s+2) (s+8) = 0.

∴ The poles are lying at s = 0, –2, –8.

Let us denote poles by, p_1, p_2 and p_3. Here, $p_1 = 0$, $p_2 = -2$ and $p_3 = -8$.

To find root locus on real axis

The segments of real axis between s = 0 and s = –2 and the segment of real axis between s = –8 and s = –∞ will be part of root locus. Because if we choose a test point in this segment then to the right of this point we have odd number of real poles and zeros.

To find angles of asymptotes and centroid

Since there are three poles, the number of root locus branches are three. There is no finite zero and so all the three root locus branches will meet the zeros at infinity. Hence the number of asymptotes required is three.

$$\text{Angles of asymptotes} = \frac{\pm 180°(2q+1)}{n-m}; \qquad q = 0, 1, 2, \ldots\ldots n-m.$$

Here n = 3 and m = 0. ∴ q = 0, 1, 2, 3 .

$$\text{When, } q = 0, \text{ Angles} = \frac{\pm 180°}{3} = \pm 60°$$

$$\text{When, } q = 1, \text{ Angles} = \frac{\pm 180°(2+1)}{3} = \pm 180°$$

∴ Angles of asymptotes are, +60°, –60° and ±180°

$$\text{Centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m} = \frac{0-2-8}{3} = -3.33$$

To find breakaway point

$$\text{Closed loop transfer function, } \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\dfrac{K}{s(s+2)(s+8)}}{1+\dfrac{K}{s(s+2)(s+8)}} = \frac{K}{s(s+2)(s+8)+K}$$

The characteristic equation is, s(s+2) (s+8) + K = 0.

$$\therefore K = -s(s+2)(s+8) = -(s^3 + 10s^2 + 16s)$$

On differentiating K with respect to s we get,

$dK/ds = -(3s^2 + 20s + 16)$

Put $dK/ds = 0$, $\therefore 3s^2 + 20s + 16 = 0$

$$s = \frac{-20 \pm \sqrt{20^2 - 4 \times 3 \times 16}}{2 \times 3} = -0.9 \text{ or } -5.7.$$

When $s = -0.9$, $K = -(s^3+10s^2+16s) = -((-0.9)^3 + 10(-0.9)^2 + 16(-0.9)) = 7$

When $s = -5.7$, $K = -(s^3+10s^2+16s) = -((-5.7)^3 + 10(-5.7)^2 + 16(-5.7)) = -48$

For $s = -0.9$, the value of K is positive and real and so it is actual breakaway point.

To find crossing point on imaginary axis

The characteristic equation is, $s(s+2)(s+8) + K = 0$.

$\therefore s^3 + 10s^2 + 16s + K = 0$

Put $s = j\omega$.

$(j\omega)^3 + 10(j\omega)^2 + 16(j\omega) + K = 0 \quad \Rightarrow \quad -j\omega^3 - 10\omega^2 + j16\omega + K = 0$

On equating imaginary part to zero we get,

$-j\omega^3 + j16\omega = 0 \quad \Rightarrow \quad -j\omega^3 = -j16\omega \quad \Rightarrow \quad \omega^2 = 16$

$\therefore \omega = \pm\sqrt{16} = \pm 4$

Hence the root locus crosses the imaginary axis at +j4 and −j4. The complete root locus sketch is shown in fig 6.3.1.

Step-2 : Determine the dominant pole s_d.

Given that, $\%M_p = 16\%$

We know that, $\%M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}}$; $\therefore e^{-\zeta\pi/\sqrt{1-\zeta^2}} = 0.16$

On taking natural log we get, $\frac{-\zeta\pi}{\sqrt{1-\zeta^2}} = ln\ 0.16 = -1.83$

On squaring we get, $\frac{\zeta^2\pi^2}{1-\zeta^2} = 3.3489$

On cross multiplication we get,

$\zeta^2\pi^2 = 3.3489(1-\zeta^2) \Rightarrow \zeta^2\pi^2 + 3.3489\zeta^2 = 3.3489 \Rightarrow \zeta^2(\pi^2 + 3.3489) = 3.3489$

$\therefore \zeta = \sqrt{\frac{3.3489}{\pi^2 + 3.3489}} = 0.5$

$\therefore \cos^{-1}\zeta = \cos^{-1}0.5 = 60°$

Draw a straight line at an angle of 60° with respect to real axis as shown in fig 6.3.1. The meeting point of this line with root locus is the dominant pole, s_d.

From fig 6.3.1. we get, $s_d = -0.75 \pm j6.35$ (Dominant poles occur as conjugate poles).

Step 3 : To find gain K, at $s = s_d$

$$K = \frac{\text{Product of vector lengths from open loop poles to } s_d}{\text{Product of vector lengths from open loop zeros to } s_d}$$

Product of vector lengths from poles $= l_1 \times l_2 \times l_3$

From root locus plot of fig 6.3.1. we get, $l_1 = 1.5$; $l_2 = 1.8$ and $l_3 = 7.35$

Note : Vector lengths are measured to scale.

Since there is no finite zero, the product of vector lengths from zeros is unity.

$$\therefore K = l_1 \times l_2 \times l_3 = 1.5 \times 1.8 \times 7.35 = 19.845 \approx 20$$

Step 4 : To find parameter, β

Given that, $G(s) = K/s(s+2)(s+8) = 20/s(s+2)(s+8)$

Velocity error constant of uncompensated system $\Big\}$ $K_{vu} = \underset{s\to 0}{Lt}\ s\,G(s) = \underset{s\to 0}{Lt}\ s\,\frac{20}{s(s+2)(s+8)} = \frac{20}{2\times 8} = 1.25$

It is given that steady state error, $e_{ss} \leq 0.125$ for ramp input.

$$\therefore \text{The desired velocity error constant, } K_{vd} = \frac{1}{e_{ss}} = \frac{1}{0.125} = 8$$

Let A be the factor by which K_v is increased.

$$\therefore A = \frac{K_{vd}}{K_{vu}} = \frac{8}{1.25} = 6.4$$

Let, $\beta = 1.2 \times A = 1.2 \times 6.4 = 7.68$

Step-5 : Determine the transfer function of lag compensator.

Zero of the compensator, $z_c = -1/T = 0.1 \times$ second pole of G(s)

$$= 0.1 \times (-2) = -0.2$$

Now, $T = 1/0.2 = 5$

Pole of the compensator, $P_c = \frac{-1}{\beta T} = \frac{-1}{7.68 \times 5} = \frac{-1}{38.4} = -0.026$

$$\therefore \frac{1}{\beta T} = 0.026 \text{ and } \beta T = 38.4$$

Transfer function of lag compensator $\Big\}$ $G_c(s) = \frac{s + 1/T}{s + 1/\beta T} = \frac{(s+0.2)}{(s+0.026)}$

Step-6 : Transfer function of compensated system.

The lag compensator is connected in series with G(s) as shown in fig 6.3.2.

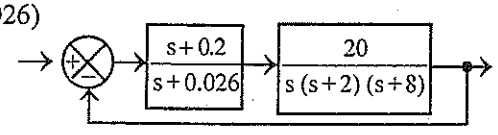

Fig 6.3.2 : Block diagram of lag compensated system.

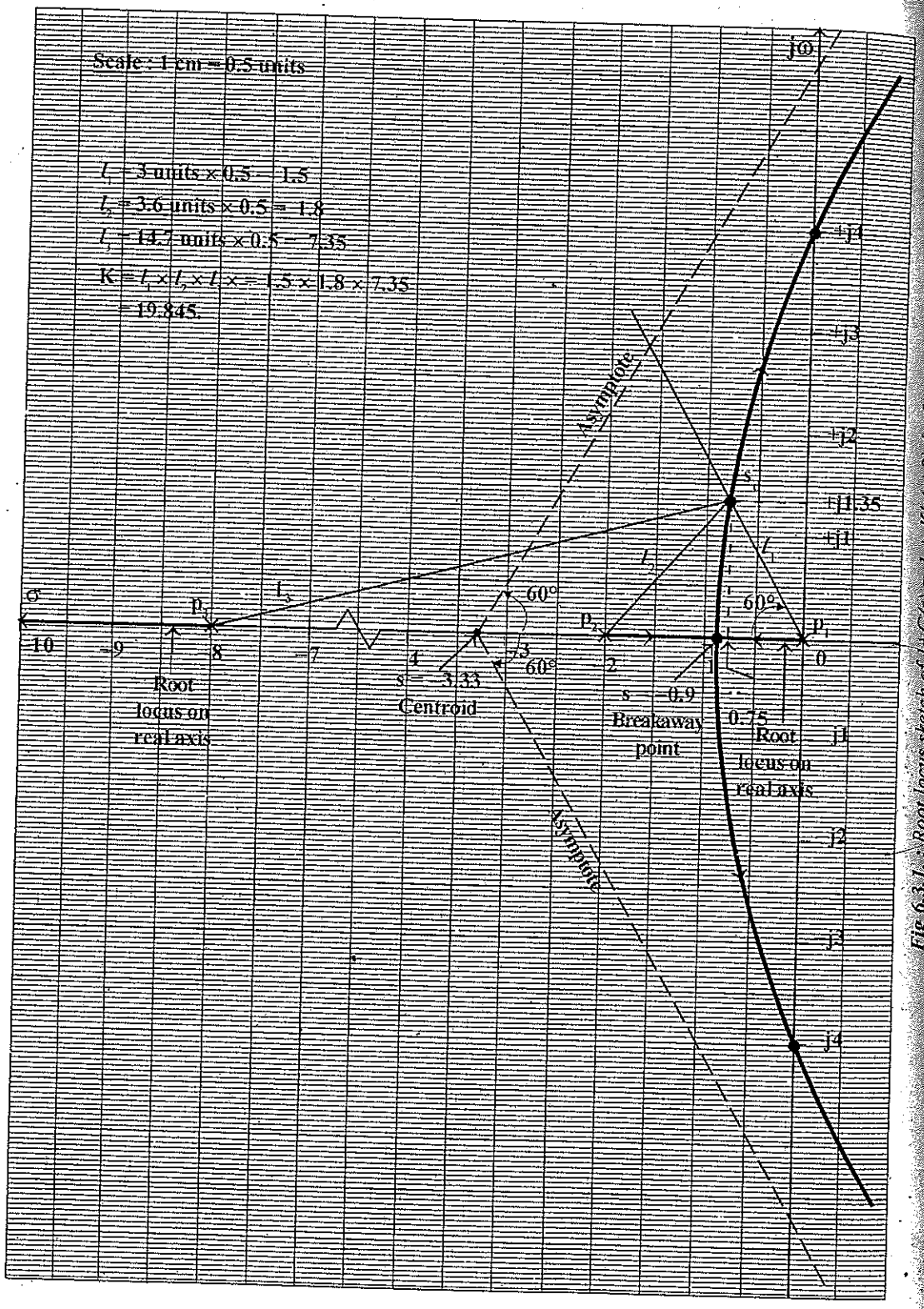

Fig. 6.3.1. : Root locus sketch of 1 + G(s) = 1 + K/s(s+2) (s+8)

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Open loop transfer function of compensated system $\Big\}$ $G_o(s) = \dfrac{(s+0.2)}{(s+0.026)} \times \dfrac{20}{s(s+2)(s+8)}$

$$= \frac{20(s+0.2)}{s(s+2)(s+8)(s+0.026)}$$

Step-7 : Check steady state error of compensated system

Velocity error constant of compensated system $\Big\}$ $K_{vc} = \underset{s\to 0}{Lt}\ s\,G_o(s) = \underset{s\to 0}{Lt}\ s\dfrac{20(s+0.2)}{s(s+2)(s+8)(s+0.026)}$

$$= \frac{20 \times 0.2}{2 \times 8 \times 0.026} = 9.615$$

Steady state error of compensated system for unit ramp input $\Big\}$ $e_{ss} = \dfrac{1}{K_{vc}} = \dfrac{1}{9.615} = 0.104$

CONCLUSION

Since the steady state error of the compensated system is less than 0.125, the design is acceptable.

RESULT

Transfer function of lag compensator, $G_c(s) = (s+0.2)/(s+0.026)$.

Open loop transfer function of lag compensated system $\Big\}$ $G_o(s) = \dfrac{20(s+0.2)}{s(s+2)(s+8)(s+0.026)}$

EXAMPLE 6.4

The controlled plant of a unity feedback system is $G(s) = K/s(s+10)^2$. It is specified that velocity error constant of the system be equal to 20, while the damping ratio of the dominant roots be 0.707. Design a suitable cascade compensation scheme to meet the specifications.

SOLUTION

Step-1 : Sketch the root locus of uncompensated system

To find poles of open loop system

Given that, $G(s) = K/s(s+10)^2$

The poles of open loop transfer function are the roots of the equation $s(s+10)^2 = 0$.

$\therefore$ The poles are lying at $s = 0, -10, -10$.

Let us denote poles by, p_1, p_2 and p_3. Here, $p_1 = 0$, $p_2 = -10$ and $p_3 = -10$.

To find root locus on real axis

The entire negative real axis will be a part of root locus. Because if we choose a test point on negative real axis then to the right of this point we have odd number of real poles and zeros.

To find angles of asymptotes and centroid

Since there are three poles, the number of root locus branches are three. There is no finite zero and so all the three root locus branches will meet the zeros at infinity. Hence the number of asymptotes required is three.

Fig 6.3.1. : Root locus sketch of $1+G(s) = 1+K/s(s+2)(s+8)$.

Angles of asymptotes $= \frac{\pm 180°(2q+1)}{n-m}$; $q = 0,1,2,.....n-m.$

Here $n = 3$ and $m = 0$. $\therefore q = 0, 1, 2, 3$.

When, $q = 0$, Angles $= \frac{\pm 180°}{3} = \pm 60°$

When, $q = 1$, Angles $= \frac{\pm 180°(2+1)}{3} = \pm 180°$

$\therefore$ Angles of asymptotes are, $+60°, -60°$ and $\pm 180°$

$$\text{Centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n - m} = \frac{0 - 10 - 10}{3} = \frac{-20}{3} = -6.6$$

To find breakaway point

$$\text{Closed loop transfer function, } \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)} = \frac{\frac{K}{s(s+10)^2}}{1 + \frac{K}{s(s+10)^2}} = \frac{K}{s(s+10)^2 + K}$$

The characteristic equation is, $s(s + 10)^2 + K = 0$

$\therefore K = -s(s+10)^2 = -(s^3 + 20s^2 + 100s)$

On differentiating K with respect to s we get, $dK/ds = -(3s^2 + 40s + 100)$.

Put $dK / ds = 0$, $\therefore 3s^2 + 40s + 100 = 0$

$$s = \frac{-40 \pm \sqrt{40^2 - 4 \times 3 \times 100}}{2 \times 3} = \frac{-40 \pm 20}{6} = -3.33, -10.$$

When $s = -3.33$, $K = -[(-3.33)^3 + 20(-3.33)^2 + 100(-3.33)] = 148.14$

When $s = -10$, $K = -[(-10)^3 + 20(-10)^2 + 100(-10)] = 0$

For $s = -3.33$, the value of K is positive and real and so it is actual breakaway point.

To find crossing point of imaginary axis

The characteristic equation is, $s(s + 10)^2 + K = 0$

$\therefore s^3 + 20s^2 + 100s + K = 0$

Put $s = j\omega$.

$\therefore (j\omega)^3 + 20(j\omega)^2 + 100(j\omega) + K = 0 \Rightarrow -j\omega^3 - 20\omega^2 + j100\omega + K = 0$

On equating imaginary part to zero we get,

$-j\omega^3 + j100\omega = 0 \Rightarrow -j\omega^3 = -j100\omega \Rightarrow \omega^2 = 100$

$\therefore \omega = \pm\sqrt{100} = \pm 10$

Hence the root locus crosses the imaginary axis at $+j10$ and $-j10$. The complete root locus sketch is shown in fig 6.4.1.

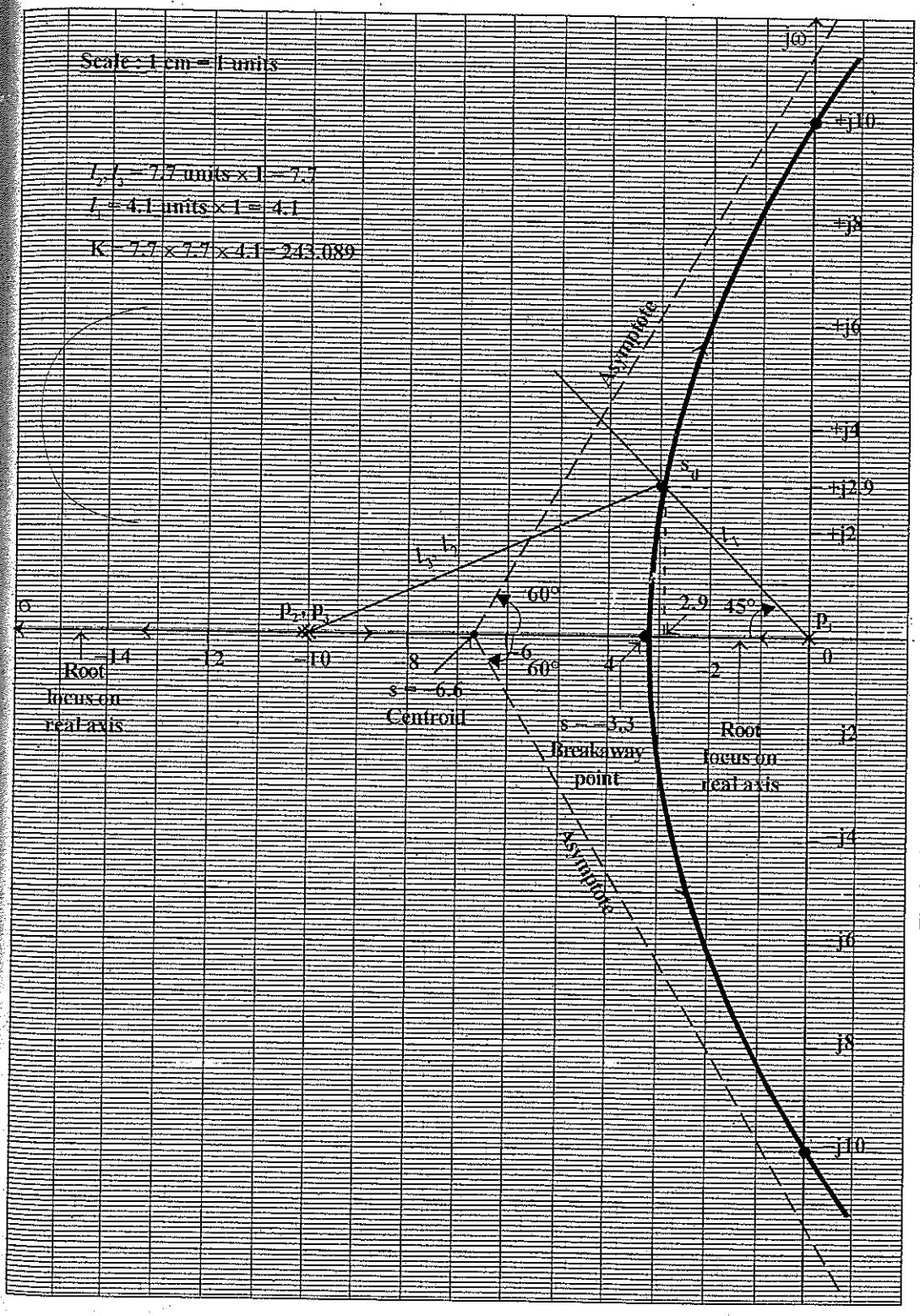

Fig 6.4.1. : Root locus sketch of 1+G(s) = 1 + K/s(s+10)².

Step-2 : Determine the dominant pole s_d.

Given that, $\zeta = 0.707$

$$\therefore \cos^{-1}\zeta = \cos^{-1}0.707 = 45.008 \approx 45^\circ$$

Draw a straight line at an angle of 45° with respect to real axis as shown in fig 6.4.1. The point of this line with root locus is the dominant pole, s_d.

From fig 6.4.1. we get, $s_d = -2.9 \pm j2.9$ (Dominant poles occur as conjugate poles).

Step-3 : To find gain K, at $s = s_d$

$$K = \frac{\text{Product of vector lengths from open loop poles to } s_d}{\text{Product of vector lengths from open loop zeros to } s_d}$$

Product of vector lengths from poles $= l_1 \times l_2 \times l_3$

From root locus plot of fig 6.4.1. we get,

$l_1 = 4.1$; $l_2 = l_3 = 7.7$

Note : Lengths should be measured to

Since there is no finite zero, the product of vector lengths from zeros is unity.

$$\therefore K = l_1 \times l_2 \times l_3 = 4.1 \times 7.7 \times 7.7 = 243.089 \approx 240$$

Step-4 : To find parameter, β

Given that, $G(s) - K/s(s + 10)^2 = 240/s(s + 10)^2$

Velocity error constant of uncompensated system $\Big\}$ $K_{vu} = \underset{s \to 0}{Lt}\ s\,G(s) = \underset{s \to 0}{Lt}\ s \dfrac{240}{s(s + 10)^2} = \dfrac{240}{10^2} = 2.4$

It is given that desired velocity error constant, K_{vd} should be 20. Let A be the factor by which is increased.

$$\therefore A = \frac{K_{vd}}{K_{vu}} = \frac{20}{2.4} = 8.33$$

Let, $\beta = 1.2 \times A = 1.2 \times 8.33 = 9.996 \approx 10$

Step-5 : Determine the transfer function of lag compensator

Zero of the compensator, $z_c = -1/T = 0.1 \times$ second pole of $G(s) = 0.1 \times (-10) = -1.0$

$$\therefore 1/T = 1.0 \text{ and } T = 1.0$$

Pole of the compensator, $p_c = \dfrac{-1}{\beta T} = \dfrac{-1}{10 \times 1} = -0.1$

$$\therefore \frac{1}{\beta T} = 0.1 \text{ and } \beta T = 10$$

Transfer function of lag compensator $\Big\}$ $G_c(s) = \dfrac{s + \frac{1}{T}}{s + \frac{1}{\beta T}} = \dfrac{(s + 1)}{(s + 0.1)}$

Step-6 : Determine the open loop transfer function of compensated system.

The block diagram of lag compensated system is shown in fig 6.4.2 and in this, the lag compensator is connected in series with G(s).

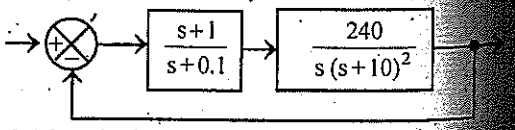

Fig 6.4.2 : Block diagram of lag compensated syst

$$\left.\begin{array}{l}\text{Open loop transfer function}\\ \text{of compensated system}\end{array}\right\} G_o(s) = \frac{(s+1)}{(s+0.1)} \times \frac{240}{s(s+10)^2} = \frac{240(s+1)}{s(s+10)^2(s+0.1)}$$

...p-7 : Check K_v of compensated system

$$\left.\begin{array}{l}\text{Velocity error constant}\\ \text{of compensated system}\end{array}\right\} K_{vc} = \lim_{s \to 0} s\, G_o(s)$$

$$= \lim_{s \to 0} s \frac{240(s+1)}{s(s+10)^2(s+0.1)} = \frac{240}{10^2 \times 0.1} = 24.$$

ONCLUSION

Since the velocity error constant of the compensated system is greater than the desired value, the design is accepted.

ESULT

$$\left.\begin{array}{l}\text{Transfer function of}\\ \text{lag compensator}\end{array}\right\} G_c(s) = \frac{(s+1)}{(s+0.1)}$$

$$\left.\begin{array}{l}\text{Open loop transfer function}\\ \text{of lag compensated system}\end{array}\right\} G_o(s) = \frac{240(s+1)}{s(s+10)^2(s+0.1)}$$

3 LEAD COMPENSATOR

A compensator having the characteristics of a lead network is called a lead compensator. If a usoidal signal is applied to the lead network, then in steady state the output will have a phase lead with pect to the input.

The lead compensation increases the bandwidth, which improves the speed of response and also uces the amount of overshoot. Lead compensation appreciably improves the transient response, whereas e is a small change in steady state accuracy. Generally, lead compensation is provided to make an able system as a stable system.

A lead compensator is basically a high pass filter and so it amplifies high frequency noise signals. If pole introduced by the compensator is not cancelled by a zero in the system, then lead compensation eases the order of the system by one.

LANE REPRESENTATION OF LEAD COMPENSATOR

The lead compensator has a zero at $s = -1/T$ and a pole at $-1/\alpha T$. The pole-zero plot of lead compensator is shown in 6.9. Here, $\alpha < 1$, so the zero is closer to the origin than the pole. e general form of lead compensator transfer function is given by uation (6.19),

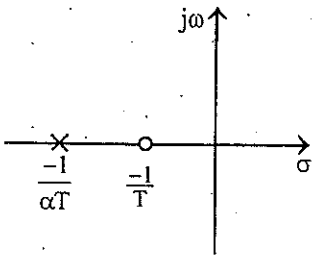

Fig 6.9 : Pole-zero plot of lead compensator.

$$G_c(s) = \frac{s + z_c}{s + p_c} = \frac{\left(s + \frac{1}{T}\right)}{\left(s + \frac{1}{\alpha T}\right)} \qquad \text{.....(6.19)}$$

where, $T > 0$ and $\alpha < 1$

Also we can express ϕ_m in terms of α and α in terms of ϕ_m as shown below.

$$\phi_m = \tan^{-1}\left(\frac{1-\alpha}{2\sqrt{\alpha}}\right) \quad \text{.....(6.34)}$$

$$\alpha = \frac{1-\sin\phi_m}{1+\sin\phi_m} \quad \text{.....(6.35)}$$

Note : The equations (6.33), (6.34) and (6.35) can be derived by a similar analysis shown in n 6.2 for lag compensator after replacing β by α.

EDURE FOR DESIGN OF LEAD COMPENSATOR USING BODE PLOT

The following steps may be followed to design a lead compensator using bode plot and to be connected in series with transfer function of uncompensated system, G(s).

The open loop gain K of the given system is determined to satisfy the requirement of the error constant.

The bode plot is drawn for the uncompensated system using the value of K, determined from the previous step. *[Refer Chapter-4 for the procedure to sketch bode plot].*

The phase margin of the uncompensated system is determined from the bode plot.

Determine the amount of phase angle to be contributed by the lead network by using the formula given below,

$$\phi_m = \gamma_d - \gamma + \epsilon$$

where,

ϕ_m = Maximum phase lead angle of the lead compensator

γ_d = Desired phase margin

γ = Phase margin of the uncompensated system

ϵ = Additional phase lead to compensate for shift in gain crossover frequency

Choose an initial choice of ϵ as 5^0

Note : If ϕ_m is more than 60° then realize the compensator as cascade of two lead compensator ach compensator contributing half of the required angle).

Determine the transfer function of lead compensator

Calculate α using the equation, $\alpha = \dfrac{1-\sin\phi_m}{1+\sin\phi_m}$

From the bode plot, determine the frequency at which the magnitude of $G(j\omega)$ is $-20\log 1/\sqrt{\alpha}$ db. This frequency is ω_m.

Calculate T from the relation, $\omega_m = \dfrac{1}{T\sqrt{\alpha}}$ $\quad \therefore T = \dfrac{1}{\omega_m\sqrt{\alpha}}$

Transfer function of lead compensator $\Big\}$ $G_c(s) = \dfrac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}} = \dfrac{\alpha(1+sT)}{(1+\alpha sT)}$

Step-6 : Determine the open loop transfer function of compensated system.

The lag compensator is connected in series with G(s) as shown in fig 6.12. When the lead network is inserted in series with the plant, the open loop gain of the system is attenuated by the factor α ($\because \alpha<1$), so an amplifier with the gain of $1/\alpha$ has to be introduced in series with the compensator to nullify the attenuation caused by the lead compensator.

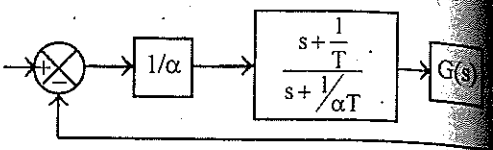

Fig 6.12 : Block diagram of lead compensated system.

$$\text{Open loop transfer function of the overall system } G_0(s) = \frac{1}{\alpha} \times \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}} \times G(s)$$

$$= \frac{1}{\alpha} \times \frac{\alpha(1 + sT)}{(1 + s\alpha T)} \times G(s) = \frac{(1 + sT)}{(1 + s\alpha T)} \times G(s)$$

Step-7 : Verify the design.

Finally the Bode plot of the compensated system is drawn and verify whether it satisfies given specifications. If the phase margin of the compensated system is less than the requ phase margin then repeat step 4 to 10 by taking ϵ as 5° more than the previous design

PROCEDURE FOR DESIGN OF LEAD COMPENSATOR USING ROOT LOCUS

The following steps may be followed to design a lead compensator using root locus and to connected in series with transfer function of uncompensated system, G(s).

Step-1 : Determine the dominant pole, s_d from the given specifications.

Dominant pole, $s_d = -\zeta \omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$

Note : If ζ alone is specified and ω_n is not available, then draw the root locus and from the roo locus find the dominant pole. Refer example 6.8.

Step-2 : Mark the poles and zeros of open loop transfer function and the dominant pole on the s-plan Let the dominant pole be point P.

Step-3 : Determine the angle to be contributed by lead network.

Let, ϕ = Angle to be contributed by lead network to make the point P as a point on root locus.

Draw vectors from all open loop poles and zeros to point P. Measure the angle contributed the vectors. *[For procedure to find angle contribution by vectors refer root locus in Chapter-*

$$\text{Now, } \phi = \begin{pmatrix}\text{Sum of angles}\\ \text{contributed by poles}\\ \text{of uncompensated system}\end{pmatrix} - \begin{pmatrix}\text{Sum of angles}\\ \text{contributed by zeros}\\ \text{of uncompensated system}\end{pmatrix} \pm n\,180°$$

where n is an odd integer so that n180° is nearest to the difference between angles contribute by poles and zeros.

Note : If the angle to be contributed is more than 60° then realise the compensator as cascade of two lead compensators with each compensator contributing half of the required angle.

Step-4 : Determine the pole and zero of the lead compensator.

Let point O be the origin of s-plane and point P be the dominant pole. Draw straight lines OP and AP such that AP is parallel to x-axis as shown in fig 6.13.

Draw a line PC so as to bisect the angle APO [∠APO] where the point C is on the real axis. With line PC as reference, draw angles BPC and CPD such that each equal to $\phi/2$. Here the points B and D are located on the real axis.

Now the point B is the location of the pole of the compensator ($-1/\alpha T$) and the point D is the location of the zero of the compensator ($-1/T$). From the values of point D and B compute T and α.

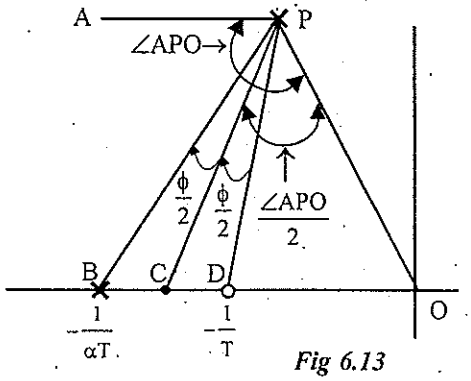

Fig 6.13

Step-5 : Determine the transfer function of lead compensator

$$\text{Transfer function of lead compensator } \Big\} \; G_c(s) = \frac{\left(s + \frac{1}{T}\right)}{\left(s + \frac{1}{\alpha T}\right)} = \alpha \frac{(1 + sT)}{(1 + s\alpha T)}$$

Step-6 : Determine open loop transfer function of lead compensated system.

The lead compensator is connected in series with the plant as shown in fig 6.14.

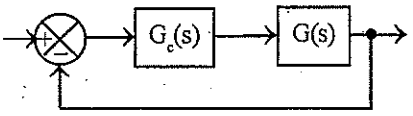

***Fig 6.14 :** Block diagram of lead compensated system.*

Open loop transfer function of compensated system, $G_0(s) = G_c(s)\, G(s)$.

The open loop gain K is given by the value of gain of $s = s_d$. The value of gain, K is determined from pole-zero plot of lead compensated system and by using the magnitude condition given below.

$$K = \frac{\text{Product of vector lengths from all poles to } s = s_d}{\text{Product of vector lengths from all zeros to } s = s_d}$$

Note : The length of vectors should be measured to scale. For details of magnitude condition, refer root locus in chapter-5.

Step-7 : Check whether the compensated system satisfies the error requirement. If the error requirement is satisfied then the design is accepted. Otherwise repeat the design by altering the location of poles and zeros by trial and error, without changing the value of ϕ.

Note : *If the open loop gain K is specified in the problem, then take the gain at $s = s_d$ as K_c. Find a parameter, A where $A = K_1/K$. Now introduce an amplifier with gain, A in cascade with compensator to account for reduction in gain due to attenuation by parameter, α. Now, $G_0(s) = A\, G_c(s)\, G(s)$.*

EXAMPLE 6.5

Design a phase lead compensator for the system shown in fig 6.5.1 to satisfy the following specifications. (i) The phase margin of the system $\geq 45°$. (ii) Steady state error for a unit ramp input $\leq 1/15$. (iii) The gain crossover frequency of the system must be less than 7.5 rad/sec.

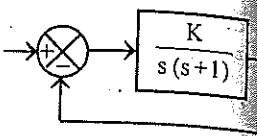

Fig 6.5.1

SOLUTION

Step-1 : Determine K .

Given that, steady state error, $e_{ss} \leq 1/15$ for unit ramp input

When the input is unit ramp, $e_{ss} = 1/K_v = 1/15$. $\therefore K_v = 15$

By definition of velocity error constant we get, $K_V = \underset{S\to 0}{Lt}\ s\,G(s)\,H(s)$

Here, $G(s) = \dfrac{K}{s(s+1)}$ and $H(s) = 1$. $\therefore K_v = \underset{s\to 0}{Lt}\ s\dfrac{K}{s(s+1)} = K$

Step-2 : Draw bode plot .

Given that, $G(s) = \dfrac{K}{s(s+1)} = \dfrac{15}{s(s+1)}$

Let $s = j\omega$, $\therefore G(j\omega) = \dfrac{15}{j\omega(1+j\omega)}$

MAGNITUDE PLOT

The corner frequency is, $\omega_{cl} = 1$ rad/sec.

The various terms of $G(j\omega)$ are listed in table-6. Also the table shows the slope contributed by each term and the change in slope at the corner frequency.

TABLE-1

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\dfrac{15}{j\omega}$	–	–20	–
$\dfrac{1}{1+j\omega}$	$\omega_{cl} = 1$	–20	–20 + (–20) = –40

Choose a low frequency ω_l such that $\omega_l < \omega_{cl}$ and choose a high frequency ω_h such that $\omega_h > \omega_{cl}$.

Let, $\omega_l = 0.1$ rad/sec and $\omega_h = 10$ rad/sec.

Let, $A = |G(j\omega)|$ in db

Let us calculate A at ω_l, ω_{cl} and ω_h.

$$\text{At } \omega = \omega_l = 0.1 \text{ rad/sec}, \quad A = 20 \log\left|\frac{15}{j\omega}\right| = 20 \log \frac{15}{0.1} = 43.5 \text{ db} \approx 44 \text{ db}$$

$$\text{At } \omega = \omega_{c1} = 1 \text{ rad/sec}, \quad A = 20 \log\left|\frac{15}{j\omega}\right| = 20 \log \frac{15}{1} = 23.5 \text{ db} \approx 24 \text{ db}$$

$$\text{At } \omega = \omega_h = 10 \text{ rad/sec}, \quad A = \left[\text{slope from } \omega_{c1} \text{ to } \omega_h \times \log \frac{\omega_h}{\omega_{c1}}\right] + A_{(\text{at } \omega = \omega_{c1})}$$

$$= -40 \times \log \frac{10}{1} + 24 = -16 \text{ db}$$

Let the points a, b and c be the points corresponding to frequencies ω_l, ω_{c1}, and ω_h respectively on the magnitude plot. In a semilog graph sheet choose appropriate scales and fix the points a, b and c. Join the points by straight lines and mark the slope on respective region. Magnitude plot is shown in fig 6.5.2.

PHASE PLOT

The phase angle of G(jω) as a function of ω is given by,

$$\phi = \angle G(j\omega) = -90° - \tan^{-1}\omega$$

The phase angle of G(jω) are calculated for various values of ω and listed in table-2.

TABLE-2

ω rad/sec	0.1	0.5	1	2	5	10
ϕ deg	−96	−117	−135	−153	−169	−174

On the same semilog sheet take another y-axis, choose appropriate scale and draw phase plot as shown in fig 6.5.2.

Step-3 : Determine the phase margin of uncompensated system.

Let, ϕ_{gc} = Phase of G(jω) at gain crossover frequency (ω_{gc}).

γ = Phase margin of uncompensated system.

From the bode plot of uncompensated system we get, $\phi_{gc} = -167°$.

Now, $\gamma = 180° + \phi_{gc} = 180° - 167° = 13°$

The system requires a phase margin of 45°, but the available phase margin is 13° and so lead compensation should be employed to improve the phase margin.

Step-4 : Find ϕ_m

The desired phase margin, $\gamma_d \geq 45°$

Let additional phase lead required, $\epsilon = 5°$

Maximum lead angle, $\phi_m = \gamma_d - \gamma + \epsilon = 45° - 13° + 5° = 37°$

Step-5 : Determine the transfer function of lead compensator.

$$\alpha = \frac{1 - \sin\phi_m}{1 + \sin\phi_m} = \frac{1 - \sin 37°}{1 + \sin 37°} = 0.2486 \approx 0.25$$

The db magnitude corresponding to $\omega_m = -20 \log \frac{1}{\sqrt{\alpha}} = -20 \log \frac{1}{\sqrt{0.25}} = -6$ db.

From the bode plot of uncompensated system the frequency ω_m corresponding to a d of –6 db is found to be 5.6 rad/sec.

$\therefore \omega_m = 5.6$ rad/sec.

Now, $T = \frac{1}{\omega_m \sqrt{\alpha}} = \frac{1}{5.6\sqrt{0.25}} = 0.357 \approx 0.36$

Transfer function of the lead compensator $G_c(s) = \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}} = \alpha \frac{(1 + sT)}{(1 + s\alpha T)} = 0.25 \frac{(1 + 0.36s)}{(1 + 0.09s)}$

Step-6 : Open loop transfer function of compensated system.

The block diagram of the lead compensated system is shown in fig 6.5.3.

The compensator will provide an attenuation of α. To compensate for that, an amplifier of gain $1/\alpha$ is introduced in series with compensator.

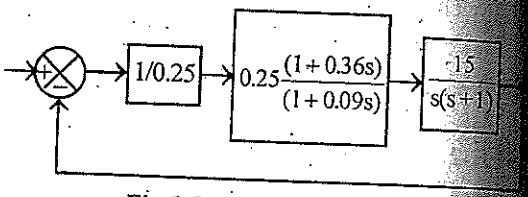

Fig 1.5.3 : Block diagram of lead compensated system.

Open loop transfer function of compensated system $G_0(s) = \frac{1}{0.25} \times \frac{0.25\,(1 + 0.36s)}{(1 + 0.09s)} \times \frac{15}{s\,(s + 1)}$

$= \frac{15\,(1 + 0.36s)}{s\,(1 + 0.09s)\,(1 + s)}$

Step-7 : Draw the bode plot of compensated system to verify the design.

Put $s = j\omega$ in $G_0(s)$, $\therefore G_0(j\omega) = \frac{15\,(1 + j0.36\omega)}{j\omega\,(1 + j0.09\omega)\,(1 + j\omega)}$

MAGNITUDE PLOT

The corner frequencies are ω_{c1}, ω_{c2} and ω_{c3}.

$\omega_{c1} = \frac{1}{1} = 1$ rad / sec ; $\omega_{c2} = \frac{1}{0.36} = 2.8$ rad / sec ; $\omega_{c3} = \frac{1}{0.09} = 11.1$ rad / sec

The various terms of $G_0(j\omega)$ are listed in table-3. Also the table shows the slope contributed by each term and the change in slope at the corner frequency.

Choose a low frequency ω_l such that $\omega_l < \omega_{c1}$ and choose a high frequency ω_h such that $\omega_h > \omega_{c3}$.

Let $\omega_l = 0.1$ rad/sec and $\omega_h = 50$ rad/sec

Let $A_0 = |G_0(j\omega)|$ in db

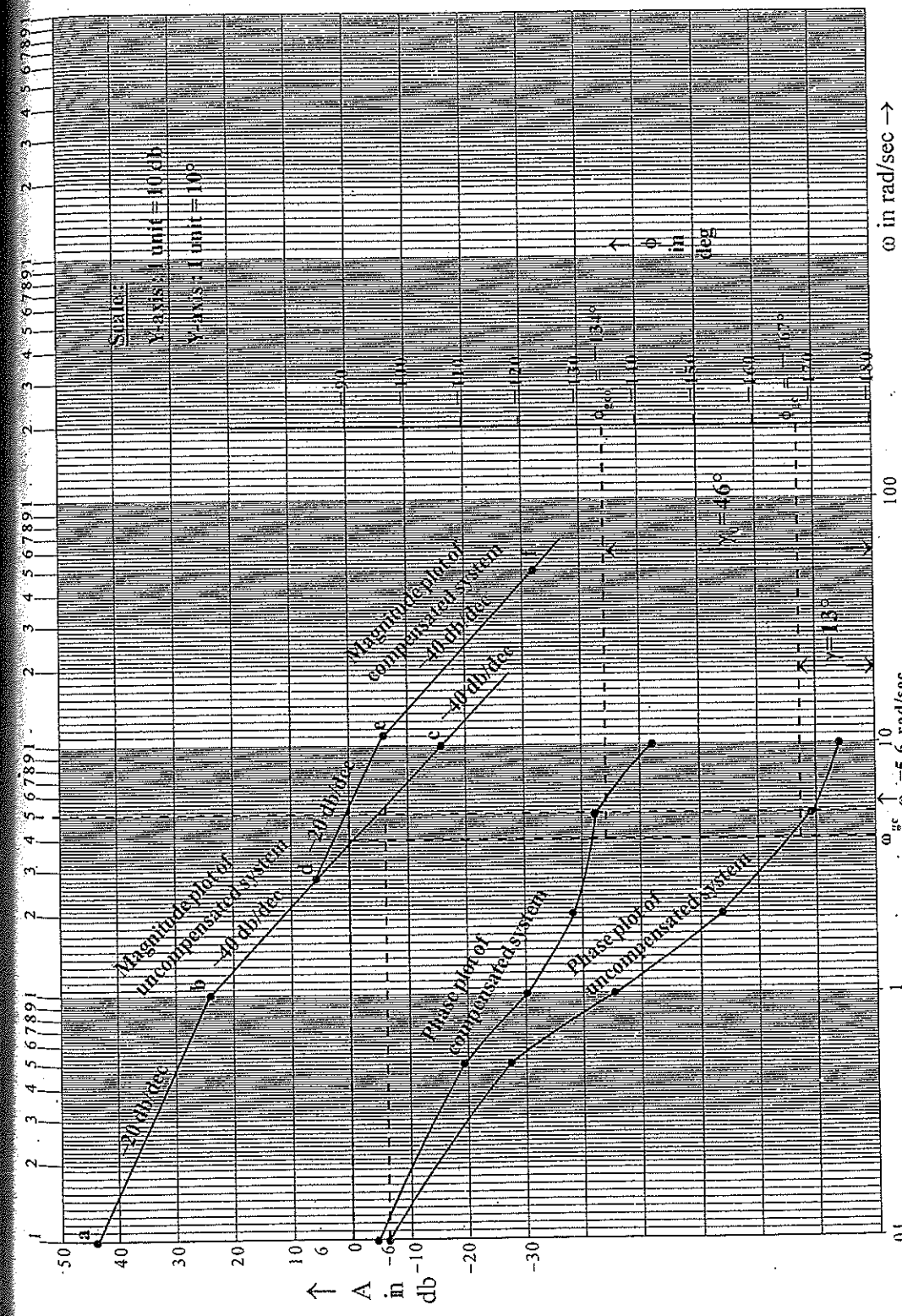

Fig 6.5.2 : Bode plot of $G(j\omega) = 15/j\omega(1 + j\omega)$.

$$\text{At } \omega = \omega_l = 0.1 \text{ rad/sec}, \quad A_0 = 20 \log\left|\frac{15}{j\omega}\right| = 20 \log \frac{15}{0.1} = 43.5 \text{ db} \approx 44 \text{ db}$$

$$\text{At } \omega = \omega_{c1} = 1 \text{ rad/sec}, \quad A_0 = 20 \log\left|\frac{15}{j\omega}\right| = 20 \log \frac{15}{1} = 23.5 \text{ db} \approx 24 \text{ db}$$

$$\text{At } \omega = \omega_{c2} = 2.8 \text{ rad/sec}, \quad A_0 = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + \begin{pmatrix}\text{gain at} \\ \omega = \omega_{c1}\end{pmatrix}$$

$$= -40 \times \log\frac{2.8}{1} + 24 = 6 \text{ db}$$

$$\text{At } \omega = \omega_{c3} = 11.1 \text{ rad/sec}, \quad A_0 = \left[\text{slope from } \omega_{c2} \text{ to } \omega_{c3} \times \log\frac{\omega_{c3}}{\omega_{c2}}\right] + \begin{pmatrix}\text{gain at} \\ \omega = \omega_{c2}\end{pmatrix}$$

$$= -20 \times \log\frac{11.1}{2.8} + 6 = -6 \text{ db}$$

$$\text{At } \omega = \omega_h = 50 \text{ rad/sec}, \quad A_0 = \left[\text{slope from } \omega_{c3} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c3}}\right] + \begin{pmatrix}\text{gain at} \\ \omega = \omega_{c3}\end{pmatrix}$$

$$= -40 \times \log\frac{50}{11.1} + (-6) = -32 \text{ db}$$

TABLE-3

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\frac{15}{j\omega}$	–	–20	–
$\frac{1}{1+j\omega}$	$\omega_{c1} = 1$	–20	–20 –20 = –40
$1+j0.36\omega$	$\omega_{c2} = \frac{1}{0.36} = 2.8$	+20	–40+20 = –20
$\frac{1}{1+j0.09\omega}$	$\omega_{c3} = \frac{1}{0.09} = 11.1$	–20	–20–20=–40

Let the points a, b, d, e and f be the points corresponding to frequencies ω_l, ω_{c1}, ω_{c2}, ω_{c3} and ω_h respectively on the magnitude plot of compensated system. The magnitude plot of compensated system is drawn on the same semilog graph sheet by using the same scales as shown in fig 6.5.2.

PHASE PLOT

The phase angle of $G_0(j\omega)$ as a function of ω is given by,

$$\phi_0 = \angle G_0(j\omega) = \tan^{-1}0.36\omega - 90° - \tan^{-1}0.09\omega - \tan^{-1}\omega.$$

The phase angle of $G_0(j\omega)$ are calculated for various values of ω and listed in table- 4.

TABLE-4

ω rad/sec	0.1	0.5	1	2	5	10
ϕ_0 deg	−94	−109	−120	−128	−132	−142

In the same semilog sheet and by using the same scales, the phase plot of compensated system is sketched as shown in fig 6.5.2.

Let, ϕ_{gc0} = Phase of $G_0(j\omega)$ at new gain crossover frequency.

γ_0 = Phase margin of compensated system.

From the bode plot of compensated system we get, $\phi_{gc0} = -134°$.

Now, $\gamma_0 = 180° + \phi_{gc0} = 180° - 134° = 46°$

CONCLUSION

The phase margin of the compensated system is satisfactory. Hence the design is acceptable.

RESULT

The transfer function of lead compensator, $G_c(s) = \dfrac{0.25\,(1 + 0.36s)}{(1 + 0.09s)} = \dfrac{(s + 2.78)}{(s + 11.11)}$

$$\left.\begin{array}{l}\text{Open loop transfer function}\\ \text{of lead compensated system}\end{array}\right\} G_0(s) = \frac{15(1 + 0.36s)}{s\,(1 + 0.09s)\,(1 + s)}$$

EXAMPLE 6.6

Design a lead compensator for a unity feedback system with open loop transfer function, $G(s) = K/s\,(s+1)(s+5)$ to satisfy the following specifications (i) Velocity error constant, $K_v \geq 50$ (ii) Phase margin is $\geq 20°$.

SOLUTION

Step-1 : Determine K

Given that, $K_v \geq 50$, Let $K_v = 50$

By definition of velocity error constant, K_v we get,

$$K_v = \underset{s\to 0}{\text{Lt}}\; s\,G(s) = \underset{s\to 0}{\text{Lt}}\; s\frac{K}{s(s+1)\,(s+5)} = \frac{K}{5}$$

$\therefore\; K = 5 \times K_v = 5 \times 50 = 250.$

Step-2 : Draw bode plot

$$\text{Given that, } G(s) = \frac{K}{s\,(s+1)\,(s+5)} = \frac{250}{s\,(s+1)\,(s+5)}$$

$$= \frac{250}{s\,(1+s) \times 5 \times (1 + s/5)} = \frac{50}{s(1+s)\,(1+0.2s)}$$

Let $s = j\omega$, $\therefore\ G(j\omega) = \dfrac{50}{j\omega\,(1 + j\omega)\,(1 + j0.2\omega)}$

MAGNITUDE PLOT

The corner frequency are, $\omega_{c1} = 1$ rad/sec and $\omega_{c2} = 1/0.2 = 5$ rad/sec.

The various terms of $G(j\omega)$ are listed in table-6. Also the table shows the slope contribute each term and the change in slope at the corner frequencies.

TABLE-1

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\dfrac{50}{j\omega}$	–	–20	–
$\dfrac{1}{1+j\omega}$	$\omega_{c1} = 1$	–20	–20 + (–20) = –40
$\dfrac{1}{1+j0.2\omega}$	$\omega_{c2} = \dfrac{1}{0.2} = 5$	–20	–40 – 20 = –60

Choose a low frequency ω_l such that $\omega_l < \omega_{c1}$ and choose a high frequency ω_h such that $\omega_h > \omega$

Let $\omega_l = 0.5$ rad/sec and $\omega_h = 10$ rad/sec

Let $A = |G(j\omega)|$ in db

$$\text{At } \omega = \omega_l,\quad A = 20 \log\left|\frac{50}{j\omega}\right| = 20 \log \frac{50}{0.5} = 40 \text{ db}$$

$$\text{At } \omega = \omega_{c1},\quad A = 20 \log\left|\frac{50}{j\omega}\right| = 20 \log \frac{50}{1} = 34 \text{ db}$$

$$\text{At } \omega = \omega_{c2},\quad A = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log \frac{\omega_{c2}}{\omega_{c1}}\right] + A_{(\text{at } \omega = \omega_{c1})}$$

$$= -40 \times \log \frac{5}{1} + 34 = 6 \text{ db}$$

$$\text{At } \omega = \omega_h,\quad A = \left[\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log \frac{\omega_h}{\omega_{c2}}\right] + A_{(\text{at } \omega = \omega_{c2})}$$

$$= -60 \times \log \frac{10}{5} + 6 = -12 \text{ db}$$

Let the points a, b, c and d be the points corresponding to frequencies ω_l, ω_{c1}, ω_{c2} and ω_h respectively on the magnitude plot. In a semilog graph sheet choose appropriate scales and fix the points a, b, c and d. Join the points by straight lines and mark the slope on the respective region. The magnitude plot is shown in fig 6.6.2.

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HASE PLOT

The phase angle of G(jω) as a function of ω is given by,

$$\phi = \angle G(j\omega) = -90° - \tan^{-1}\omega - \tan^{-1}0.2\omega$$

The phase angle of G(jω) are calculated for various values of ω and listed in table-2.

ABLE-2

ω rad/sec	0.1	0.5	1.0	5	10
φ deg	−96°	−122	−146	−214	−238

On the same semilog sheet take another y-axis, choose appropriate scale and draw phase plot as shown in fig 6.6.2.

tep-3 : Determine the phase margin

Let, ϕ_{gc} = Phase of G(jω) at gain crossover frequency.

γ = Phase margin of uncompensated system.

From the bode plot of uncompensated system we get, $\phi_{gc} = -224°$.

Now, $\gamma = 180° + \phi_{gc} = 180° - 224° = -44°$

The phase margin of the system is negative and so the system is unstable. Hence lead compensation is required to make the system stable and to have a phase margin of 20°.

tep-4 : Find ϕ_m

The desired phase margin, $\gamma_d \geq 20°$

Let additional phase lead required, $\epsilon = 5°$

Maximum lead angle, $\phi_m = \gamma_d - \gamma + \epsilon = 20° - (-44°) + 5° = 69°$

Since the lead angle required is greater than 60°, we have to realise the lead compensator as cascade of two lead compensators with each compensator providing half of the required phase lead angle.

$$\therefore \phi_{in} = \frac{69°}{2} = 34.5°$$

tep-5 : Determine the transfer function of lead compensator

$$\alpha = \frac{1 - \sin\phi_{in}}{1 + \sin\phi_m} = \frac{1 - \sin 34.5°}{1 + \sin 34.5°} = 0.28$$

The db magnitude corresponding to ω_m $= -20 \log \frac{1}{\sqrt{a}} = -20 \log \frac{1}{\sqrt{0.28}} = -5.5$ db.

From the bode plot of uncompensated system the frequency, ω_m corresponding to a db gain of −5.5 db is found to be 7.8 rad/sec.

$$\therefore \omega_m = 7.8 \text{ rad/sec.}$$

$$\text{Now, } T = \frac{1}{\omega_m \sqrt{\alpha}} = \frac{1}{7.8\sqrt{0.28}} = 0.24$$

$$\left.\begin{matrix}\text{Transfer function of} \\ \text{the lead compensator}\end{matrix}\right\} G_c(s) = \frac{\left(s + \frac{1}{T}\right)^2}{\left(s + \frac{1}{\alpha T}\right)^2} = \alpha^2 \frac{(1 + sT)^2}{(1 + s\alpha T)^2}$$

$$= (0.28)^2 \frac{(1 + 0.24s)^2}{(1 + 0.28 \times 0.24s)^2} = 0.0784 \frac{(1+0.24s)^2}{(1+0.067s)^2}$$

Step-6 : Open loop transfer function of compensated system.

The block diagram of the compensated system is shown in fig 6.6.1.

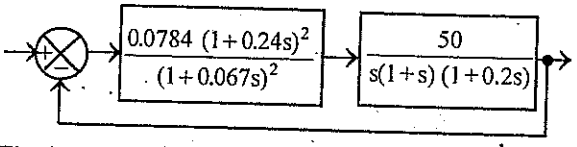


Fig 6.6.1 : *Block diagram of lead compensated system.*

The attenuation provided by the compensator can be retained to reduce the large value of open loop gain, so that the unstable system can be easily brought to stable region.

Let $G_0(s)$ be open loop transfer function of compensated system.

$$G_0(s) = \frac{0.0784\ (1 + 0.24s)^2}{(1 + 0.067s)^2} \times \frac{50}{s\ (1 + s)\ (1 + 0.2s)} = \frac{4\ (1 + 0.24s)^2}{s\ (1 + s)\ (1 + 0.2s)\ (1 + 0.067s)^2}$$

Step-7 : Draw the bode plot of compensated system to verify the design.

$$\text{Put, } s = j\omega \text{ in } G_0(s), \quad \therefore G_0(j\omega) = \frac{4\ (1 + j0.24\omega)^2}{j\omega\ (1 + j\omega)\ (1 + j0.2\omega)\ (1 + j0.067\omega)^2}$$

MAGNITUDE PLOT

The corner frequencies are ω_{c1}, ω_{c2}, ω_{c3} and ω_{c4}.

$$\omega_{c1} = 1\ ;\ \omega_{c2} = \frac{1}{0.24} = 4.2\ ;\ \omega_{c3} = \frac{1}{0.2} = 5\ ;\ \omega_{c4} = \frac{1}{0.06}$$

The various terms of $G_0(j\omega)$ are listed in table-3. Also the table shows the slope contributed each term and the change in slope at the corner frequency.

Choose a low frequency ω_l such that $\omega_l < \omega_{c1}$ and choose a high frequency ω_h such that $\omega_h >$

Let $\omega_l = 0.5$ rad/sec and $\omega_h = 30$ rad/sec

Let $A_0 = |G_0(j\omega)|$ in db

$$\text{At } \omega = \omega_l, \quad A_0 = 20 \log \left|\frac{4}{j\omega}\right| = 20 \log \frac{4}{0.5} = 18 \text{ db}$$

$$\text{At } \omega = \omega_{c1},\quad A_0 = 20\log\left|\frac{4}{j\omega}\right| = 20\log\frac{4}{1} = 12 \text{ db}$$

$$\text{At } \omega = \omega_{c2},\quad A_0 = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{ci}}\right] + A_0 \text{ at } (\omega = \omega_{c1})$$

$$= -40 \times \log\frac{4.2}{1} + 12 = -13 \text{ db}$$

$$\text{At } \omega = \omega_{c3},\quad A_0 = \left[\text{slope from } \omega_{c2} \text{ to } \omega_{c3} \times \log\frac{\omega_{c3}}{\omega_{c2}}\right] + A_0 \text{ at } (\omega = \omega_{c2})$$

$$= 0 \times \log\frac{5}{4.2} + (-13) = -13 \text{ db}$$

$$\text{At } \omega = \omega_{c4},\quad A_0 = \left[\text{slope from } \omega_{c3} \text{ to } \omega_{c4} \times \log\frac{\omega_{c4}}{\omega_{c3}}\right] + A_0 \text{ at } (\omega = \omega_{c3})$$

$$= -20 \times \log\frac{15}{5} + (-13) = -22.5 \text{ db} \approx -23 \text{ db}$$

$$\text{At } \omega = \omega_h,\quad A_0 = \left[\text{slope from } \omega_{c4} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c4}}\right] + A_0 \text{ at } (\omega = \omega_{c4})$$

$$= -60 \times \log\frac{30}{15} + (-23) = -41 \text{ db}$$

TABLE-3

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$4/j\omega$	–	−20	–
$\frac{1}{1+j\omega}$	$\omega_{c1} = 1$	−20	−20 −20 = −40
$(1+j0.24\omega)^2$	$\omega_{c2} = \frac{1}{0.24} = 4.2$	+40	−40 + 40 = 0
$\frac{1}{1+j0.2\omega}$	$\omega_{c3} = \frac{1}{0.2} = 5$	−20	0 −20 = −20
$\frac{1}{(1+j0.067\omega)^2}$	$\omega_{c4} = \frac{1}{0.067} = 15$	−40	−20 − 40 = −60

Let the points e, f, g, h, i and j be the points corresponding to frequencies ω_l, ω_{c1}, ω_{c2}, ω_{c3}, ω_{c4} and ω_h respectively on the magnitude plot of compensated system. The magnitude plot of compensated system is drawn on the same semilog graph sheet by using the same scales as shown in fig 6.5.2.

PHASE PLOT

The phase angle of $G_0(j\omega)$ as a function of ω is given by,

$$\phi_0 = \angle G_0(j\omega) = 2\tan^{-1}0.24\omega - 90° - \tan^{-1}\omega - \tan^{-1}0.2\omega - 2\tan^{-1}0.067\omega.$$

The phase angle of $G_0(j\omega)$ are calculated for various values of ω and listed in table - 4.

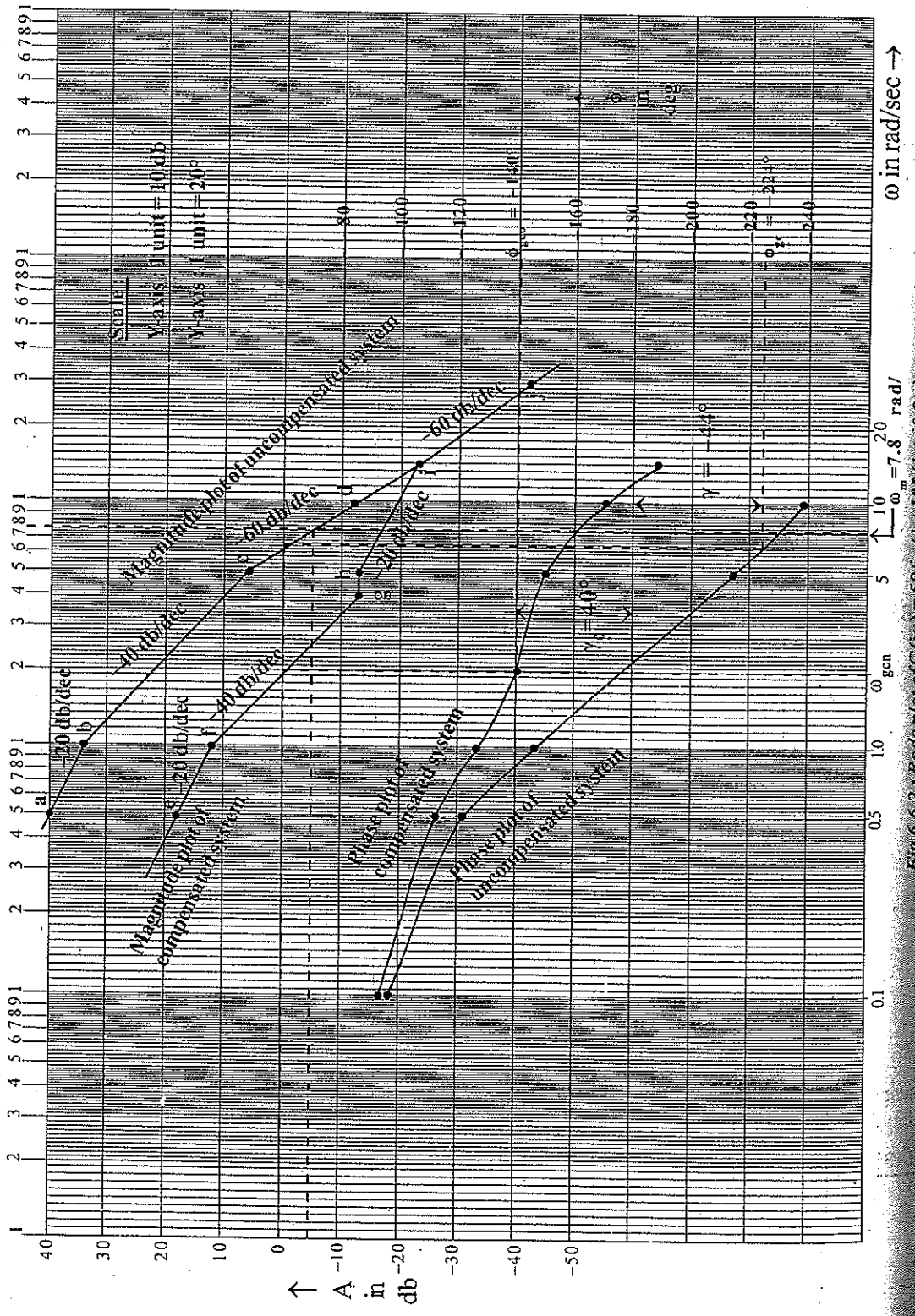


Fig 6.6.2 : Bode plot of G(jω) = 50/jω(1 + 0.5j)(1 + j0.2ω)

TABLE-4

ω rad/sec	0.1	0.5	6.0	2.0	5	10	15
$\angle G_0(j\omega)$ deg	−94	−112	−127°	−139 ≈ −140	−150	−171°	−189 ≈ −188

In the same semilog sheet and by using the same scales, the phase plot of compensated system is sketched as shown in fig 6.6.2.

Let, ϕ_{gc0} = Phase of $G_0(j\omega)$ at new gain crossover frequency (ω_{gcn}).

and γ_0 = Phase margin of compensated system.

From the bode plot of compensated system we get, $\phi_{gc0} = -140°$.

Now, $\gamma_0 = 180° + \phi_{gc0} = 180° - 140° = 40°$

CONCLUSION

The phase margin of the compensated system is satisfactory. Hence the design is acceptable.

RESULT

The transfer function of lead compensator $G_c(s) = \dfrac{0.0784\,(1 + 0.24s)^2}{(1 + 0.067s)^2} = \dfrac{(s + 4.17)^2}{(s + 14.92)^2}$.

Open loop transfer function of lead compensated system $G_0(s) = \dfrac{4\,(1 + 0.24s)^2}{s\,(1 + s)\,(1 + 0.2s)\,(1 + 0.067s)^2}$

Fig 6.6.2 : Bode plot of $G(j\omega) = 50/j\omega(1+j\omega)(1+j0.2\omega)$.

EXAMPLE 6.7

Consider a unity feedback system with open loop transfer function, G(s) = K/s(s+8). Design a lead compensator to meet the following specifications. (i) Percentage peak overshoot = 9.5%. (ii) Natural frequency of oscillation, ω_n = 12 rad/sec. (iii) Velocity error constant, $K_v \geq 10$.

SOLUTION

Step-1 : Determine the dominant pole, s_d.

Dominant pole, $s_d = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$.

Given that, ω_n = 12 rad/sec and $\%M_p = 9.5\%$.

We know that, $\%M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100 \quad \therefore e^{-\zeta\pi/\sqrt{1-\zeta^2}} = 0.095$.

On taking natural log we get, $-\zeta\pi / \sqrt{1-\zeta^2} = ln\,(0.095)$

On squaring we get, $\frac{\zeta^2\pi^2}{1-\zeta^2} = (ln\ 0.095)^2 = 5.54$.

$$\therefore \zeta^2\pi^2 = 5.54 - 5.54\zeta^2 \quad \Rightarrow \quad \zeta^2\pi^2 + 5.54\zeta^2 = 5.54 \quad \Rightarrow \quad \zeta^2(\pi^2 + 5.54) = 5.54$$

$$\therefore \zeta = \sqrt{\frac{5.54}{\pi^2 + 5.54}} = 0.6$$

$$\therefore s_d = -(0.6 \times 12) \pm j12 \times \sqrt{1-0.6^2} = -7.2 \pm j9.6$$

Step-2 : Draw the pole-zero plot

The pole-zero plot of open loop transfer function is shown fig 6.7.1. Poles are represented by the symbol "x". The pole at point P is the dominant pole, s_d.

Step-3 : To find the angle to be contributed by lead network.

Let ϕ = Angle to be contributed by lead network to make point, P as a point on root locus.

$$\text{Now, } \phi = \begin{pmatrix}\text{sum of angles}\\ \text{contributed by poles}\\ \text{of uncompensated system}\end{pmatrix} - \begin{pmatrix}\text{sum of angles}\\ \text{contributed by zeros}\\ \text{of uncompensated system}\end{pmatrix} \pm n180°$$

From fig 6.7.1, we get,

$$\left.\begin{matrix}\text{Sum of angles contributed}\\ \text{by poles of uncompensated system}\end{matrix}\right\} = \theta_1 + \theta_2 = 127° + 85° = 212°$$

Since there is no finite zero in uncompensated system, there is no angle contribution by zeros.

$\therefore \phi = 212° \pm n180°$

Let $n = 1$, $\quad \therefore \phi = 212° - 180° = 32°$.

Step-4 : To find the pole and zero of the compensator

Draw a line AP parallel to x-axis as shown in fig 6.7.1. The bisector PC is drawn to bisect the angle APO. The angles CPD and BPC are constructed as shown in fig 6.7.1. Here $\angle CPD = \angle BPC = \phi/2 = 32°/2 = 16°$.

From fig 6.7.1.

Pole of the compensator, $p_c = -16.25$

Zero of the compensator, $z_c = -9.1$

We know that, $z_c = -1/T \quad \therefore T = 1/9.1 = 0.11$

We know that, $p_c = -1/\alpha T \quad \therefore \alpha T = 1/16.25$ (or) $\alpha = 1/(T \times 16.25) = 0.56$.

Step-5 : Determine the transfer function of lead compensator

$$\left.\begin{matrix}\text{Transfer function of}\\ \text{lead compensator}\end{matrix}\right\} G_c(s) = \frac{(s + \frac{1}{T})}{(s + \frac{1}{\alpha T})} = \frac{(s + 9.1)}{(s + 16.25)}$$

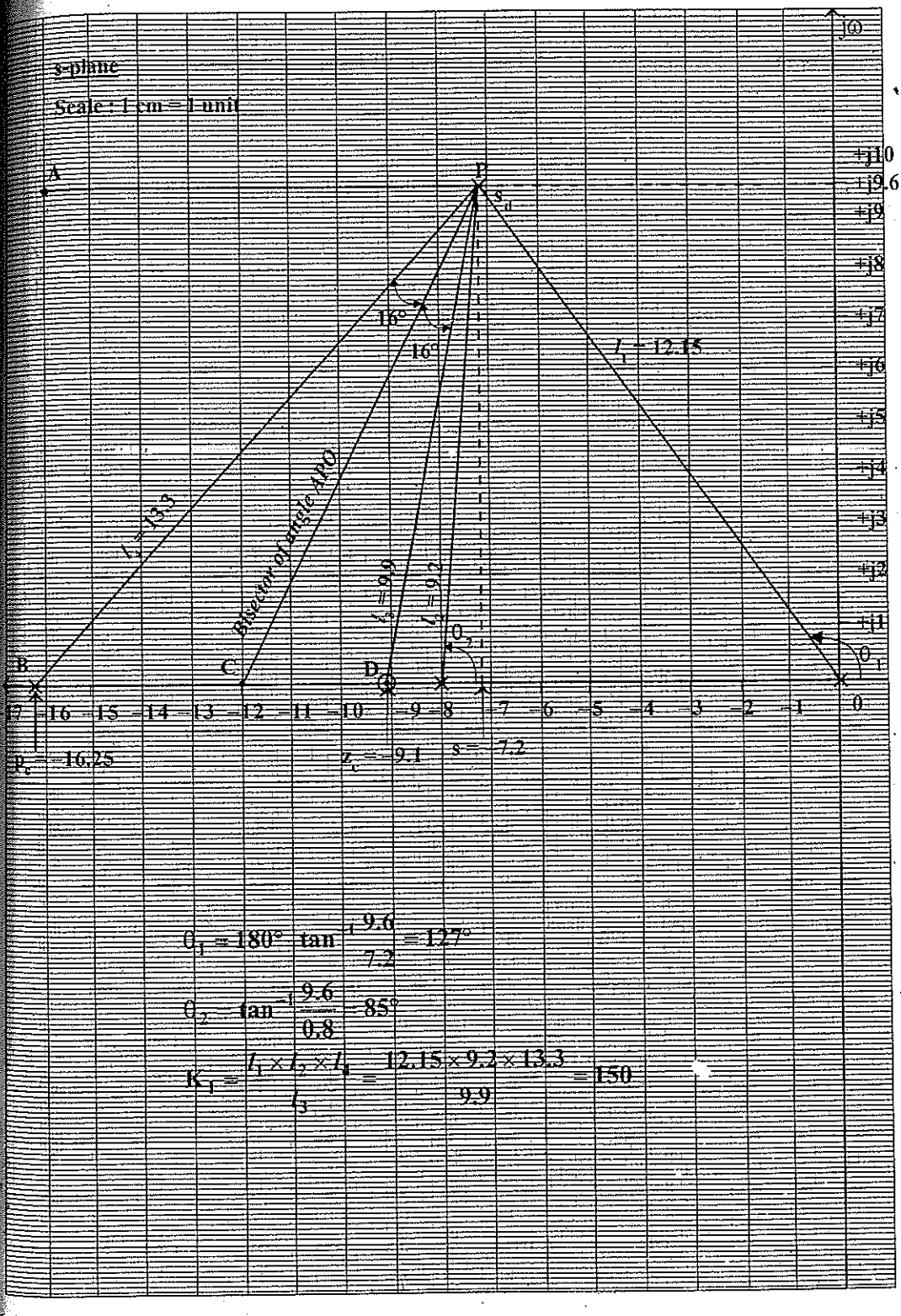


Fig 6.7.1. : Pole zero plot of open loop transfer function.

$$\theta_1 = 180° - \tan^{-1}\frac{9.6}{7.2} = 127°$$

$$\theta_2 = \tan^{-1}\frac{9.6}{0.8} = 85°$$

$$K_1 = \frac{l_1 \times l_2 \times l_4}{l_3} = \frac{12.15 \times 9.2 \times 13.3}{9.9} = 150$$

Step-6 : Determine the open loop transfer function of lead compensated system.

The block diagram of lead compensated system is shown in fig 6.7.2.

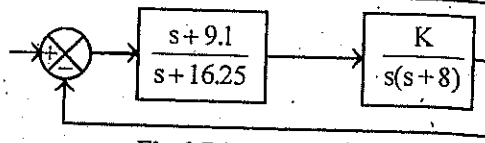


Fig 6.7.2 : Block diagram of lead compensated system.

$$\left.\begin{array}{l}\text{Open loop transfer function}\\ \text{of lead compensated system}\end{array}\right\} G_0(s) = \frac{(s+9.1)}{(s+16.25)} \times \frac{K}{s(s+8)} = \frac{K(s+9.1)}{s(s+8)(s+16.25)}$$

Here the value of K is given by the value of gain at the dominant pole, s_d on the root locus. By magnitude condition K is given by,

$$K = \frac{\text{Product of vector lengths from all poles to } s = s_d}{\text{Product of vector lengths from all zeros to } s = s_d}$$

From fig 6.7.1, we get,

$$K = \frac{l_1 \times l_2 \times l_4}{l_3} = \frac{12.15 \times 9.2 \times 13.3}{9.9} = 150$$

$$\therefore G_0(s) = \frac{150\,(s+9.1)}{s(s+8)(s+16.25)}$$

Step-7 : Check for error requirement

For the compensated system, the velocity error constant is given by,

$$K_v = \lim_{s\to 0} s\,G_0(s) = \lim_{s\to 0} s\frac{150\,(s+9.1)}{s\,(s+8)(s+16.25)} = \frac{150 \times 9.1}{8 \times 16.25} = 10.5$$

CONCLUSION

Since the velocity error constant of the compensated system, satisfies the requirement, design is acceptable.

RESULT

$$\left.\begin{array}{l}\text{Transfer function of}\\ \text{lead compensator}\end{array}\right\} G_c(s) = \frac{(s+9.1)}{(s+16.25)} = 0.56\frac{(1+0.11s)}{(1+0.06s)}$$

$$\left.\begin{array}{l}\text{Transfer function of}\\ \text{lead compensated system}\end{array}\right\} G_0(s) = \frac{150\,(s+9.1)}{s\,(s+8)(s+16.25)} = \frac{10.5\,(1+0.11s)}{s\,(1+0.125s)(1+0.06}$$

EXAMPLE 6.8

Design a lead compensator for a unity feedback system with open loop transfer funct $G(s) = K/s\,(s+4)\,(s+7)$ to meet the following specifications. (i) % Peak overshoot = 12.6 (ii) Natural frequency of oscillation, $\omega_n = 8$ rad/sec. (iii) Velocity error constant, $K_v \geq 2.5$.

SOLUTION

Step-1: Determine the dominant pole, s_d.

Dominant pole, $s_d = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$.

Given that, $\omega_n = 8$ rad/sec and $\%M_p = 12.63\%$.

We know that, $\%M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$; $\therefore e^{-\zeta\pi/\sqrt{1-\zeta^2}} = 0.1263$

On taking natural log we get, $-\zeta\pi / \sqrt{1-\zeta^2} = ln\,(0.1263)$

On squaring we get, $\frac{\zeta^2\pi^2}{1-\zeta^2} = (ln\,0.1263)^2 = 4.28$

$\therefore \zeta^2\pi^2 = 4.28 - 4.28\zeta^2 \Rightarrow \zeta^2\pi^2 + 4.28\zeta^2 = 4.28 \Rightarrow \zeta^2(\pi^2 + 4.28) = 4.28$

$\therefore \zeta = \sqrt{\frac{4.28}{\pi^2 + 4.28}} = 0.55$

$\therefore s_d = -(0.55 \times 8) \pm j8 \times \sqrt{1 - 0.55^2} = -4.4 \pm j6.68 = -4.4 \pm j6.7$

...2 : Draw the pole-zero plot

The pole-zero plot of open loop transfer function is shown fig 6.8.1. Poles are represented by the symbol "x". The pole at point P is the dominant pole, s_d.

...3 : To find the angle to be contributed by lead network.

Let ϕ = Angle to be contributed by lead network to make point, P as a point on root locus.

$$\text{Now, } \phi = \begin{pmatrix}\text{sum of angles}\\ \text{contributed by poles}\\ \text{of uncompensated system}\end{pmatrix} - \begin{pmatrix}\text{sum of angles}\\ \text{contributed by zeros}\\ \text{of uncompensated system}\end{pmatrix} \pm n180°$$

From fig 6.8.1, we get,

$$\left.\begin{matrix}\text{Sum of angles contributed}\\ \text{by poles of uncompensated system}\end{matrix}\right\} = \theta_1 + \theta_2 + \theta_3 = 123° + 93° + 69° = 285°$$

Since there is no finite zero in uncompensated system, there is no angle contribution by zeros.

$\therefore \phi = 285° \pm n180°$

Let $n = 1$,

$\therefore \phi = 285° - 180° = 105°$.

Since the angle contribution is more than 60°, the lead compensator is realised as cascade of two compensators with each compensator, contributing half of the required angle.

$\therefore \phi = 105/2 = 52.5° \approx 52°$

...p-4 : To find the poles and zeros of the compensator

Draw a line AP parallel to x-axis as shown in fig 6.8.6. The bisector PC is drawn to bisect the angle APO. The angles CPD and BPC are constructed as shown in fig 6.8.6. Here $\angle CPD = \angle BPC = \phi/2 = 52°/2 = 26°$.

From fig 6.8.1.

Pole of the compensator, $p_c = -13.55$

Zero of the compensator, $z_c = -4.65$

70

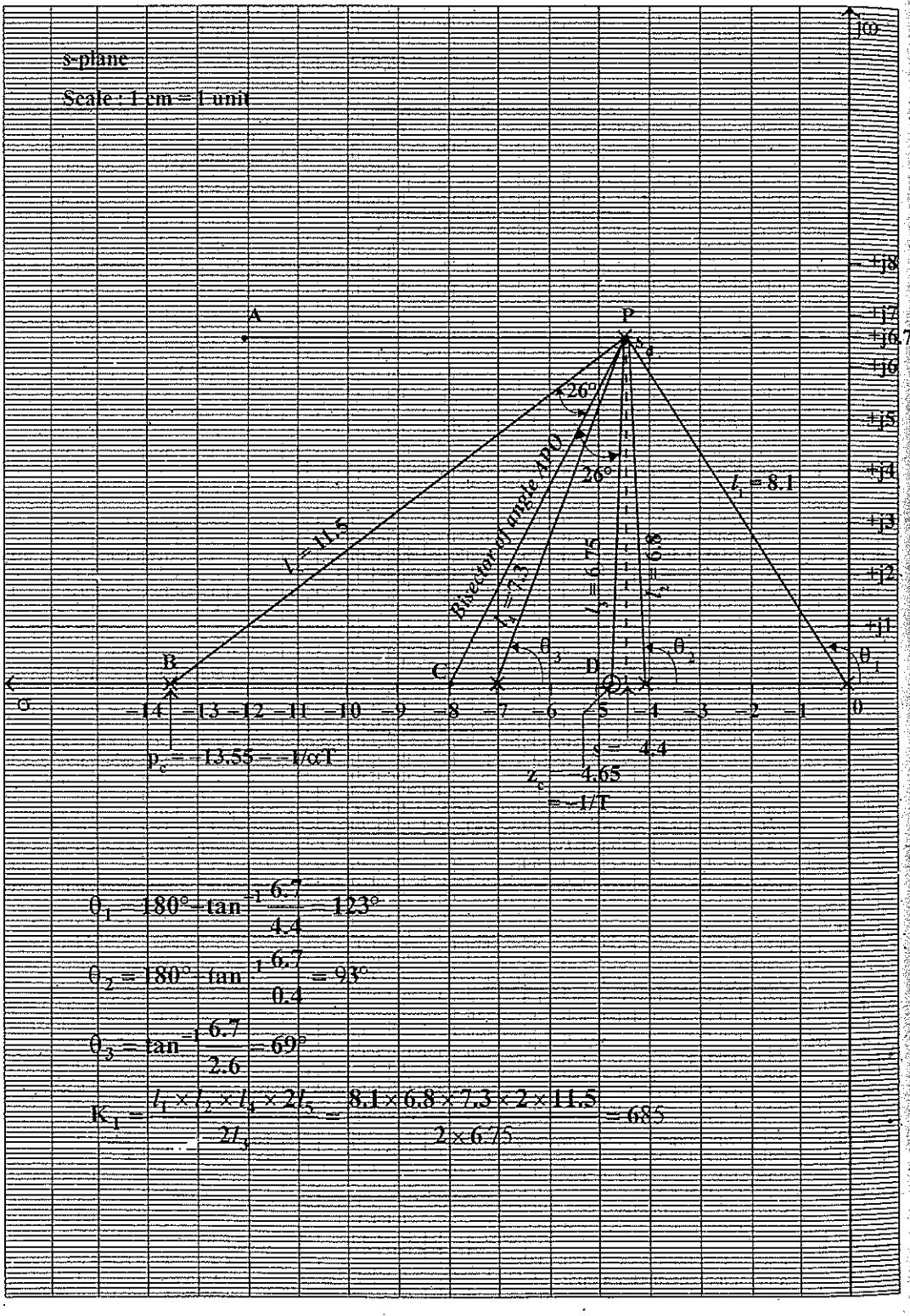


Fig 6.81 : Pole zero plot of lead compensated system

$$\theta_1 = 180° - \tan^{-1}\frac{6.7}{4.4} = 123°$$

$$\theta_2 = 180° - \tan^{-1}\frac{6.7}{0.4} = 93°$$

$$\theta_3 = \tan^{-1}\frac{6.7}{2.6} = 69°$$

$$K_1 = \frac{l_1 \times l_2 \times l_4 \times 2l_5}{2l_3} = \frac{8.1 \times 6.8 \times 7.3 \times 2 \times 11.5}{2 \times 6.75} = 685$$

We know that, $z_c = -1/T$; $\therefore T = 1/4.65 = 0.215$

We know that, $p_c = -1/\alpha T$; $\therefore \alpha T = 1/13.55$ (or) $\alpha = 1/T \times 13.55 = 0.343$.

Step-5 : Determine the transfer function of lead compensator

$$\text{Transfer function of lead compensator } G_c(s) = \frac{\left(s + \frac{1}{T}\right)^2}{\left(s + \frac{1}{\alpha T}\right)^2} = \frac{(s + 4.65)^2}{(s + 13.55)^2}$$

Step-6 : Determine the open loop transfer function of lead compensated system.

Block diagram of lead compensated system is shown in fig 6.8.2.

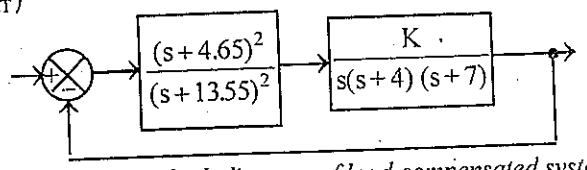


Fig 6.8.2 : Block diagram of lead compensated system.

$$\text{Open loop transfer function of lead compensated system } G_0(s) = \frac{(s + 4.65)^2}{(s + 13.55)^2} \times \frac{K}{s(s + 4)(s + 7)}$$

Here the value of K is given by the value of gain at the dominant pole, s_d on the root locus. From magnitude condition K is given by,

$$K = \frac{\text{Product of vector lengths from all poles to } s = s_d}{\text{Product of vector lengths from all zeros to } s = s_d}$$

From fig 6.8.1, we get,

$$K = \frac{l_1 \times l_2 \times l_4 \times l_5^2}{l_3^2} = \frac{8.1 \times 6.8 \times 7.3 \times 11.5^2}{6.75^2} = 1167$$

$$\therefore G_0(s) = \frac{1167 (s + 4.65)^2}{s(s + 4)(s + 7)(s + 13.55)^2}$$

Fig 6.8.1. : Pole zero plot of lead compensated system

Step-7 : Check for error requirement

For the compensated system, the velocity error constant is given by,

$$K_v = \lim_{s \to 0} s\, G_0(s) = \lim_{s \to 0} s \frac{1167 (s + 4.65)^2}{s (s + 4)(s + 7)(s + 13.55)^2} = \frac{1167 \times 4.65^2}{4 \times 7 \times c13.55^2} = 4.91$$

CONCLUSION

Since the velocity error constant of the compensated system, satisfies the requirement, the design is acceptable.

RESULT

$$\text{Transfer function of lead compensator } G_c(s) = \frac{(s + 4.65)^2}{(s + 13.55)^2} = 0.1178 \frac{(1 + 0.215s)^2}{(1 + 0.0738s)^2}$$

$$\text{Transfer function of lead compensated system } G_0(s) = \frac{1167 (s + 4.65)^2}{s (s + 4)(s + 7)(s + 13.55)^2}$$

$$= \frac{4.91 (1 + 0.215s)^2}{s (1 + 0.25s)(1 + 0.143s)(1 + 0.0738s)^2}$$

FREQUENCY RESPONSE OF LAG-LEAD COMPENSATOR

Consider the transfer function of lag-lead compensator.

$$G_c(s) = \frac{(s+1/T_1)(s+1/T_2)}{(s+1/\beta T_1)(s+1/\alpha T_2)} = \alpha\beta \frac{(1+sT_1)(1+sT_2)}{(1+s\beta T_1)(1+s\alpha T_2)} \quad \ldots\ldots$$

The sinusoidal transfer function of lag-lead compensator is obtained by le $s = j\omega$ in equation (6.46).

$$\therefore G_c(j\omega) = \alpha\beta \frac{(1+j\omega T_1)(1+j\omega T_2)}{(1+j\omega\beta T_1)(1+j\omega\alpha T_2)} \quad \ldots\ldots(6$$

For a single lag-lead compensator, $\alpha\beta = 1$. Hence from equation (6.47) we can say that the lag- compensator provides a dc gain of unity.

$$\therefore G_c(j\omega) = \frac{(1+j\omega T_1)(1+j\omega T_2)}{(1+j\omega\beta T_1)(1+j\omega\alpha T_2)} \quad \ldots\ldots(6$$

The sinusoidal transfer function shown in equation (6.48) has four corner frequencies and they ω_{c1}, ω_{c2}, ω_{c3} and ω_{c4}, where $\omega_{c1} < \omega_{c2} < \omega_{c3} < \omega_{c4}$.

Here $\omega_{c1} = \frac{1}{\beta T_1}$, ; $\omega_{c2} = \frac{1}{T_1}$; $\omega_{c3} = \frac{1}{T_2}$ and $\omega_{c4} = \frac{1}{\alpha T_2}$

By an analysis similar to that of lag compensator the bode plot of lag-lead compensator is sketc as shown in fig 6.17.

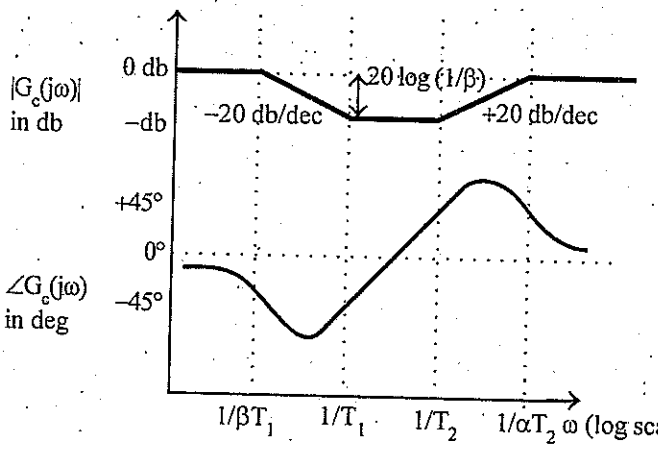


Fig 6.17 : Bode plot of lag-lead compensator.

PROCEDURE FOR DESIGN OF LAG-LEAD COMPENSATOR USING BODE PLOT

The lag-lead compensator is employed only when a large error constant and a large bandwidth a required. First design a lag section and then take $\alpha = 1/\beta$ and design a lead section. The step by st procedure for the design of lag-lead compensator is given below.

Step-1 : Determine the open loop gain K of the uncompensated system to satisfy the specified err requirement.

Step-2 : Draw the bode plot of uncompensated system.

Step-3 : From the bode plot determine the gain margin of the uncompensated system.

Let, ϕ_{gc} = Phase of G(jω) at gain crossover frequency.

γ = Phase margin of uncompensated system.

Now, $\gamma = 180° + \phi_{gc}$

If the gain margin is not satisfactory then compensation is required.

...4 : Choose a new phase margin

Let, γ_d = Desired phase margin

Now, new phase margin, $\gamma_n = \gamma_d + \epsilon$

Choose an initial value of $\epsilon = 5°$.

...5 : From the bode plot, determine the new gain crossover frequency, which is the frequency corresponding to a phase margin of γ_n.

Let, ω_{gcn} = New gain crossover frequency

ϕ_{gcn} = Phase of G(jω) at ω_{gcn}

γ_n = $180° + \phi_{gcn}$ (or) $\phi_{gcn} = \gamma_n - 180°$

In the phase plot of uncompensated system, the frequency corresponding to a phase of ϕ_{gcn} is the new gain crossover frequency ω_{gcn}.

Choose the gain crossover frequency of the lag compensator, ω_{gcl}, somewhat greater than ω_{gcn} (i.e., choose ω_{gcl} such that $\omega_{gcl} > \omega_{gcn}$).

...6 : Calculate β of lag compensator.

Let, $A_{gcl} = |G(j\omega)|$ in db at $\omega = \omega_{gcl}$

From the bode plot find A_{gcl}

Now, $A_{gcl} = 20 \log \beta$ (or) $\beta = 10^{\left(A_{gcl}/20\right)}$

...7 : Determine the transfer function of lag section

The zero of the lag compensator is placed at a frequency one-tenth of ω_{gcl}.

∴ Zero of lag compensator, $z_{c1} = 1/T_1 = \omega_{gcl}/10$

Now, $T_1 = 10/\omega_{gcl}$

Pole of lag compensator, $p_{c1} = 1/\beta T_1$

Transfer function of lag section $G_1(s) = \dfrac{(s+1/T_1)}{(s+1/\beta T_1)} = \beta\dfrac{(1+sT_1)}{(1+s\beta T_1)}$

...8 : Determine the transfer function of lead section

Take, $\alpha = 1/\beta$

From the bode plot find ω_m which is the frequency at which the db gain is $-20\log(1/\sqrt{\alpha})$

Now $T_2 = \dfrac{1}{\omega_m\sqrt{\alpha}}$

Transfer function of lead section $G_2(s) = \dfrac{(s+1/T_2)}{(s+1/\alpha T_2)} = \alpha\dfrac{(1+sT_2)}{(1+s\alpha T_2)}$

obtained by ...

can say that the lag-...

frequencies and the...

compensator is sket...

...g 6.17 : Bode plot of ...g-lead compensator

BODE PLOT

...nd a large bandwidth ...section. The step by ...

...atisfy the specified ...

...system.

Step-9 : Determine the transfer function of lag-lead compensator.

Transfer function of lag - lead compensator, $G_c(s) = G_1(s) \times G_2(s) = \beta \frac{(1+sT_1)}{(1+s\beta T_1)} \times \alpha \frac{(1+sT_2)}{(1+s\alpha T_2)}$

Since $\alpha = \frac{1}{\beta}$,

$$G_c(s) = \frac{(1+sT_1)(1+sT_2)}{(1+s\beta T_1)(1+s\alpha T_2)}$$

Step-10 : Determine the open loop transfer function of compensated system.

The lag-lead compensator is connected in series with G(s) as shown in fig 6.18.

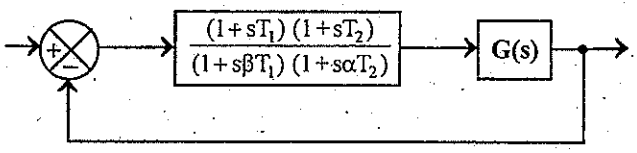


Fig 6.18 : *Block diagram of lag-lead compensated system.*

Open loop transfer function of compensated system $\Big\} G_0(s) = \frac{(1+sT_1)(1+sT_2)}{(1+s\beta T_1)(1+s\alpha T_2)} \times G(s)$

Step-11 : Draw the bode plot of compensated system and verify whether the specifications are satisfied or not. If the specifications are not satisfied then choose another choice of α such that $\alpha < 1/\beta$ and repeat the steps 8 to 11.

PROCEDURE FOR DESIGN OF LAG-LEAD COMPENSATOR USING ROOT LOCUS

The lag-lead compensation is employed to improve both the transient and steady state responses of a system. First design a lead section to realize the required ζ and ω_n for the dominant closed loop pole. Then determine the error constant of lead compensated system. If it is satisfactory then only lead compensation will meet the requirement. If the error constant has to be increased then design a lag section. The step-by-step procedure for the design of lag-lead compensator is given below.

Step-1 : Determine the dominant pole, s_d

$$s_d = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

where, ζ = Damping ratio ; ω_n = Natural frequency of oscillation, rad/sec.

Step-2 : Mark the poles and zeros of open loop transfer function and the dominant pole on the s-plane. Let the dominant pole be point P.

Step-3 : Find the angle to be contributed by lead network to make the point P as a point on root locus.

Let, ϕ = Angle to be contributed by lead network to make point, P as a point on root locus.

Draw vectors from all open loop poles and zeros to point P. Measure the angle contributed by the vectors. [For the procedure to find angle contribution by vectors refer root locus of Chapter-5].

$$\text{Now, } \phi = \left(\begin{array}{c}\text{sum of angles}\\ \text{contributed by poles}\\ \text{of uncompensated system}\end{array}\right) - \left(\begin{array}{c}\text{sum of angles}\\ \text{contributed by zeros}\\ \text{of uncompensated system}\end{array}\right) \pm n180°$$

where n is an odd integer, so that n180° is nearest to the difference between angles contributed by poles and zeros.

...: Determine the pole and zero of the lead section.

Let point O be the origin of s-plane and point P be the dominant pole. Draw straight lines OP and AP such that AP is parallel to x-axis as shown in fig.6.19. Draw a line PC so as to bisect the angle APO [∠APO] where the point C is on the real axis. With line PC as reference, draw angles BPC and CPD such that each equal to $\phi/2$. Here the points B and D are located on the real axis.

Now the point B is the location of the pole of the compensator $(-1/\alpha T_2)$ and the point D is the location of the zero of the compensator $(-1/T_2)$. Compute T_2 and α from the values of point D and B.

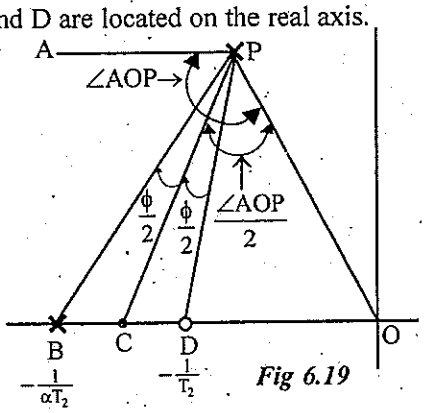


Fig 6.19

...: Determine the transfer function of lead section.

$$\text{Transfer function of lead section} \Big\} \; G_2(s) = \frac{\left(s+\frac{1}{T_2}\right)}{\left(s+\frac{1}{\alpha T_2}\right)}$$

...-6 : Determine the open loop gain, K.

The open loop gain K is the value of gain at $s = s_d$. The value of gain, K is determined from pole-zero plot of lead compensated system and by using the magnitude condition given below.

$$K = \frac{\text{Product of vector lengths from all poles to } s = s_d}{\text{Product of vector lengths from all zeros to } s = s_d}$$

Note : The length of vectors should be measured to scale. For details of magnitude condition ...er root locus in Chapter-5.

$$\text{Open loop transfer function of lead compensated system} \Big\} \; G_{02}(s) = G_2(s) \times G(s)$$

...-7 : Determine the velocity error constant of lead compensated system.

$$\text{Velocity error constant of lead compensated system} \Big\} \; K_{v2} = \lim_{s \to 0} s\, G_{02}(s)$$

If K_{v2} satisfies the requirement then only lead compensation is sufficient but if K_{v2} is less than the desired value then provide lag compensation.

...-8 : Determine the parameter, β of lag section.

Let, K_{vd} = Desired velocity error constant ; A = Factor by which K_v is increased.

Now, $A = K_{vd}/K_{v2}$. Select β, such that $\beta > A$. [i.e., $\beta = (1.1 \text{ to } 1.2) \times A$]

...-9 : Determine the transfer function of lag section. Choose the zero of lag section as 10% of the second pole of uncompensated system.

$\therefore$ Zero of lag section, $z_{c1} = 0.1 \times$ second pole of G(s)

Also, $z_{c1} = \frac{-1}{T_1}$; $\therefore T_1 = \frac{-1}{z_{c1}}$; $\therefore$ Pole of lag section, $p_{c1} = \frac{-1}{\beta T_1}$

Transfer function of lag section, $G_1(s) = \frac{\left(s+\frac{1}{T_1}\right)}{\left(s+\frac{1}{\beta T_1}\right)}$

Step-10 : Determine the transfer function of lag-lead compensator and compensated system.

$$\left.\begin{matrix}\text{Transfer function of}\\ \text{lag - lead compensator}\end{matrix}\right\} G_c(s) = G_1(s) \times G_2(s) = \frac{\left(s+\frac{1}{T_1}\right)\left(s+\frac{1}{T_2}\right)}{\left(s+\frac{1}{\beta T_1}\right)\left(s+\frac{1}{\alpha T_2}\right)}$$

The lag-lead compensator is connected in series with G(s) as shown in fig 6.20.

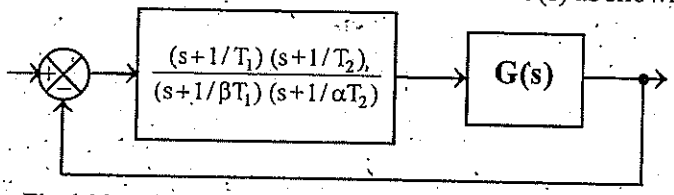


Fig 6.20 : Block diagram of lag-lead compensated system.

$$\left.\begin{matrix}\text{Open loop transfer function of}\\ \text{lag - lead compensated system}\end{matrix}\right\} G_0(s) = \frac{\left(s+\frac{1}{T_1}\right)\left(s+\frac{1}{T_2}\right)}{\left(s+\frac{1}{\beta T_1}\right)\left(s+\frac{1}{\alpha T_2}\right)} \times G(s)$$

Step-11 : Check the velocity error constant of compensated system, if it is satisfactory then the desi accepted, otherwise repeat the design by modifying the locations of poles and zeros of compensator.

EXAMPLE 6.9

Consider the unity feedback system whose open loop trans function is $G(s) = K/s(s+3)(s$ Design a lag-lead compensator to meet the following specifications. (i) Velocity error constant, K (ii) Phase margin, $\gamma \geq 35°$.

SOLUTION

Step-1 : Determine K

For unity feedback system,

Velocity error constant, $K_v = \lim_{s\to 0} sG(s)$

Given that, $K_v = 80$.

$$\therefore \lim_{s\to 0} s\,G(s) = \lim_{s\to 0} s\cdot\frac{K}{s(s+3)(s+6)} = 80 \quad \Rightarrow \quad \frac{K}{3\times 6} = 80 \quad \Rightarrow \quad K = 80\times 3\times 6 = 144$$

$$\therefore G(s) = \frac{1440}{s(s+3)(s+6)} = \frac{1440}{s\times 3(1+s/3)\times 6(1+s/6)} = \frac{80}{s(1+0.33s)(1+0.167s)}$$

Step-2: Bode plot of uncompensated system.

In G(s), put $s = j\omega$

$$\therefore G(j\omega) = \frac{80}{j\omega\,(1+j0.33\omega)(1+j0.167\omega)}$$

...NITUDE PLOT

The corner frequencies are ω_{c1} and ω_{c2}.

Here $\omega_{c1} = 1/0.33 = 3$ rad/sec and $\omega_{c2} = 1/0.167 = 6$ rad/sec.

The various terms of G(jω) are listed in table-1. Also the table shows the slope contributed by each ...and the change in slope at the corner frequency.

...LE-1

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\frac{80}{j\omega}$	–	–20	–
$\frac{1}{1+j0.33\omega}$	$\omega_{c1} = \frac{1}{0.33} = 3$	–20	–20 –20 = –40
$\frac{1}{1+j0.167\omega}$	$\omega_{c2} = \frac{1}{0.167} = 6$	–20	–40 –20 = –60

Choose a low frequency ω_l such that $\omega_l < \omega_{c1}$ and choose a high frequency ω_h such that $\omega_h > \omega_{c2}$.

Let $\omega_l = 0.5$ rad/sec and $\omega_h = 20$ rad/sec.

Let A = |G(jω)| in db

$$\text{At } \omega = \omega_1, \quad A = 20 \log\frac{80}{\omega} = 20 \log \frac{80}{0.5} = 44 \text{ db}$$

$$\text{At } \omega = \omega_{c1}, \quad A = 20 \log\frac{80}{\omega} = 20 \log\frac{80}{3} = 28.5 \text{ db} \approx 28 \text{ db}$$

$$\text{At } \omega = \omega_{c2}, \quad A = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + A \text{ at } (\omega = \omega_{c1})$$

$$= -40 \times \log\frac{6}{3} + 28 = 16 \text{ db}$$

$$\text{At } \omega = \omega_h, \quad A = \left[\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c2}}\right] + A \text{ at } (\omega = \omega_{c2})$$

$$= -60 \times \log\frac{20}{6} + 16 = -15 \text{ db}$$

Let the points a, b, c and d be the points corresponding to frequencies ω_l, ω_{cl}, ω_{c2} and ω_h ...ctively on the magnitude plot. In a semilog graph sheet choose appropriate scales and fix the points ...c and d. Join the points by straight lines and mark the slope on the respective region. The magnitude ...is shown in fig 6.9.2.

...ASE PLOT

The phase angle of G(jω) as a function of ω is given by

$$\phi = \angle G(j\omega) = -90° - \tan^{-1}0.33\omega - \tan^{-1}0.167\omega.$$

The phase angle of G(jω) are calculated for various values of ω and listed in table-2.

TABLE-2

ω rad/sec	0.5	1.0	3.0	6	10	20
$\angle G(j\omega)$ deg	−104	−118	−161 ≈ −160	−198	−222	−244.7 ≈ −244

On the same semilog sheet take another y-axis, choose appropriate scale and draw phase shown in fig 6.9.2.

Step-3 : Find phase margin of uncompensated system.

Let, ϕ_{gc} = Phase of $G(j\omega)$ at gain crossover frequency

γ = Phase margin of uncompensated system.

From the bode plot of uncompensated system we get, $\phi_{gc} = -226°$.

Now, $\gamma = 180° + \phi_{gc} = 180° - 226° = -46°$

Step-4 : Choose a new phase margin

The desired phase margin, $\gamma_d = 35°$

The phase margin of compensated system, $\gamma_n = \gamma_d + \epsilon$

Let initial choice of $\epsilon = 5°$

$\therefore \gamma_n = \gamma_d + \epsilon = 35° + 5° = 40°$.

Step-5 : Determine new gain crossover frequency

Let, ω_{gcn} = New gain crossover frequency and ϕ_{gcn} = Phase of $G(j\omega)$ at ω_{gcn}

Now, $\gamma_n = 180° + \phi_{gcn}$, $\therefore \phi_{gcn} = \gamma_n - 180° = 40° - 180° = -140°$

From the bode plot we found that the frequency corresponding to a phase of is 1.8 rad/sec.

Let, ω_{gcl} = Gain crossover frequency of lag compensator.

Choose ω_{gcl} such that, $\omega_{gcl} > \omega_{gcn}$. Let $\omega_{gcl} = 4$ rad/sec.

Step-6 : Calculate β of lag compensator

From the bode plot we found that the db magnitude at ω_{gcl} is 23 db.

$\therefore |G(j\omega)|$ in db at $(\omega = \omega_{gcl}) = A_{gcl} = 23$ db.

Also, $A_{gcl} = 20\log\beta$; $\therefore \beta = 10^{A_{gcl}/20} = 10^{23/20} = 14$

Step-7 : Determine the transfer function of lag section.

The zero of the lag compensator is placed at a frequency one-tenth of ω_{gcl}.

$\therefore$ Zero of lag compensator, $z_{c1} = \dfrac{-1}{T_1} = \dfrac{\omega_{gcl}}{10}$

$$\text{Now, } T_1 = \frac{10}{\omega_{gcl}} = \frac{10}{4} = 2.5$$

Pole of lag compensator, $p_{c1} = \frac{1}{\beta T_1} = \frac{1}{14 \times 2.5} = \frac{1}{35}$

Transfer function of lag section $\Big\}$ $G_1(s) = \beta \frac{(1+sT_1)}{(1+s\beta T_1)} = 14 \frac{(1+2.5s)}{(1+35s)}$

8.: Determine the transfer function of lead section.

Let $\alpha = 1/\beta$; $\therefore \alpha = 1/14 = 0.07$

The db gain (magnitude) corresponding to ω_m. $\Big\}$ $= -20\log\frac{1}{\sqrt{\alpha}} = -20\log\frac{1}{\sqrt{0.07}} = -11.5 \text{ db} \approx -12 \text{ db}$

From the bode plot of uncompensated system the frequency ω_m corresponding to a db pair of −12 db is found to be 17 rad/sec.

$\therefore \omega_m = 17 \text{ rad/sec}$

$\therefore T_2 = \frac{1}{\omega_m \sqrt{\alpha}} = \frac{1}{17\sqrt{0.07}} = 0.22$

Transfer function of lead section $\Big\}$ $G_2(s) = \alpha \frac{(1+sT_2)}{(1+s\alpha T_2)} = 0.07 \frac{(1+0.22s)}{(1+0.0154s)}$

10.: Determine the transfer function of lag-lead compensator.

Transfer function of lag-lead compensator $\Big\}$ $G_c(s) = G_1(s) \times G_2(s) = 14 \frac{(1+2.5s)}{(1+35s)} \times 0.07 \frac{(1+0.22s)}{(1+0.0154s)}$

$$= \frac{(1+2.5s)(1+0.22s)}{(1+35s)(1+0.0154s)}$$

11.: Determine open loop transfer function of compensated system.

The lag-lead compensator is connected in series with G(s) as shown in fig 6.9.1.

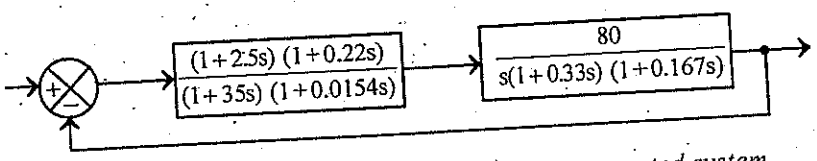


Fig 6.9.1 : Block diagram of lag-lead compensated system.

Open loop transfer function of compensated system $\Big\}$ $G_0(s) = \frac{80(1+2.5s)(1+0.22s)}{s(1+35s)(1+0.0154s)(1+0.33s)(1+0.167s)}$

: Bode plot of compensated system.

Put $s = j\omega$ in $G_0(s)$

$$\therefore G_0(j\omega) = \frac{80(1+j2.5\omega)(1+j0.22\omega)}{j\omega(1+j35\omega)(1+j0.0154\omega)(1+j0.33\omega)(1+j0.167\omega)}$$

MAGNITUDE PLOT

There are six corner frequencies, which are given below.

$$\omega_{c1} = \frac{1}{35} = 0.03\,\text{rad/sec}\ ;\ \omega_{c2} = \frac{1}{2.5} = 0.4\,\text{rad/sec}\ ;\ \omega_{c3} = \frac{1}{0.33} = 3\,\text{rad/sec}\ ;$$

$$\omega_{c4} = \frac{1}{0.22} = 4.5\,\text{rad/sec};\ \omega_{c5} = \frac{1}{0.167} = 6\,\text{rad/sec}\ ;\ \omega_{c6} = \frac{1}{0.0154} = 65\,\text{rad/sec}$$

The various terms of $G_0(j\omega)$ are listed in table-3. Also the table shows the slope contribu each term and the change in slope at the corner frequency.

TABLE-3

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\frac{80}{j\omega}$	–	–20	–
$\frac{1}{1+j35\omega}$	$\omega_{c1} = \frac{1}{35} = 0.03$	–20	–20 –20 = –40
$1+j2.5\omega$	$\omega_{c2} = \frac{1}{2.5} = 0.4$	+20	–40 +20 = –20
$\frac{1}{1+j0.33\omega}$	$\omega_{c3} = \frac{1}{0.33} = 3$	–20	–20 –20 = –40
$1+j0.22\omega$	$\omega_{c4} = \frac{1}{0.22} = 4.5$	+20	–40 +20 = –20
$\frac{1}{1+j0.167\omega}$	$\omega_{c5} = \frac{1}{0.167} = 6$	–20	–20 –20 = –40
$\frac{1}{1+j0.0154\omega}$	$\omega_{c6} = \frac{1}{0.0154} = 65$	–20	–40 –20 = –60

Choose a low frequency ω_l such that $\omega_l < \omega_{c1}$ and choose a high frequency ω_h such that ω_h

Let $\omega_l = 0.01$ rad/sec and $\omega_h = 80$ rad/sec

Let $A_0 = |G_0(j\omega)|$ in db.

$$\text{At } \omega = \omega_l,\quad A_0 = 20\log\frac{80}{0.01} = 78\text{ db}$$

$$\text{At } \omega = \omega_{c1},\quad A_0 = 20\log\frac{80}{0.03} = 68.5\text{ db} \approx 68\text{ db}$$

$$\text{At } \omega = \omega_{c2},\quad A_0 = -40\times\log\frac{0.4}{0.03} + 68 = 23\text{ db}$$

$$\text{At } \omega = \omega_{c3},\quad A_0 = -20\times\log\frac{3}{0.4} + 23 = 5\text{ db}$$

At $\omega = \omega_{c4}$, $A_0 = -40 \times \log\frac{4.5}{3} + 5 = -2$ db

At $\omega = \omega_{c5}$, $A_0 = -20 \times \log\frac{6}{4.5} + (-2) = -4$ db

At $\omega = \omega_{c6}$, $A_0 = -40 \times \log\frac{65}{6} + (-4) = -45$ db

At $\omega = \omega_h$, $A_0 = -60 \times \log\frac{80}{65} + (-45) = -50$ db

ing the values of A_0 at various frequencies the magnitude plot of compensated system is wn as shown in fig 6.9.2.

OT

e phase angle of $G_0(j\omega)$ as a function of ω is given by

$-\angle G_0(j\omega) = \tan^{-1}2.5\omega + \tan^{-1}0.22\omega - 90° - \tan^{-1}35\omega$

$-\tan^{-1}0.0154\omega - \tan^{-1}0.33\omega - \tan^{-1}0.167\omega.$

e phase angle of $G_0(j\omega)$ are calculated for various values of ω and listed in table-4.

ω ad/sec	0.01	0.03	0.1	0.4	1	4	10	65	80
$\angle G_0(j\omega)$ deg	−108	−132	−152	−138	−126	−144	−168	−221 ≈ −220	−228

sing the values of ϕ_0 listed in table-4, the phase plot of compensated system is sketched as hown in fig 6.9.2.

et ϕ_{gc0} = Phase of $G_0(j\omega)$ at the gain crossover frequency of compensated system.

nd γ_0 = Phase margin of compensated system.

rom the bode plot of compensated system, we get, $\phi_{gc0} = -144°$

ow, $\gamma_0 = 180° + \phi_{gc0} = 180° - 144° = 36°$

SION

he phase margin of the compensated system is satisfactory. Hence the design is acceptable.

Transfer function of lag-lead compensator, $G_c(s) = \frac{(1+2.5s)(1+0.22s)}{(1+35s)(1+0.0154s)}$

Open loop transfer function of compensated system } $G_0(s) = \frac{80(1+2.5s)(1+0.22s)}{s(1+35s)(1+0.0154s)(1+0.33s)(1+0.167s)}$

82

Scales:
y-axis: 1 unit = 10 db
y-axis: 1 unit = 20°
Magnitude plot of uncompensated system
Magnitude plot of compensated system
-20 db/dec
-40 db/dec
-60 db/dec
-40 db/dec
-20 db/dec
-40 db/dec
A_{gc} = 23 db
ϕ_{gc} = -143°
ϕ_{gc} = -226°
Phase plot of uncompensated system
Phase plot of compensated system
deg

IPLE 6.10

esign a lag-lead compensator for a system with open loop transfer function G(s) = K/s (s + 0.5) y the following specifications. (i) Damping ratio of dominant closed-loop poles, $\zeta = 0.5$. mped natural frequency of dominant closed loop poles, $\omega_n = 5$ rad/sec. (iii) Velocity error constant, sec^{-1}.

TION

Determine dominant pole, s_d.

Dominant pole, $s_d = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$.

Given that, $\zeta = 0.5$ and $\omega_n = 5$ rad/sec.

$$\therefore s_d = -0.5\times 5 \pm j5\times\sqrt{1-0.5^2} = -2.5 \pm j4.3$$

Draw the pole-zero plot

The pole-zero plot of open loop transfer function is shown fig 6.10.1. Poles are represented by the symbol "x". The pole at point P is the dominant pole, s_d.

To find the angle to be contributed by lead network.

Let ϕ = Angle to be contributed by lead network to make point, P as a point on root locus.

$$\text{Now, } \phi = \begin{pmatrix}\text{sum of angles}\\ \text{contributed by poles}\\ \text{of uncompensated system}\end{pmatrix} - \begin{pmatrix}\text{sum of angles}\\ \text{contributed by zeros}\\ \text{of uncompensated system}\end{pmatrix} \pm n180^\circ$$

From fig 6.10.1, we get,

$$\left.\begin{matrix}\text{Sum of angles contributed}\\ \text{by poles of uncompensated system}\end{matrix}\right\} = \theta_1 + \theta_2 = 120^\circ + 115^\circ = 235^\circ$$

Since there is no finite zero in uncompensated system, there is no angle contribution by zeros.

$$\therefore \phi = 235^\circ \pm n180^\circ$$

Let n = 1, $\quad \therefore \phi = 235^\circ - 180^\circ = 55^\circ$.

To find the pole and zero of the lead section

Draw a line AP parallel to x-axis as shown in fig 6.10.1. The bisector PC is drawn to bisect the angle APO. The angles CPD and BPC are constructed as shown in fig 6.10.1. Here $\angle CPD = \angle BPC = \phi/2 = 55^\circ/2 = 27.5^\circ \approx 27^\circ$.

From fig 6.10.1., Pole of the lead section, $p_{c2} = -10$

Zero of the lead section, $z_{c2} = -2.65$.

We know that, $z_{c2} = -1/T_2 \quad \therefore T_2 = 1/2.65 = 0.377$

We know that, $p_{c2} = -1/\alpha T_2 \quad \therefore \alpha T_2 = 1/10$ (or) $\alpha = 1/(T_2\times 10) = 0.265$.

Transfer function of lead compensator

Transfer function of lead section, $G_2(s) = \dfrac{(s+1/T_2)}{(s+1/\alpha T_2)} = \dfrac{(s+2.65)}{(s+10)}$

Step-6 : To find gain, K

$$\left.\begin{matrix}\text{Open loop transfer function}\\ \text{of lead compensated system}\end{matrix}\right\} G_{02}(s) = G_2(s) \times G(s) = \frac{(s+2.65)}{(s+10)} \times \frac{K}{s(s+0.5)} = \frac{K(s+2.65)}{s(s+0.5)(s+10)}$$

Here the value of K is given by the value of gain at the dominant pole s_d on the root locus. From magnitude condition K is given by,

$$K = \frac{\text{Product of vector lengths from all poles to } s = s_d}{\text{Product of vector lengths from all zeros to } s = s_d}$$

From fig 6.10.1, we get,

$$K = \frac{l_1 \times l_2 \times l_4}{l_3} = \frac{5 \times 4.75 \times 7.75}{4.3} = 42.8$$

$$G_{02}(s) = \frac{42.8(s+2.65)}{s(s+0.5)(s+10)}$$

Step-7 : To find velocity error constant of lead compensated system.

Let, K_{v2} = Velocity error constant of lead compensated system.

$$\therefore K_{v2} = \lim_{s \to 0} s.\, G_{02}(s) = \lim_{s \to 0} s \frac{42.8(s+2.65)}{s(s+0.5)(s+10)} = \frac{42.8 \times 2.65}{0.5 \times 10} = 22.684$$

Step-8 : To find the parameter, β

Let K_{vd} = Desired velocity error consta.

A = The factor by which K_v is increased

Now, $A = K_{vd} / K_{v2} = 80 / 22.684 = 3.5267$

Select β, such that $\beta > A$. Let $\beta = 4$.

Step-9 : To find the transfer function of lag section.

Let zero of lag section, $z_{c1} = 0.1 \times$ second pole of $G(s) = 0.1 \times (-0.5) = -0.05$

Also, $z_{c1} = \frac{-1}{T_1}$; $\therefore T_1 = \frac{1}{0.05} = 20$

Pole of lag section, $p_{c1} = \frac{-1}{\beta T_1} = \frac{-1}{4 \times 20} = \frac{-1}{80} = -0.0125$

Transfer function of lag section, $G_1(s) = \frac{(s+1/T_1)}{(s+1/\beta T_1)} = \frac{(s+0.05)}{(s+0.0125)}$

Step-10 : Transfer function of compensated system

$$\left.\begin{matrix}\text{Transfer function of}\\ \text{lag - lead compensator}\end{matrix}\right\} G_c(s) = G_1(s) \times G_2(s) = \frac{(s+0.05)(s+2.65)}{(s+0.0125)(s+10)}$$

The lag-lead compensator is connected in series with G(s) as shown in fig 6.10.2.

$$\left.\begin{matrix}\text{Open loop transfer function}\\ \text{of compensated system}\end{matrix}\right\} G_0(s) = \frac{42.8(s+0.05)(s+2.65)}{s(s+0.0125)(s+0.5)(s+10)}$$

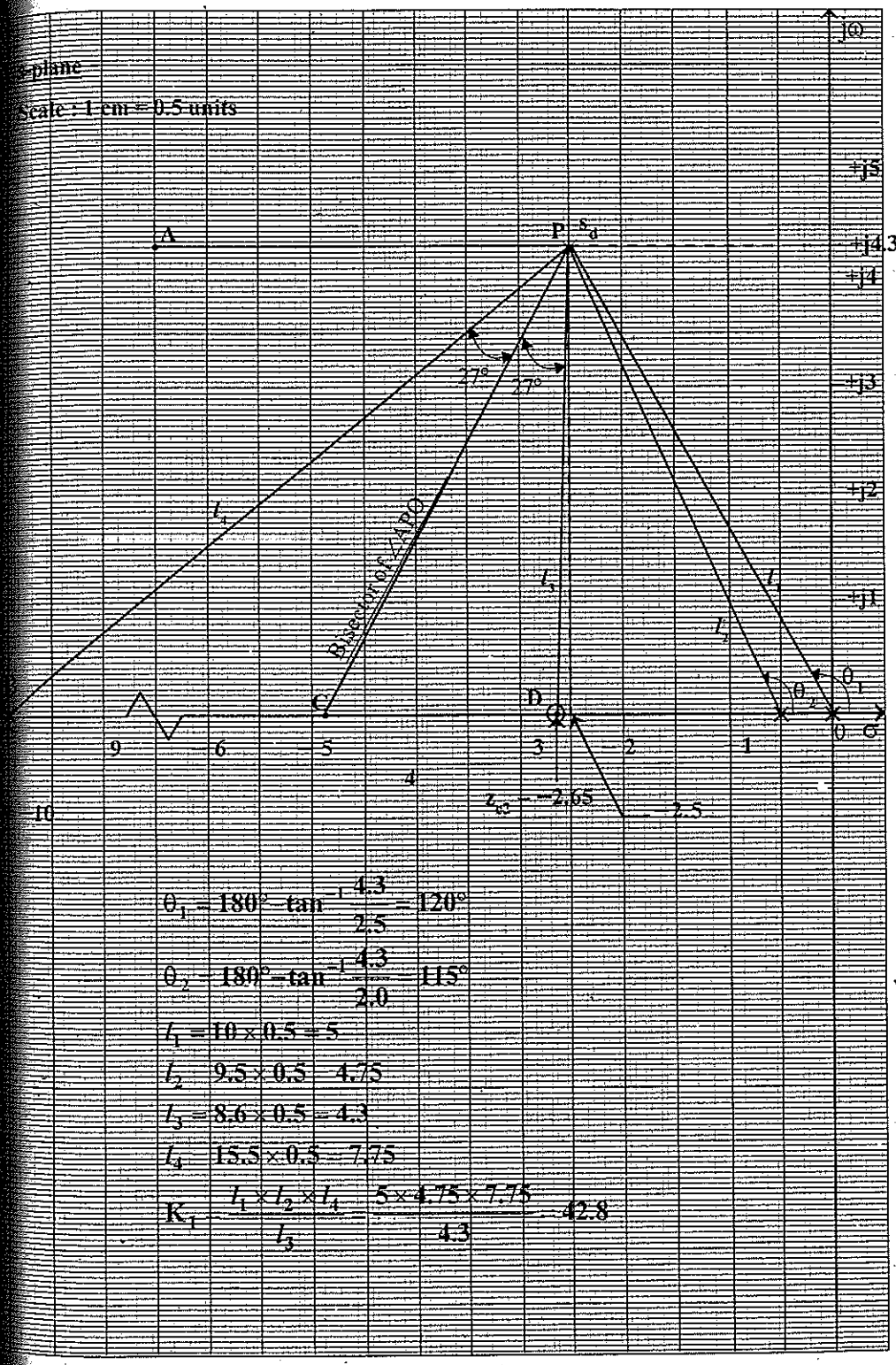


Fig 6.10.1. : Pole zero plot of lead compensated system.

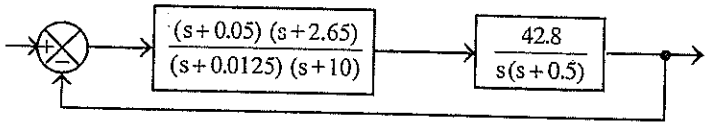


Fig 6.10.2 : Block diagram of lag-lead compensated system.

Step-11 : Check velocity error constant of compensated system.

$$\left.\begin{array}{l}\text{Velocity error}\\ \text{constant of}\\ \text{compensated system}\end{array}\right\} K_{vc} = \lim_{s\to 0} s\,G_0(s) = \lim_{s\to 0} s\,\frac{42.8\,(s+0.05)\,(s+2.65)}{s\,(s+0.0125)\,(s+0.5)\,(s+10)}$$

$$= \frac{42.8\times 0.05\times 2.65}{0.0125\times 0.5\times 10} = 90.7$$

CONCLUSION

The velocity error constant of the compensated system satisfies the requirement. Hence the design is accepted.

RESULT

Transfer function of lag - lead compensator, $G_c(s) = \dfrac{(s+0.05)\,(s+2.65)}{(s+0.0125)\,(s+10)}$

$$\left.\begin{array}{l}\text{Open loop transfer function of}\\ \text{lag - lead compensated system}\end{array}\right\} G_0(s) = \frac{42.8\,(s+0.05)\,(s+2.65)}{s\,(s+0.0125)\,(s+0.5)\,(s+10)}$$

6.5 PI, PD AND PID CONTROLLERS

A controller with transfer function, $G_c(s)$ can be introduced in cascade with open loop transfer function, G(s) as shown in fig 6.21, to modify the transient and steady state response of the system.

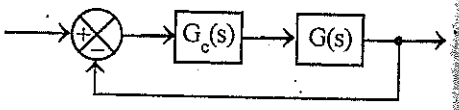


Fig 6.21 : Block diagram of a system with a controller in cascade.

The different types of controllers employed in control system are the following

1. *Proportional controller (P-controller)*
2. *Proportional-plus-integral controller (PI-controller)*
3. *Proportional-plus-derivative controller (PD-controller)*
4. *Proportional-plus-derivative-plus-integral controller (PID-controller)*

The proportional controller is a device that produces an output signal, u(t), which is proportional to the input signal, e(t).

In P-controller, $u(t) \propto e(t)$

$$\therefore u(t) = K_p\, e(t) \qquad \text{.....(6.49)}$$

where, K_p = Proportional gain or constant.

On taking laplace transform of equation (6.49) we get,

$$U(s) = K_p\, E(s)$$

Transfer function of P - controller, $G_c(s) = \dfrac{U(s)}{E(s)} = K_p$ .....(6.50)

PROCEDURE FOR DESIGN USING ROOT LOCUS TECHNIQUE

Step-1 : Determine the dominant pole, s_d.

Dominant pole, $s_d = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$

where, ζ = Damping ratio

ω_n = Natural frequency of oscillation, rad / sec

Step-2 : Draw the pole-zero plot of G(s)

In an ordinary graph sheet, take suitable scales and mark the poles by "x" and zeros by "o". Also mark the dominant pole, s_d. Let point P be the dominant pole.

Step-3 : Determine the new loop transfer function.

Choose a compensation scheme as shown in fig 6.29. Determine the closed loop transfer function C(s)/R(s). The new loop transfer function is obtained by reconstructing the characteristic equation of C(s)/R(s) in the form $1+ [G(s)H(s)]_{new}$. In the new loop transfer function, there will be an additional term, $(s + K_A / K_t)$ which contributes a zero, $z_c = -K_A / K_t$.

Step-4 : Locate the zero introduced by the feedback compensator

Let, ϕ = Angle contributed by zero of compensator to make point, P as a point on root locus

Now, $\phi = \begin{pmatrix}\text{Sum of angles contributed}\\ \text{by poles of G(s)}\end{pmatrix} - \begin{pmatrix}\text{Sum of angles contributed}\\ \text{by zeros of G(s)}\end{pmatrix} \pm n180°$

where n is an odd integer.

With line AP as reference draw a line PC such that $\angle APC = \phi$, as shown in fig 6.30. This line will intersect the real axis at point C. The value corresponding to point C is the zero of the compensator, z_c.

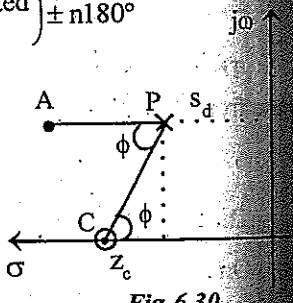


Fig 6.30

Step-5 : Determine the parameter K_A and K_t.

From $[G(s)H(s)]_{new}$ we get,

$z_c = -K_A / K_t$

Choose a suitable value for K_t and calculate K_A.

Step-6 : Determine the gain K of G(s)

Let, $[G(s)H(s)]_{new} = \dfrac{K K_t (s + K_A / K_t)}{s (s + 1/T_1) (s + 1/T_2)}$

The value of $K K_t$ can be obtained from the magnitude condition.

$\therefore K K_t = \dfrac{\text{Product of vector lengths from poles to } s = s_d}{\text{Product of vector lengths from zeros to } s = s_d}$

Note : *Vector lengths should be measured to sc*

Step-7 : Verify the design.

Check the velocity error constant, K_v of the new loop transfer function.

$K_v = \underset{s\to 0}{Lt}\ s\,[G(s)H(s)]_{new}$

If the velocity error constant is satisfactory then the design is accepted otherwise this proce calls for modifications in time domain specifications.

MPLE 6.17

Design a feedback compensation scheme for a unity feedback system with open loop transfer ion, G(s) = 3/s(s+1) to satisfy the following specifications. (1) phase margin of system should be st 45°. (2) Velocity error constant, $K_v \geq 20$.

UTION

The feedback compensation scheme is shown in fig 6.17.1. The feedback transfer function, H(s) sts of a rate device in cascade with a high pass filter. The minor-loop feedback can be eliminated and ced by an equivalent transfer function, $G_{eq}(s)$ as shown in fig 6.17.2.

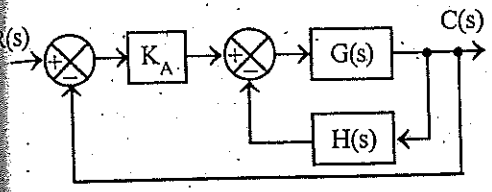


Fig 6.17.1

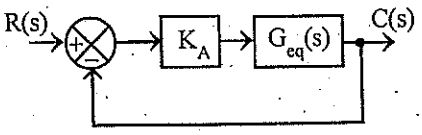


Fig 6.17.2

$$H(s) = K_t s\frac{sT}{1+sT} = \frac{K_t Ts^2}{(1+sT)} \quad \text{and} \quad G_{eq}(s) = \frac{G(s)}{1+G(s)\,H(s)}$$

1 : To draw the magnitude plot of G(jω).

Given that, $G(s) = \dfrac{3}{s(s+1)}$

Put $s = j\omega$, $\therefore\ G(j\omega) = \dfrac{3}{j\omega(1+j\omega)}$

The corner frequency is, $\omega_c = 1$ rad/sec.

The various terms of G(jω) are listed in table-1. Also the table shows the slope contributed by each term and the change in slope at the corner frequency.

BLE-1

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\dfrac{3}{j\omega}$	–	–20	–
$\dfrac{1}{1+j\omega}$	$\omega_c = 1$	–20	–20 –20 = –40

Choose a low frequency, ω_l such that $\omega_l < \omega_{c1}$ and choose a high frequency, ω_h such that $\omega_h > \omega_c$.

Let $\omega_l = 0.5$ rad/sec and $\omega_h = 10$ rad/sec.

Let $A = |G(j\omega)|$ in db

$$\text{At } \omega = \omega_l, \quad A = 20\log\frac{3}{\omega} = 20\log\frac{3}{0.5} = 15.5 \approx 16 \text{ db}$$

"x" and zeros by "o". Al...

...osed loop transfer funct...
the characteristic equati...
function, there will be ...

...P as a point on root loc...

$\pm n180°$

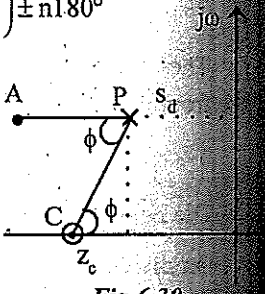


Fig 6.30

...uld be measured f...

...on.

...l otherwise this p...

$$\text{At } \omega = \omega_c, \quad A = 20 \log \frac{3}{\omega} = 20 \log \frac{3}{1} = 9.5 \text{ db} \approx 10 \text{ db}$$

$$\text{At } \omega = \omega_h, \quad A = \left[\text{slope from } \omega_c \text{ to } \omega_h \times \log \frac{\omega_h}{\omega_c}\right] + A \text{ at } (\omega = \omega_c) = -40 \times \log \frac{10}{1} + 10 = -30 \text{ db}$$

Let the points a, b and c be the points corresponding to frequencies ω_l, ω_c and ω_h respectively on the magnitude plot. In a semilog graph sheet choose appropriate scales and fix the points a, b and c. Join the points by straight lines and mark the slope on the respective region. The magnitude plot is shown in fig 6.17.3.

Step-2 : Draw the magnitude plot of $1/H(j\omega)$.

Transfer function of feedback compensator $\Big\}$ $H(s) = \dfrac{K_t T s^2}{1 + sT}$

Put $s = j\omega$, $\therefore H(j\omega) = \dfrac{K_t T (j\omega)^2}{(1 + j\omega T)}$; Now, $\dfrac{1}{H(j\omega)} = \dfrac{1}{K_t T} \cdot \dfrac{1 + j\omega T}{(j\omega)^2}$

Let $K_t T = 1$, $\therefore \dfrac{1}{H(j\omega)} = \dfrac{1 + j\omega T}{(j\omega)^2}$; Let $T = 1.25$ sec, $\therefore \dfrac{1}{H(j\omega)} = \dfrac{1 + j1.25\omega}{(j\omega)^2}$

The corner frequency, $\omega_{ch} = \dfrac{1}{1.25} = 0.8$ rad / sec

The various terms of $1/H(j\omega)$ are listed in table-2.

TABLE-2

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\dfrac{1}{(j\omega)^2}$	–	–40	–
$1 + j1.25\omega$	$\omega_{ch} = \dfrac{1}{1.25} = 0.8$	+20	–40 + 20 = –20

Choose a low frequency, ω_{lh} such that $\omega_{lh} < \omega_{ch}$ and choose a high frequency, ω_{hh} such that $\omega_{hh} > \omega_{ch}$.

Let $\omega_{lh} = 0.1$ rad/sec and $\omega_{hh} = 2.0$ rad/sec.

Let $H = |1/H(j\omega)|$ in db

$$\text{At } \omega = \omega_{lh}, \quad H = 20 \log \frac{1}{\omega^2} = 20 \log \frac{1}{0.1^2} = 40 \text{ db}$$

$$\text{At } \omega = \omega_{ch}, \quad H = 20 \log \frac{1}{\omega^2} = 20 \log \frac{1}{0.8^2} = 4 \text{ db}$$

$$\text{At } \omega = \omega_{hh}, \quad H = \left[\text{slope from } \omega_{ch} \text{ to } \omega_{hh} \times \log \frac{\omega_{hh}}{\omega_{ch}}\right] + H \text{ at } (\omega = \omega_{ch}) = -20 \times \log \frac{2}{0.8} + 4 = -4 \text{ db}$$

Let the points d, e and f be the points corresponding to frequencies ω_{lh}, ω_{ch} and ω_{hh} respectively on the magnitude plot. In the same semilog graph sheet and using the same scales draw the magnitude plot as shown in fig 6.17.3.

$= -40 \times \log\frac{10}{1} + 10 = -30$ ...

ω_p ω_c and ω_h respectiv...
scales and fix the point...
ective region. The magn...

$\frac{\omega T}{)^2}$

$= \frac{1+ j1.25\omega}{(j\omega)^2}$.

slope
:
= –20

high frequency, $\omega_{...}$ ...

$_1) = -20 \times \log\frac{2}{0.8} + 4 =$ ...

ω_{lh}, ω_{ch} and ω_{hh} respect...
iles draw the magnitude...

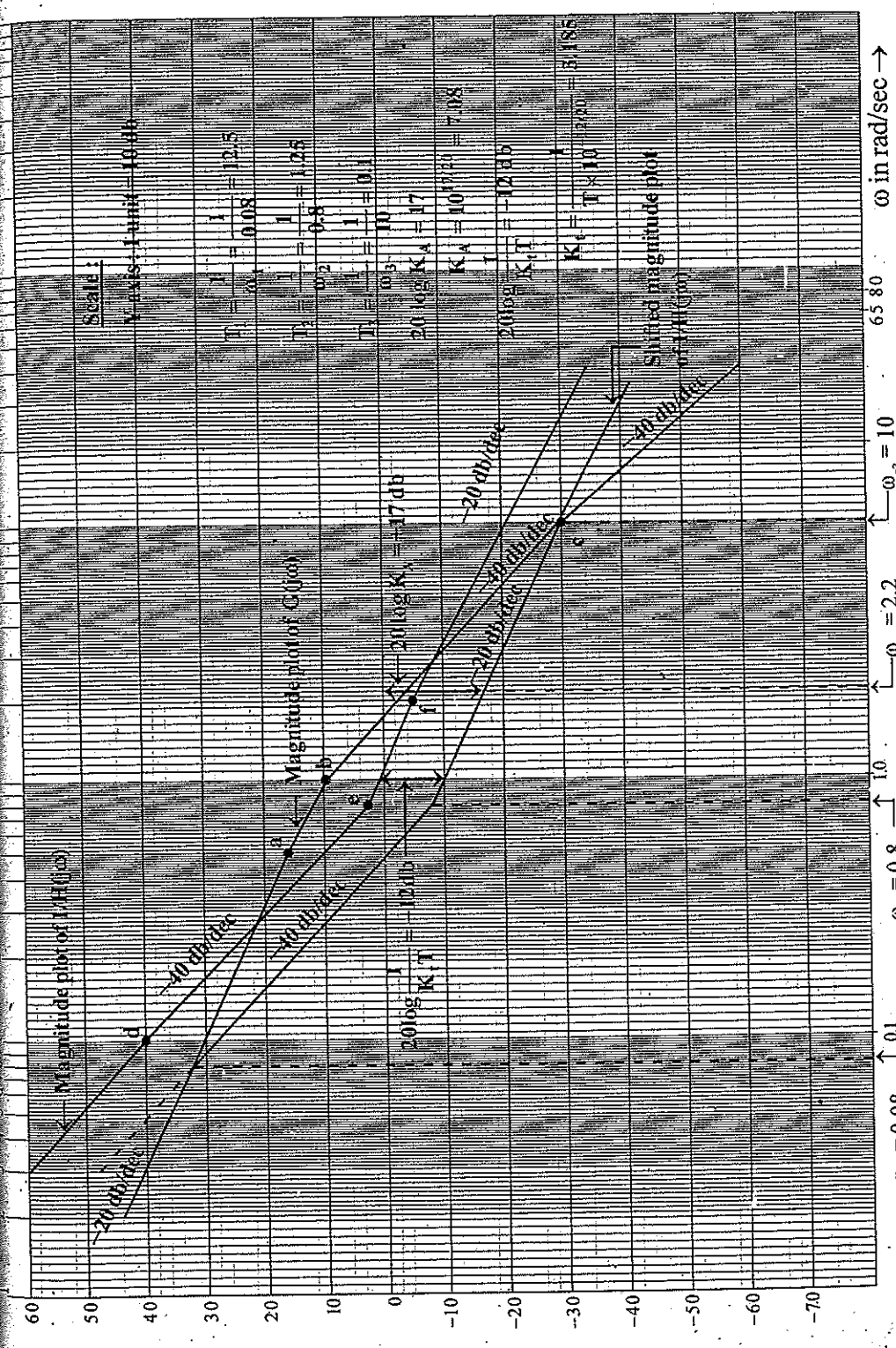


Fig 6.17.3 :Magnitude plots of $G(j\omega)$, $1/H(j\omega)$ and $G_{eq}(j\omega)$

Step-3 : Determine the gain cross-over frequency.

Let ω_{gc} = Desired gain crossover frequency

γ_d = Desired phase margin

Choose a new value of phase margin, γ_n such that, $\gamma_n = \gamma_d + \epsilon$.

Let, $\epsilon = 25°$, $\therefore \gamma_n = \gamma_d + \epsilon = 45° + 25° = 70°$.

Let, $\phi_{hgc} = \angle (1/H(j\omega))$ at $\omega = \omega_{gc}$

From the transfer function of $1/H(j\omega)$ we get,

$$\phi_{hgc} = \tan^{-1}\omega_{gc}T - 180° = \tan^{-1}1.25\omega_{gc} - 180°$$

Here, $\gamma_n = 180° + \phi_{hgc}$

$$\therefore 70° = 180° + \tan^{-1}1.25\,\omega_{gc} - 180° \quad \Rightarrow \quad 1.25\omega_{gc} = \tan 70° \quad \Rightarrow \quad \omega_{gc} = \frac{\tan 70°}{1.25} = 2.2 \text{ rad}$$

Step-4 : Determine the parameter, K_t

The $|1/H(j\omega)|$ curve is shifted down to cross $|G(j\omega)|$ curve at $\omega = 10$ rad/sec. (here $4\,\omega_{gc} = 2.2 = 8.8$, hence frequency corresponding to crossing point is chosen to be 10 rad/sec).

From the bode plots we found that $|1/H(j\omega)|$ curve is shifted by 12 db downwards.

$$\therefore 20\log\frac{1}{K_tT} = -12\text{db} \quad \Rightarrow \quad \log\frac{1}{K_tT} = \frac{-12}{20} \quad \Rightarrow \quad \frac{1}{K_tT} = 10^{-12/20} = 0.2512$$

$$\therefore K_t = \frac{1}{T \times 0.2512} = \frac{1}{1.25 \times 0.2512} = 3.185$$

Step-5 : Determine the new transfer function, $G_{eq}(j\omega)$

From the bode plots it is observed that the shifted $|1/H(j\omega)|$ curve crosses the $|G(j\omega)|$ curve at $\omega = \omega_{c1} = 0.08$ rad/sec and $\omega = \omega_{c3} = 10$ rad/sec.

In the frequency range, $0 < \omega < \omega_{c1}$, $|G_{eq}(j\omega)| = |G(j\omega)|$

In the frequency range, $\omega_{c1} < \omega < \omega_{c2}$, $|G_{eq}(j\omega)| = |1/H(j\omega)|$ (Shifted curve)

In the frequency range, $\omega_{c2} < \omega < \infty$, $|G_{eq}(j\omega)| = |G(j\omega)|$

The bode magnitude plot of $G_{eq}(j\omega)$ is shown as a bold curve in fig 6.17.3.

The bode magnitude plot of $G_{eq}(j\omega)$ is interpreted as follows,

1. The slope in the starting portion of the curve is –20 db/dec and it is due to an integral factor $K/j\omega$, where K is given by $G(j\omega)$ when $\omega = 0$,

   $|G(j\omega)|_{\omega=0} = 3$, $\therefore K = 3$

2. At $\omega = \omega_{c1}$, the slope changes from –20 db/dec to –40 db/dec. This is due to a first factor $1/(1 + j\omega T_1)$ where, $T_1 = 1/\omega_{c1} = 1/0.08 = 12.5$ sec.

3. At $\omega = \omega_{c2}$, the slope changes from –40 db/dec to –20 db/dec. This is due to a first factor $(1 + j\omega T_2)$ where, $T_2 = 1/\omega_{c2} = 1/0.8 = 1.25$ sec.

4. At $\omega = \omega_{c3}$, the slope changes from –20 db/dec to –40 db/dec. This is due to a first factor, $1/(1 + j\omega T_3)$ where, $T_3 = 1/\omega_{c3} = 1/10 = 0.1$ sec.

Hence the transfer function, $G_{eq}(j\omega)$ can be written as,

$$G_{eq}(j\omega) = \frac{K(1+j\omega T_2)}{j\omega(1+j\omega T_1)(1+j\omega T_3)} = \frac{3(1+j1.25\omega)}{j\omega(1+j12.5\omega)(1+j0.1\omega)}$$

Put $j\omega = s$, $\therefore G_{eq}(s) = \frac{3(1+1.25s)}{s(1+12.5s)(1+0.1s)}$

Step-6 : Determine the parameter, K_A

From the bode plot it is observed that $|G_{eq}(j\omega)|$ curve should be shifted 17 db upwards to make $\omega_{gc} = 2.2$ rad/sec as the gain crossover frequency of $G_{eq}(j\omega)$ (i.e., to make $|G_{eq}(j\omega)|$ in db = 0 at $\omega = \omega_{gc} = 2.2$ rad/sec).

Now, $20\log K_A = 17$ db $\Rightarrow$ $\log K_A = \frac{17}{20}$ $\Rightarrow$ $K_A = 10^{17/20} = 7.08$

Step-7 : Verify the design

Open loop transfer function of feedback compensated system $= K_A G_{eq}(s) = 7.08 \times \frac{3(1+1.25s)}{s(1+12.5s)(1+0.1s)} = \frac{21.24(1+1.25s)}{s(1+12.5s)(1+0.1s)}$

Velocity error constant, $K_v = \underset{s\to 0}{Lt}\ s K_A G_{eq}(s) = \underset{s\to 0}{Lt}\ s \frac{21.24(1+1.25s)}{s(1+12.5s)(1+0.1s)} = 21.24$

CONCLUSION

Since the velocity error constant of the compensated system satisfies the requirement, the design is accepted.

RESULT

$K_A = 7.08$, $K_t = 3.185$ and $T = 6.25$ sec.

Open loop transfer function of feedback compensated system $= \frac{21.24(1+1.25s)}{s(1+12.5s)(1+0.1s)}$

EXAMPLE 6.18

Consider a unity feedback system with open loop transfer function, $G(s) = K/s(s+2)(s+5)$. Design a feedback compensator to satisfy the following specifications.

(a) Maximum overshoot, $M_p \le 12\%$, (b) Settling time, $t_s \le 3$ sec for 2% error and (c) Velocity error constant, $K_v \ge 4$.

SOLUTION

Step-1 : To find dominant pole, s_d

Given that $M_p \le 12$ %

Let $M_p = 10\%$, $\therefore M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100 = 10\%$ $\Rightarrow$ $e^{-\zeta\pi/\sqrt{1-\zeta^2}} = 0.1$

On taking natural log we get, $-\zeta\pi/\sqrt{1-\zeta^2} = \ln(0.1) = -2.3$

On squaring we get, $\frac{\zeta^2\pi^2}{1-\zeta^2} = 5.29$

$$\therefore \zeta^2\pi^2 = 5.29 - \zeta^2 5.29 \quad \Rightarrow \quad \zeta^2\pi^2 + 5.29\zeta^2 = 5.29 \quad \Rightarrow \quad \zeta = \sqrt{\frac{5.29}{\pi^2 + 5.29}} = 0.59 \approx 0.6$$

Given that, $t_s \le 3$ sec for 2% error. Let, $t_s = 3$ sec for 2% error.

Hence for 2% error, $e^{-\zeta\omega_n t_s} = \frac{2}{100}$

On taking natural log, we get,

$$-\zeta\omega_n t_s = ln(0.02) = -4 \quad \Rightarrow \quad \omega_n = \frac{-4}{-\zeta t_s} = \frac{4}{0.6\times 3} = 2.22 \text{ rad / sec}$$

Dominant pole, $s_d = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2} = -(0.6\times 2.22) + j2.22\sqrt{1-0.6^2} = -1.3 + j1.8$

Step-2 : Pole-zero plot of G(s).

The poles of G(s) are located at s = 0, –2 and –5. The poles are marked by "x" as shown in fig 6.18.1. The dominant pole, s_d is also marked in fig 6.18.1. and it is denoted by point P.

Step-3 : Determine the new transfer function.

Let us choose a feedback compensation scheme as shown in fig 6.18.2.

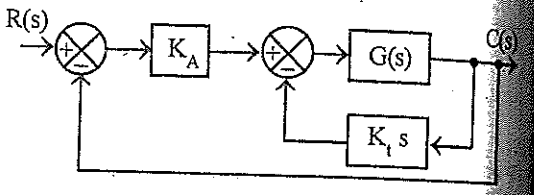


Fig 6.18.2

Given that, $G(s) = \frac{K}{s(s+2)(s+5)}$

$$G_{eq}(s) = \frac{G(s)}{1+G(s)K_t s} = \frac{\frac{K}{s(s+2)(s+5)}}{1+\frac{K}{s(s+2)(s+5)}K_t s} = \frac{K}{s(s+2)(s+5)+KK_t s}$$

Closed loop transfer function $\left\{ \frac{C(s)}{R(s)} = \frac{K_A G_{eq}(s)}{1+K_A G_{eq}(s)} \right.$

$$= \frac{K_A \frac{K}{s(s+2)(s+5)+KK_t s}}{1+K_A \frac{K}{s(s+2)(s+5)+KK_t s}} = \frac{KK_A}{s(s+2)(s+5)+KK_t s+KK_A}$$

The characteristic equation is, $s(s+2)(s+5)+KK_t s+KK_A = 0$ .....(6.18.1)

On dividing the equation (6.18.1) by, $s(s+2)(s+5)$ we get,

$$1+\frac{KK_t(s+K_A/K_t)}{s(s+2)(s+5)} = 0 \qquad \text{.....(6.18.2)}$$

The equation (6.18.2) is in the form of 1 + G(s)H(s) = 0

$$\therefore [G(s)H(s)]_{new} = \frac{KK_t(s+K_A/K_t)}{s(s+2)(s+5)} \qquad \text{.....(6.18.3)}$$

From the equation (6.18.3) we get,

Zero introduced by the feedback compensator, $z_c = -K_A / K_t$ .....(6.18.4)

$$\zeta = \sqrt{\frac{5.29}{\pi^2 + 5.29}} = 0.59 \approx 0.6$$

ad / sec

$$\sqrt{1-0.6^2} = -1.3 + j1.8$$

e marked by "x" as sho
l it is denoted by point

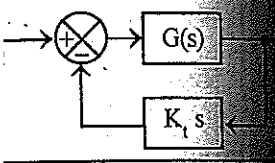


Fig 6.18.2

$$\frac{K}{s+5) + K K_t s}$$

$$\frac{K K_A}{s+5) + K K_t s + K K_A}$$

K_t

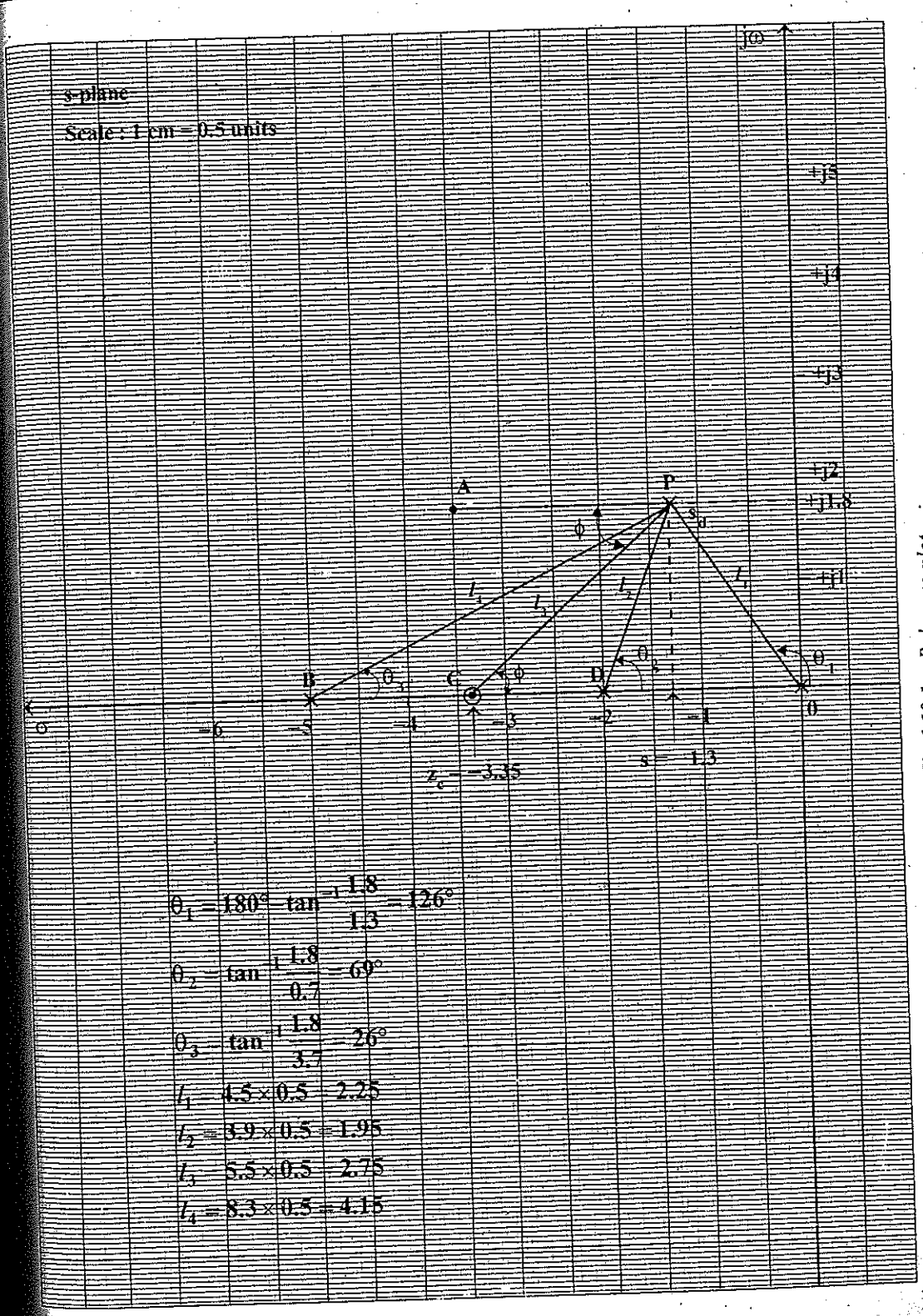


Fig 6.18.1. : Pole zero plot

Step-4 : To locate the zero of the compensator

Let ϕ = Angle contributed by the zero, z_c to make point P as a point on root locus.

$$\text{Now, } \phi = \begin{pmatrix}\text{Sum of angles contributed by} \\ \text{poles of uncompensated system}\end{pmatrix} \pm n180°$$

From pole-zero plot of fig 6.18.1, we get,

$$\left.\begin{matrix}\text{Sum of angles contributed} \\ \text{by poles of uncompensated system}\end{matrix}\right\} = \theta_1 + \theta_2 + \theta_3 = 126° + 69° + 26° = 221°$$

Let $n = 1$, $\quad \therefore \phi = 221° - 180° = 41°$.

With line AP as reference draw a line PC such that $\angle APC = \phi$ as shown in fig 6.18.1. It intersects the real axis at point C.

From the pole-zero plot of fig 6.18.1, we get,

Zero of the compensator, $z_c = -3.35$ .....(6

Step-5 : To find K_A and K_t

On equating equations (6.18.4) and (6.18.5) we get, $K_A/K_t = 3.35$

Let $K_t = 1.2$, $\quad \therefore K_A = K_t \times 3.35 = 1.2 \times 3.35 = 4.02$

Step-6 : To determine the gain K of G(s)

$$\text{Here, } K K_t = \frac{\text{Product of vector lengths from poles to } s = s_d}{\text{Product of vector lengths from zeros to } s = s_d}$$

From fig 6.18.1, we get,

$$K K_t = \frac{l_1 \times l_2 \times l_4}{l_3} = \frac{2.25 \times 1.95 \times 4.15}{2.75} = 6.62 \qquad \therefore K = \frac{6.62}{K_t} = \frac{6.62}{1.2} = 5.5$$

Step-7 : To verify the design

$$\text{Velocity error constant, } K_v = \lim_{s \to 0} s\,[G(s)H(s)]_{new}$$

$$= \lim_{s \to 0} s \frac{K K_t (s + K_A/K_t)}{s(s+1)(s+5)} = \frac{K K_A}{1 \times 5} = \frac{5.5 \times 4.02}{5} = 4.422$$

CONCLUSION

Since the velocity error constant of the compensated system is satisfactory, the design is accep

RESULT

$K = 5.5$; $K_t = 6.2$; $K_A = 4.02$

SHORT QUESTIONS AND ANSWERS

What are the time domain specifications needed to design a control system?

The time domain specifications needed to design a control system are

1. Rise time, t_r
2. Peak overshoot, M_p
3. Settling time, t_s
4. Damping ratio, ζ
5. Natural frequency of oscillation, ω_n

Write the necessary frequency domain specifications for design of a control system.

The frequency domain specifications required to design a control system are,

1. Phase margin
2. Gain margin
3. Resonant peak
4. Bandwidth

What are the two methods of designing a control system?

Two methods of designing a control system are design using root locus and design using bode plot. In design using root locus, the system is designed to satisfy the specified time domain specifications. In design using bode plot, the system is designed to satisfy the specified frequency domain specifications.

What is compensation?

The compensation is the design procedure in which the system behaviour is altered to meet the desired specifications, by introducing additional device called compensator.

What is compensator? What are the different types of compensator?

A device inserted into the system for the purpose of satisfying the specifications is called compensator. The different types of compensators are lag compensator, lead compensator and lag-lead compensator.

What are the two types of compensation schemes?

The two types of compensation schemes employed in control system are series compensation and feedback or parallel compensation.

What is series compensation?

The series compensation is a design procedure in which a compensator is introduced in series with plant to alter the system behaviour and to provide satisfactory performance (i.e., to meet the desired specifications). The block diagram of series compensation scheme is shown in fig Q6.7.

$G_c(s)$ = Transfer function of series compensator
$G(s)$ = Open loop transfer function of the plant.
$H(s)$ = Feedback path transfer function.

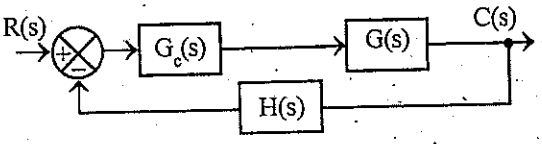


Fig Q6.7 : Series compensation.

What is feedback compensation?

The feedback compensation is a design procedure in which a compensator is introduced in the feedback path so as to meet the desired specifications. It is also called parallel compensation. The block diagram of feedback compensation scheme is shown in fig Q6.8.

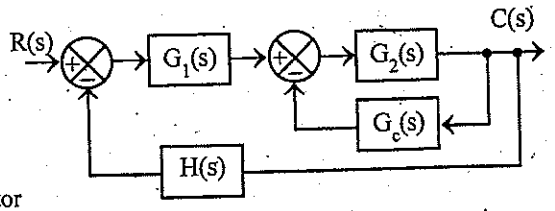


Fig Q6.8 : Feedback compensation

$G_c(s)$ = Transfer function of feedback compensator
$H(s)$ = Feedback path transfer function.
$G_1(s), G_2(s)$ = Open loop transfer function of the components of the plant.

Q6.9. ***What are the factors to be considered for choosing series or shunt/feedback compensation?***

The choice between series, shunt or feedback compensation depends on the following

1. Nature of signals in the system.
2. Power levels at various points.
3. Components available.
4. Designer's experience.
5. Economic considerations.

Q6.10. ***When lag/lead/lag-lead compensation is employed?***

Lag compensation is employed for a stable system for improvement in steady state performance.

Lead compensation is employed for stable/unstable system for improvement in transient state performance.

Lag-lead compensation is employed for stable/unstable system for improvement in both steady state and transient state performance.

Q6.11. ***Why compensation is necessary in feedback control system?***

In feedback control systems compensation is required in the following situations.

1. When the system is absolutely unstable, then compensation is required to stabilize the system and also to meet the desired performance.
2. When the system is stable, compensation is provided to obtain the desired performance.

Q6.12. ***Discuss the effect of adding a pole to open loop transfer function of a system.***

The addition of a pole to open loop transfer function of a system will reduce the steady-state error closer the pole to origin lesser will be the steady-state error. Thus the steady-state performance of system is improved.

Also the addition of pole will increase the order of the system, which inturn makes the system stable than the original system.

Q6.13. ***Discuss the effect of adding a zero to open loop transfer function of a system.***

The addition of a zero to open loop transfer function of a system will improve the transient response. The addition of zero reduces the rise time. If the zero is introduced close to origin then the overshoot will be larger. If the zero is introduced far away from the origin in the left half of s-plane the effect of zero on the transient response will be negligible.

Q6.14. ***How root loci are modified when a zero is added to open loop transfer function?***

The addition of a zero to open loop transfer function will pull the root locus to the left which makes system more stable and reduce the settling time.

Q6.15. ***How the root loci are modified when a pole is added to open loop transfer function of the***

The addition of a pole to the open-loop transfer function has the effect of pulling the root locus right, which reduce the relative stability of the system and increase the settling time.

Q6.16. ***How control system design is carried using root locus?***

In design using root locus the transient response specifications are translated into desired for a pair of dominant closed loop poles.

In order to satisfy the performance specifications, the root loci should pass through the Hence a compensator is introduced in open loop transfer function which will reshape the root force them to pass through the points where the dominant closed loop poles are located.

17. ***What is the advantage in design using root locus.***

The advantage in design using root locus technique is that the information about closed loop transient response and frequency response are directly obtained from the pole-zero configuration of the system in the s-plane.

18. ***What are the informations that can be obtained from frequency response plots?***

The low frequency region of frequency response plot provides information regarding the steady-state performance and high frequency region provides information regarding the transient performance of the system. The mid-frequency region provides information regarding relative stability.

19. ***What are the advantages and disadvantages in frequency domain design.***

The advantages of frequency domain design are the following.

1. The effect of disturbances, sensor noise and plant uncertainities are easy to visualize and asses in frequency domain.
2. The experimental information can be used for design purposes.

The disadvantage of frequency response design is that it gives the information on closed loop system's transient response indirectly.

20. ***What is lag compensation?***

The lag compensation is a design procedure in which a lag compensator is introduced in the system so as to meet the desired specifications.

21. ***What is lag compensator? Give an example.***

A compensator having the characteristics of a lag network is called lag compensator. If a sinusoidal signal is applied to a lag compensator, then in steady state the output will have a phase lag with respect to input. An electrical lag compensator can be realised by a R-C network. The R-C network shown in fig Q6.21 is an example of electrical lag compensator.

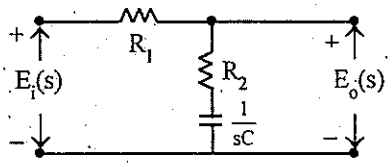


Fig Q6.21 : Lag compensator

22. ***Write the transfer function of lag compensator and draw its pole-zero plot.***

$$\left.\begin{array}{l}\text{Transfer function of}\\ \text{lag compensator}\end{array}\right\} g_c(s) = \frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}$$

The lag compensator has a pole at $s = -1/\beta T$ and a zero at $s = -1/T$. Since $\beta > 1$ and $T > 0$, the pole of lag compensator is nearer to origin. The pole-zero plot of lag compensator is shown in fig Q6.22.

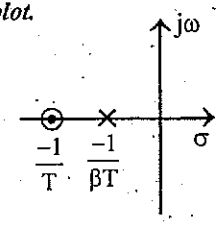


Fig Q 6.22 : Pole-zero plot of lag compensator

23. ***What are the characteristics of lag compensation? When lag compensation is employed?***

The lag compensation improves the steady state performance, reduce the bandwidth and increases the rise time. (The increase in rise time results in slower transient response). If the pole introduced by the compensator is not cancelled by a zero in the system then the lag compensator increases the order of the system by one.

When a system is stable and does not satisfy the steady-state performance specifications then lag compensation can be employed so that the system is redesigned to satisfy the steady-state requirements.

Q6.24. ***Draw the bode plot of lag compensator.***

Let, $G_c(j\omega)$ = Sinusoidal transfer function of lag compensator.

The approximate magnitude plot and phase plot at $G_c(j\omega)$ are shown in fig Q6.24.

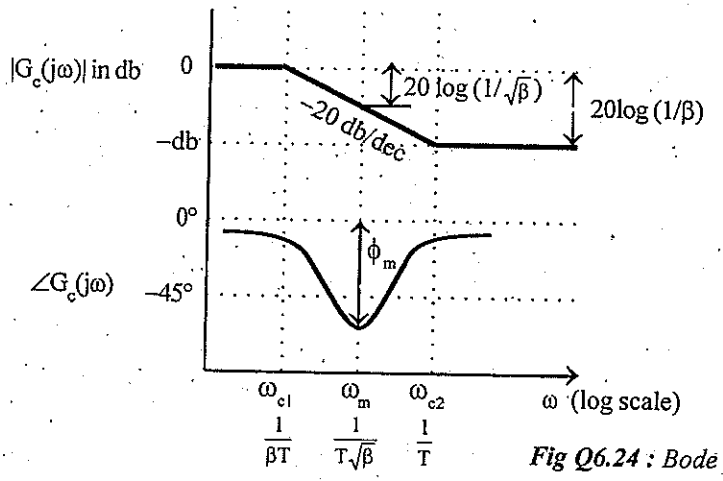


Fig Q6.24 : Bode plot of lag compensator

Q6.25. ***When maximum phase lag occurs in lag compensator? Give the expressions for maximum angle and the corresponding frequency.***

The maximum phase lag occurs at the geometric mean of two corner frequencies of the lag compen

Maximum phase lag angle, $\phi_m = \tan^{-1}\dfrac{(1-\beta)}{2\sqrt{\beta}}$

Frequency corresponding to maximum phase lag angle $\Big\}\ \omega_m = \sqrt{\omega_{c1}\,\omega_{c2}} = \sqrt{\dfrac{1}{T}\,\dfrac{1}{\beta T}} = \dfrac{1}{T\sqrt{\beta}}$

Q6.26. ***What is the relation between ϕ_m and β in lag compensator?***

In lag compensator the ϕ_m and β are related by the expressions,

$$\phi_m = \tan^{-1}\left(\frac{1-\beta}{2\sqrt{\beta}}\right) \qquad \text{or} \qquad \beta = \frac{1-\sin\phi_m}{1+\sin\phi_m}$$

Since $\beta > 1$, from the above expressions we can conclude that, larger the value of β the larger will b value of ϕ_m.

Q6.27. ***Write the two equations that relates β and ϕ_m of lag compensator.***

The following two equations relates ϕ_m and β of lag compensator.

$$\phi_m = \tan^{-1}\frac{1-\beta}{2\sqrt{\beta}} \quad ; \quad \beta = \frac{1-\sin\phi_m}{1+\sin\phi_m}$$

Q6.28. ***What is lead compensation?***

The lead compensation is a design procedure in which a lead compensator is introduced in the so as to meet the desired specifications.

g Q6.24.

lag compensator.

pressions for maximum

iencies of the lag compens

alue of β the larger will

tor is introduced in the

29. *What is lead compensator? Given an example.*

A compensator having the characteristic of a lead network is called a lead compensator. If a sinusoidal signal is applied to a lead compensator, then in steady state the output will have a phase lead with respect to input. An electrical lead compensator can be realised by a R-C network. The R-C network shown in fig Q6.29. is an example of electrical lead compensator.

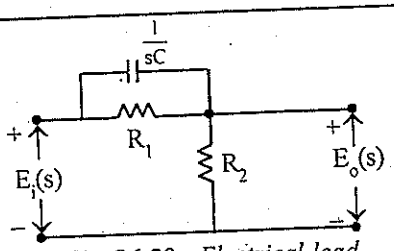


Fig Q6.29 : Electrical lead compensator

30. *Write the transfer function of lead compensator and draw its pole-zero plot.*

Transfer function of lead compensator, $g_c(s) = \dfrac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}}$

The lead compensator has a pole at $s = -1/\alpha T$ and a zero at $s = -1/T$. Since $\alpha < 1$ and $T > 0$, the zero of lead compensator is nearer to origin. The pole-zero plot of lead compensator is shown in fig Q6.30.

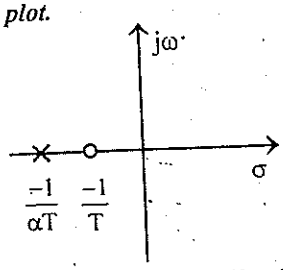


Fig Q6.30 : Pole-zero plot of lead compensator

31. *What are the characteristics of lead compensation? When lead compensation is employed?*

The lead compensation increases the bandwidth and improves the speed of response. It also reduces the peak overshoot. If the pole introduced by the compensator is not cancelled by a zero in the system, then lead compensation increases the order of the system by one. When the given system is stable/ unstable and requires improvement in transient state response then lead compensation is employed.

32. *Draw the bode plot of lead compensator.*

Let, $G_c(j\omega)$ = Sinusoidal transfer function of lead compensator. The approximate magnitude plot and phase plot of $G_c(j\omega)$ are shown in fig Q6.32.

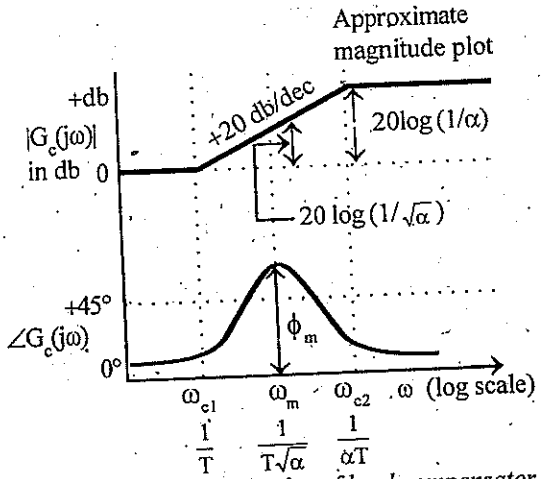


Fig Q6.32 : Bode plot of lead compensator

33. *What is the relation between ϕ_m and α in lead compensator?*

In lead compensator the ϕ_m and α are related by the expression,

$$\phi_m = \tan^{-1}\left(\frac{1-\alpha}{2\sqrt{\alpha}}\right) \quad \text{or} \quad \alpha = \frac{1-\sin\phi_m}{1+\sin\phi_m}$$

Since $\alpha < 1$ from the above expressions we can conclude that, smaller the value of α the larger will be the value of ϕ_m.

34. *When maximum phase lead occurs in lead compensator? Give the expressions for maximum lead angle and the corresponding frequency.*

The maximum phase lead occurs at the geometric mean of two corner frequencies of the lead compensator.

Maximum phase lead angle, $\phi_m = \tan^{-1}\dfrac{(1-\alpha)}{2\sqrt{\alpha}}$

Frequency corresponding to maximum phase lead angle $\omega_m = \sqrt{\omega_{c1}\,\omega_{c2}} = \sqrt{\frac{1}{T}\,\frac{1}{\alpha T}} = \frac{1}{T\sqrt{\alpha}}$

Q6.35. *Write the two equations that relates α and ϕ_m of lead compensator*

$$\phi_m = \tan^{-1}\left(\frac{1-\alpha}{2\sqrt{\alpha}}\right) \quad ; \quad \alpha = \frac{1-\sin\phi_m}{1+\sin\phi_m}$$

Q6.36. *What is lag-lead compensation?*

The lag-lead compensation is a design procedure in which a lag-lead compensator is introduced in the system so as to meet the desired specifications.

Q6.37. *What is lag-lead compensator? Given an example.*

A compensator having the characteristics of lag-lead network is called lag-lead compensator. If a sinusoidal signal is applied to a lag-lead compensator then the output will have both phase lag and lead with respect to input, but in different frequency regions.

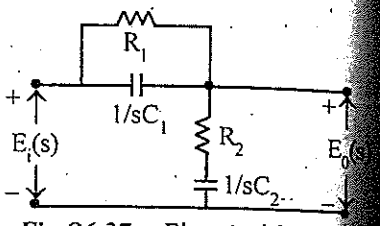


Fig Q6.37 : Electrical lag-lead compensator

An electrical lag-lead compensator can be realised by a R-C network. The R-C network shown in fig Q6.37 is an example of electrical lag-lead compensator.

Q6.38. *Write the transfer function of lag-lead compensator and draw its pole-zero plot.*

$$\text{Transfer function of lag-lead compensator} \Big\} \; G_c(s) = \underbrace{\frac{(s+1/T_1)}{(s+1/\beta T_1)}}_{\text{lag section}} \; \underbrace{\frac{(s+1/T_2)}{(s+1/\alpha T_2)}}_{\text{lead section}}$$

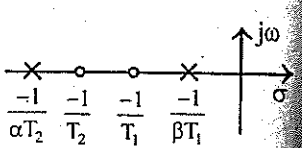


Fig Q 6.38 : Pole-zero plot of lag-lead compensator

Q6.39. *What are the characteristics of lag-lead compensation? When lag-lead compensation is emplo...*

The lag-lead compensation has the characteristics of both lag compensation and lead compens... The lag compensation improves the steady state performance and decreases the bandwidth. The... compensation increases the bandwidth and improves the speed of response. It also reduces the... overshoot. If the poles introduced by the compensator is not cancelled by zeros in the system the... lag-lead compensator increases the order of the system by two.

The lag-lead compensation is employed when improvements in both steady-state and transient r... are required.

Q6.40. *Draw the bode plot of lag-lead compensator.*

Let, $G_c(j\omega)$ = Sinusoidal transfer function of lag-lead compensator.

The approximate magnitude plot and phase plot of $G_c(j\omega)$ are shown in fig Q6.40.

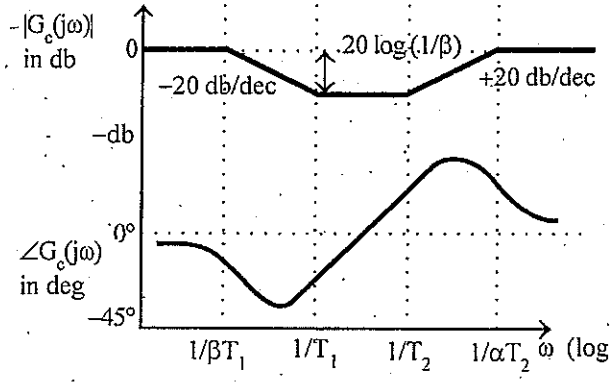
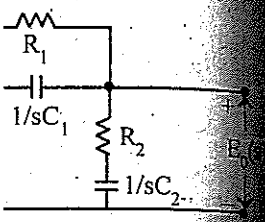
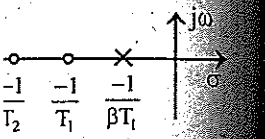


Fig Q6.40 : Bode ... lag-lead compen...

...pensator is introduced in...



37 : Electrical lag-lead compensator



Q 6.38 : Pole-zero plot ... lag-lead compensator

compensation is employ...

...tion and lead compensat... ...ses the bandwidth. The ... ...se. It also reduces the ... ...zeros in the system then...

...-state and transient resp...

...; Q6.40.

...g Q6.40 : Bode plot ... ...ag-lead compensator

What is P-controller and what are its characteristics?

The proportional controller is a device that produces an output signal which is proportional to the input signal.

The proportional controller improves the steady state tracking accuracy, disturbance signal rejection and relative stability. It also decreases the sensitivity of the system to parameter variations.

What is PI-controller and what are its effect on system performance?

The PI-controller is a device that produces an output signal consisting of two terms-one proportional to input signal and the other proportional to the integral of input signal.

The introduction of PI-controller in the system reduces the steady state error and increases the order and type number of the system by one.

Write the transfer function of PI-controller.

$$\left.\begin{matrix}\text{Transfer function}\\ \text{of PI - controller}\end{matrix}\right\} G_c(s) = K_p + \frac{K_i}{s} = \frac{K_p\,(s + K_i/K_p)}{s}$$

What is PD-controller and what are its effect on system performance?

The PD-controller is a device that produces an output signal consisting of two terms-one proportional to input signal and the other proportional to the derivative of input signal.

The PD-controller increases the damping of the system which results in reducing the peak overshoot.

Write the transfer function of PD-controller.

Transfer function of PD - controller, $G_c(s) = K_p + K_d s = K_d\,(s + K_p/K_d)$

What is PID controller and what are its effect on system performance?

The PID controller is a device which produces an output signal consisting of three terms-one proportional to input signal, another one proportional to integral of input signal and the third one proportional to derivative of input signal.

The PID controller stabilises the gain, reduces the steady state error and peak overshoot of the system.

Write the transfer function of PID controller.

$$\left.\begin{matrix}\text{Transfer function}\\ \text{of PID - controller}\end{matrix}\right\} G_c(s) = K_p + \frac{K_i}{s} + K_d s = \frac{K_d s^2 + K_p s + K_i}{s}$$

What is feedback compensation?

The feedback compensation is a design procedure in which a compensator is placed in an internal feedback path around one or more components of the forward path so as to meet the desired specifications.

Draw the block diagram of a feedback compensation scheme.

The block diagram of a popular feedback scheme employed in control systems is shown in fig Q6.49.

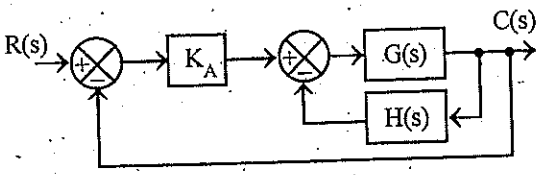


Fig Q6.49 : Feedback compensation scheme

Here, H(s) = Transfer function of feedback compensator

K_A = A parameter to adjust the velocity error constant of the system.

Q6.50. What is the disadvantage in rate feedback and how it is eliminated?

The disadvantage in the rate feedback is that the system velocity error constant, K_v is reduced. undesirable effect can be eliminated by reducing the feedback signal in the low frequency introducing a high pass filter in cascade with rate device as shown in fig Q6.50.

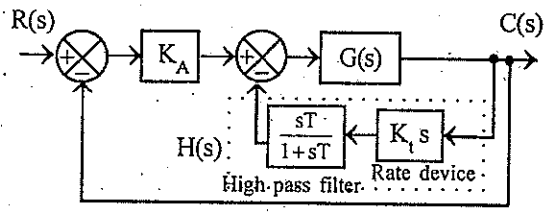


Fig Q.6.50 : Feedback compensation with high-pass filter in cascade with rate device

6.8 EXERCISES

1. Design a phase lag network for a system having $G(s) = K/s(1+0.2s)^2$ to have a phase margin of 3
2. The open loop transfer function of a certain unity feedback control system is given by G(s) = K/s(is desired to have the velocity error constant, $K_v = 10$ and the phase margin to be atleast 60°. D phase lag series compensator.
3. The controlled plant of a unity feedback system is G(s) = K/s(s+5). It is specified that veloci constant of the system be equal to 15, while the damping ratio is 0.6 and velocity error is less than per unit ramp input. Design a suitable lag compensator.
4. Consider the unity feedback system with an open loop transfer G(s) = K/s(s+2)². Design a suitable lead compensator so that phase margin is atleast 50° and velo constant is 20 s^{-6}.
5. The open loop transfer function of certain unity feedback control system is given by G(s) = K/ (0.2s+1). It is desired to have the phase margin to be atleast 30°. Design a suitable phase compensator.
6. A unity feedback system with open loop transfer function $G(s) = K/s^2(s+1.5)$ is to be lead com satisfy the following specifications.
   (i) Damping ratio = 0.45
   (ii) Undamped natural frequency, $\omega_n = 2.2$ rad/sec
   (ii) Velocity error constant, $K_v = 30$.
7. Consider a unity feedback control system whose forward transfer G(s) = K/s(s+2)(s+8). Design a lag-lead compensator so that $K_v = 80\ s^{-1}$ and dominant clo are located at $-2 \pm j2\sqrt{3}$.
8. The open loop transfer function of uncompensated system is G(s) = K/s (s +1)(s + 4). Des compensator to meet the following specifications.
   (i) Velocity error constant ≥ 5
   (ii) Damping ratio = 0.4.

Consider a unity feedback system with open loop transfer function, G(s) = K/s(2s+1)(0.5s+1). Design a suitable lag-lead compensator to meet the following specifications.

(i) $K_v = 30$ (ii) Phase margin ≥ 50

Consider a unity feedback system with open loop transfer function, G(s)= 1/s(s+1). Design a PD controller so that the phase margin of the system is 30° at a frequency of 2 rad/sec.

A unity feedback system has an open loop transfer function as G(s) = 50/(s +3) (s +1). Design a PI controller so that phase margin of the system is 35° at a frequency of 1.2 rad/sec.

Consider a unity feedback system with open loop transfer function, G(s)=20/(s+0.5)(s+2)(s+4). Design a PID controller so that the phase margin of the system is 30° at a frequency of 2 rad/sec and steady state error for unit ramp input is 0.1.

Consider a unity feedback system with open loop transfer function G(s) = 10/s(s + 4). The dominant poles are $-2 \pm j\sqrt{6}$. Design a suitable PD controller.

Consider a unity feedback system with open loop transfer function G(s) = 10/(s +1) (s +2). Design a PI controller so that the closed loop has damping ratio of 0.707 and natural frequency of oscillation as 1.2 rad/sec.

Consider a unity feedback system with G(s) = 50/(s+2) (s+10). Design a PID controller to satisfy the following specifications.

(i) $K_v \geq 2$

(ii) Damping ratio = 0.6

(iii) Natural frequency of oscillations = 2 rad/sec.

Design a feedback compensation scheme for a unity feedback system with open loop transfer function G(s) = 1/s² (s + 5) to satisfy the following specifications. (i) phase margin of the system should be atleast 50°. (ii) Velocity error constant, $K_v \geq 20$.

Consider a unity feedback system with open loop transfer function, G(s) = K/s(s+1)(s+4). Design a feedback compensator to satisfy the following specifications.

(a) Maximum overshoot, $M_p \leq 12\%$

(b) Settling time, $t_s = 10s$

(c) Velocity error constant, $K_v = 10$.

5. State Variable Analysis and Concept of State Variables.

The state variable approach is a powerful tool or technique for the analysis and design of Control systems. The analysis and design of the following systems Can be Carried using state space Method.

1. Linear system
2. Non-Linear system.
3. Time invariant system.
4. Time varying system.
5. Multiple input and Multiple output system.

The state space analysis is a modern approach and also easier for analysis using digital Computers.

The Conventional (or old) methods of analysis employs the transfer function of the system. The drawbacks in the Transfer function Model and analysis are: 1) Transfer function is defined under zero initial Conditions.

2) Transfer function is applicable to linear time invariant systems.

3) Transfer function analysis is restricted to single input and single output system.

4. Does not provide information regarding the internal state of the system.

The state variable analysis can be applied for any type of systems. The analysis can be carried with initial conditions and can be carried on multiple input and multiple output systems.

State:-

It is the group of variables, which summarizes the history of the system in order to predict the future values.

"State is because of past (history) and is used to predict future value"

State Variables :-

The smallest set of variables, that determines the state of the system. are known as state variable.

State vector :-

It is the vector, which contains the state variables as elements.

Advantages of state space Analysis

1. Analysis is done by considering initial conditions
2. More accurate than Transfer function.
3. Analysis of multi input and multi output systems

Disadvantages of state space Analysis.

1. Complex techniques
2. Many computations are required.

4. Gives information about the Controllability.

5. It is applicable to all dynamic systems.

In the state variable formulation of a system, in general, a system consists of m- inputs, p-outputs and n- state variables.

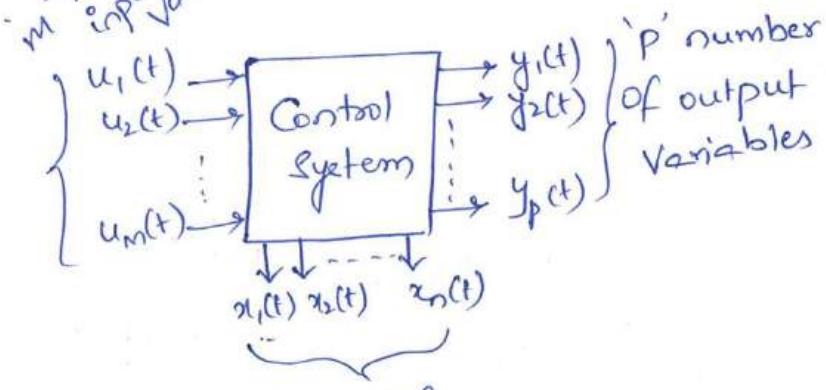
State-space formulation :-

The state space representation of the system may be visualized as:

Let. State variables = $x_1(t), x_2(t), x_3(t) \cdots x_n(t)$

Input variables = $u_1(t), u_2(t), u_3(t) \cdots u_m(t)$

Output variables = $y_1(t), y_2(t), y_3(t) \cdots y_p(t)$.



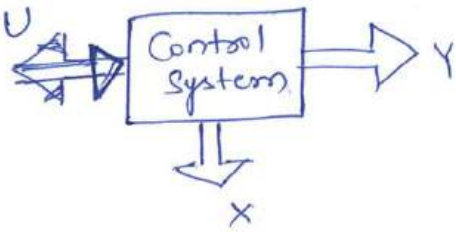


fig: state space representation of system.

The different variables represented by the vectors (column matrix)

Input vector $u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}$; Output vector $Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}$

State variable vector $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$.

State Equation :-

The state variable representation can be arranged in the form of n number of first order differential equations as shown in below

$$\frac{dx_1}{dt} = \dot{x}_1 = f_1(x_1, x_2 \cdots x_n ; u_1, u_2 \cdots u_m)$$

$$\frac{dx_2}{dt} = \dot{x}_2 = f_2(x_1, x_2 \cdots x_n ; u_1, u_2 \cdots u_m)$$

$$\vdots$$

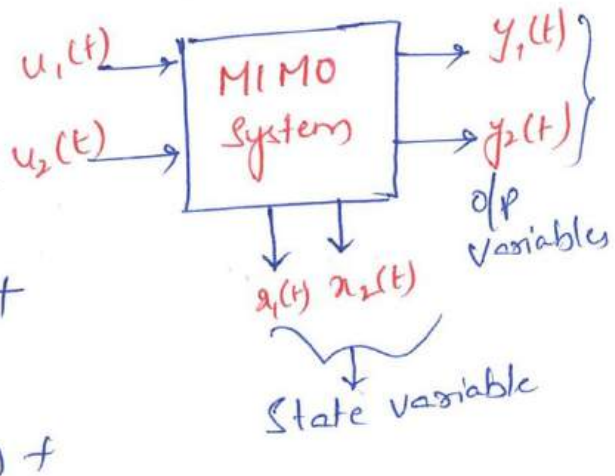
$$\frac{dx_n}{dt} = \dot{x}_n = f_n(x_1, x_2 \cdots x_n ; u_1, u_2 \cdots u_m)$$

The n numbers of differential equations may be written in vector notation as.

$$\dot{x}(t) = f(x(t), u(t)).$$

State Model :

When we are Analysing Multi input Multi Output System with the State Space Analysis, the output Equation and State equation Collectively known as "State Model".



Output equation

Consider output variables

$$y_1(t) = C_{11} x_1(t) + C_{12} x_2(t) + D_{11} u_1(t) + D_{12} u_2(t)$$

$$y_2(t) = C_{21} x_1(t) + C_{22} x_2(t) + D_{21} u_1(t) + D_{22} u_2(t).$$

Consider a Matrix

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$\therefore$ o/p equation.

$$\boxed{Y(t) = C\,x(t) + D\,U(t)} \quad \text{---(1)}$$

o/p equation can be written with → state variables & input variables.

$\therefore$ State equation is

$$\frac{dx_1}{dt} = \dot{x}_1(t) = a_{11}x_1(t) + a_{12}x_2(t) + b_{11}u_1(t) + b_{12}u_2(t)$$

$$\frac{dx_2}{dt} = \dot{x}_2(t) = a_{21}x_1(t) + a_{22}x_2(t) + b_{21}u_1(t) + b_{22}u_2(t)$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$\boxed{\dot{X}_1(t) = A\,X(t) + B\,U(t)} \rightarrow \text{state equation.} \quad \text{---(2)}$$

"The Combination of eq (1) & (2) is called "state Model".

State Diagram:

The pictorial Representation of the state model of the system is called "state Diagram".

The state Diagram of the system can be either in Block diagram or in the signal flow graph form.

The s-domain state diagram can be obtained from the transfer function of the system. The state diagram provides a direct relation between time domain and s-domain.

The state diagram (Block diagram or signal flow graph) of a state model is constructed using three basic elements

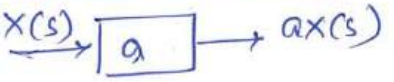
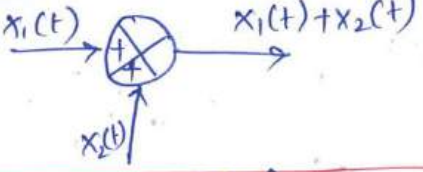
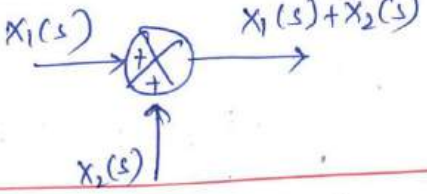
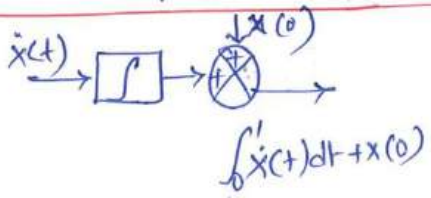
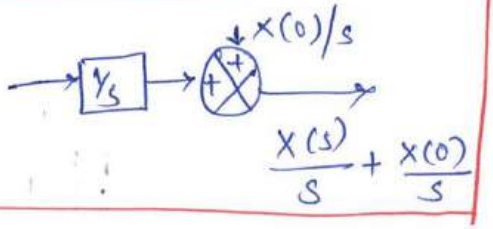
1. Scalar
2. Adder
3. Integrator.

Scalar: The scalar is used to multiply a signal by a constant. The input signal $x(t)$ is multiplied by the scalar a to give the output, $a\,x(t)$.

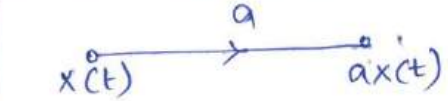
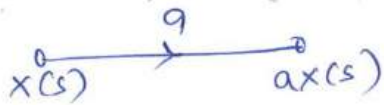
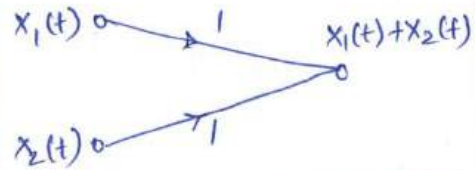
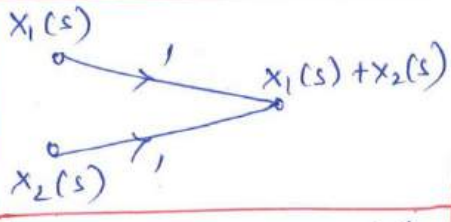
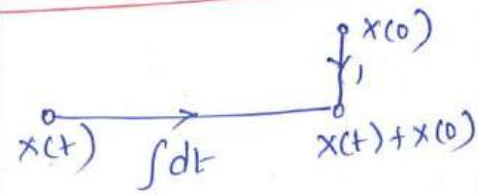
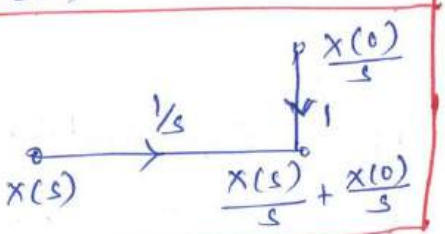
Adder: The adder is used to ~~multiply~~ add two or more signals. The output of the adder is the sum of incoming signals.

Integrator: The integrator is used to integrate the signal. They are used to integrate the derivatives of state variables to get the state variables. The initial conditions of the state variable can be added by using an adder after integrator.

Elements of Block diagram:-

Element	Time domain	s-domain
Scalar		
Adder		
Integrator		

Elements of signal flow graph:-

Element	Time domain	s-domain
Scalar		
Adder		
Integrator		

The state Model of LTI system is given by the equations.

$\dot{X}(t) = AX(t) + BU(t)$ ⟶ state equation

$Y(t) = CX(t) + DU(t)$ ⟶ output equation.

The time domain block diagram representation of

the state model is shown in fig 1 and the time domain signal flow graph representation of the system shown in fig 2.

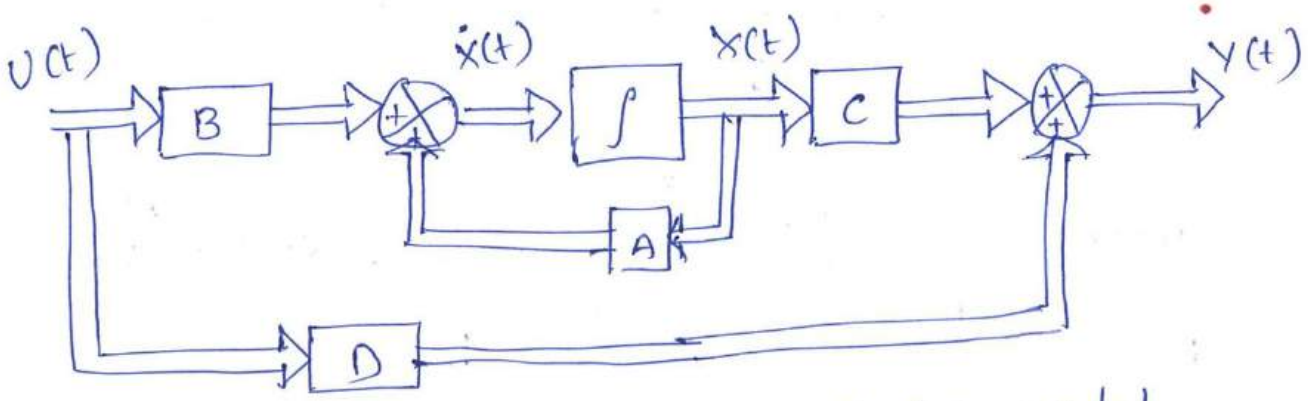


fig: Block diagram of state model.

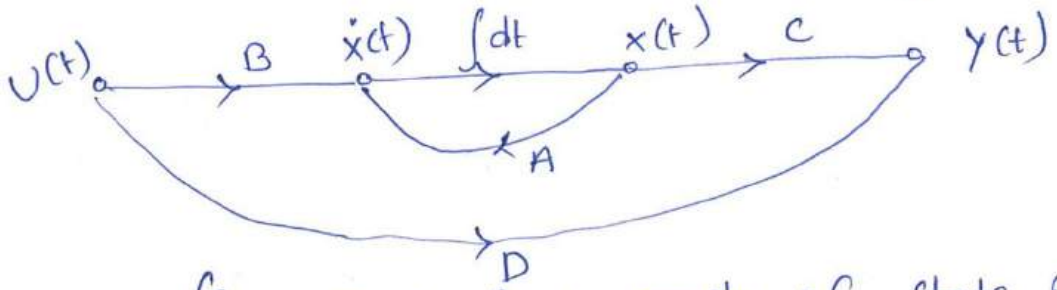


fig: signal flow graph of state model.

State Transition Matrix:

"It is the Matrix which satisfy linear homogeneous state equation called "state transition Matrix".

we have state equation $\dot{x}(t) = \frac{dx(t)}{dt} = Ax(t) + Bu(t)$ → Dynamic —— (1)

Linear homogenous state Equation $\dot{x}(t) = \frac{dx(t)}{dt} = Ax(t)$ —— (2)

A matrix which satisfy the (2) eq is known as State Transition Matrix.

Consider State Transition Matrix = $\phi(t)$

$\phi(t)$ definately need to satisfy eq ② then we Can Say $\phi(t)$ is a state transition Matrix. So that we can replace $x(t)$ as $\phi(t)$

$[\because x(t) = \phi(t)]$

$$\dot{x}(t) = \frac{d\phi(t)}{dt} = A\phi(t) \quad \text{——} (3)$$

$\therefore x(t) = \phi(t)$ $[\because x(0)$ is initial Condition at $t=0]$.

$$x(t) = \phi(t) \cdot x(0) \quad \text{——} (4)$$

To get $\phi(t)$ we need to take Laplace Transform of eq ④.

$$\dot{x}(t) = A\frac{dx(t)}{dt} = Ax(t)$$

$$L\left[\frac{dx(t)}{dt}\right] = L[Ax(t)]$$

$$sX(s) - x(0) = AX(s)$$

$$sX(s) - AX(s) - X(0)$$

$[\because I$ Identity Matrix $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ Multiplying with I, the result is same)

$$IsX(s) - AX(s) = X(0)$$

$$X(s)[Is - A] = X(0)$$

$$X(s) = [sI - A]^{-1}X(0) \quad \text{——} (5)$$

To get $x(t)$ need to take inverse Laplace transform of eq ⑤

$$L^{-1}[X(s)] = L^{-1}[sI - A]^{-1}X(0)$$

$$\boxed{x(t) = L^{-1}[sI - A]^{-1}x(0)}$$

$$\therefore \boxed{\phi(t) = L^{-1}\left[sI - A\right]^{-1}} \qquad \left[\because x(t) = \phi(t)\right]$$

↳ State Transition Matrix

→ State Transition Matrix is not unique but Transfer function is unique.

→ $|sI - A| = 0$ gives the characteristic equation so that we can obtain poles and Eigen values.

→ $\phi(s) = \left[sI - A\right]^{-1}$ Called as "reluctant Matrix".

$\therefore \phi(t) = L^{-1}[\phi(s)] \longrightarrow$ "Reluctant Matrix".

→ $\phi(0) = I$

→ $\phi^{-1}(t) = \phi(-t)$

→ $\left[\phi(t)\right]^k = \phi(kt)$

→ $\phi(t_2 - t_1) - \phi(t_1 - t_0) = \phi(t_2 - t_0)$

} properties of State Transition Matrix.

Problem:

Determine the State Transition Matrix of the Matrix

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 0 \\ 1 & 1\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}1 \\ 1\end{bmatrix}\begin{bmatrix}u_1 \\ u_2\end{bmatrix}, \quad y(t) = \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix}$$

Sol: Given problem is a state model we can compare with State equation and output equation.

$$\dot{X}(t) = AX + BU \qquad Y(t) = CX + DU$$

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 0 \\ 1 & 1\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}1 \\ 1\end{bmatrix}\begin{bmatrix}u_1 \\ u_2\end{bmatrix} \qquad = \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + 0.$$

∴ State Transition Matrix $\phi(t) = L^{-1}\left[(sI-A)^{-1}\right]$

$$sI - A = s\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$$

$$= \begin{bmatrix}s-1 & 0\\ -1 & s-1\end{bmatrix}$$

$$\left[sI-A\right]^{-1} = \frac{Adj\left[sI-A\right]}{|sI-A|}$$

$$|sI-A| = \begin{vmatrix} s-1 & 0 \\ -1 & s-1 \end{vmatrix}$$

$$= (s-1)(s-1) - (-1)(0)$$

$$= s^2 - 2s + 1 - 0$$

$$= s^2 - 2s + 1$$

$$Adj\left[sI-A\right] = \begin{bmatrix}s-1 & 0\\ -1 & s-1\end{bmatrix}$$

$$= \begin{bmatrix}s-1 & -0\\ 1 & s-1\end{bmatrix}$$

$$\left[sI-A\right]^{-1} = \begin{bmatrix}s-1 & 0\\ 1 & s-1\end{bmatrix} \cdot \frac{1}{s^2-2s+1} \Rightarrow \begin{bmatrix}s-1 & 0\\ 1 & s-1\end{bmatrix} \cdot \frac{1}{(s-1)^2}$$

$$= \begin{bmatrix}\frac{1}{s-1} & 0\\ \frac{1}{(s-1)^2} & \frac{1}{s-1}\end{bmatrix}$$

∴ $\phi(t) = L^{-1}\left[sI-A\right]^{-1}$

from problem given

$\left[\because A = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\right.$

$I = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$B = \begin{bmatrix}1\\ 1\end{bmatrix}$

$C = \begin{bmatrix}1 & 1\end{bmatrix}$

$\left[\because |sI-A| = ad-bc\right.$

$\left[\because Adj\begin{bmatrix}a & b\\ c & d\end{bmatrix}\right.$

$= \begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$

$$\phi(t) = L^{-1}\begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{s-1} \end{bmatrix}$$

$\because L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

$L^{-1}\left[\frac{1}{(s-a)^2}\right] = t \cdot e^{at}$

$$\phi(t) = \begin{bmatrix} e^t & 0 \\ t \cdot e^t & e^t \end{bmatrix}$$ → state Transition Matrix for the given model

Problem:

Determine the state Transition Matrix of a Model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \; ; \; y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Sol: Equate the Model with the state Model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \; ; \; y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\dot{x}(t) = Ax(t) + Bu(t) \; ; \; Y(t) = (x(t) + 1)u(t)$$

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$\phi(t) = L^{-1}\left[(sI - A)^{-1}\right]$$

$$sI - A = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} s+1 & -1 \\ 0 & s+2 \end{bmatrix}$$

$$[SI-A]^{-1} = \frac{\text{Adj}[SI-A]}{|SI-A|}$$

$$|SI-A| = \begin{vmatrix} S+1 & -1 \\ 0 & S+2 \end{vmatrix}$$

$$= (S+1)(S+2) + 0$$

$$= (S+1)(S+2)$$

$$\text{Adj}[SI-A] = \text{Adj}\begin{bmatrix} S+1 & -1 \\ 0 & S+2 \end{bmatrix} \qquad \left[\because \text{Adj}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right]$$

$$= \begin{bmatrix} S+2 & 1 \\ 0 & S+1 \end{bmatrix}$$

$$[SI-A]^{-1} = \begin{bmatrix} S+2 & 1 \\ 0 & S+1 \end{bmatrix} \left\{ \frac{1}{(S+1)(S+2)} \right.$$

$$= \begin{bmatrix} \frac{S+2}{(S+1)(S+2)} & \frac{1}{(S+1)(S+2)} \\ 0 & \frac{1}{S+2} \end{bmatrix}$$

$$L^{-1}[SI-A]^{-1} = L^{-1}\begin{bmatrix} \frac{S+2}{(S+1)(S+2)} & \frac{1}{(S+1)(S+2)} \\ 0 & \frac{1}{S+2} \end{bmatrix}$$

$$L^{-1}\left[\frac{S+2}{(S+1)(S+2)}\right] \Rightarrow \frac{S+2}{(S+1)(S+2)} = \frac{A}{(S+1)} + \frac{B}{(S+2)}$$

$$\frac{S+2}{(S+1)(S+2)} = \frac{A(S+2)+B(S+1)}{(S+1)(S+2)}$$

$$S+2 = A(S+2) + B(S+1)$$

Put $S=-1 \rightarrow 1 = A$

Put $S=-2 \rightarrow 0 = -B$

$\therefore A=1, \quad B=0$

$$\therefore \frac{S+2}{(S+1)(S+2)} = \frac{1}{S+1} + 0$$

$$L^{-1}\left[\frac{1}{(S+1)(S+2)}\right] = \frac{A}{S+1} + \frac{B}{S+2}$$

$$\frac{1}{(S+1)(S+2)} = \frac{A(S+2)+B(S+1)}{(S+1)(S+2)}$$

$$1 = A(S+2) + B(S+1)$$

$S \Rightarrow -1 \Rightarrow 1 = A$

$S \Rightarrow -2 \Rightarrow 1 = -B$

$L^{-1}\left[\frac{1}{S+1} \oplus \frac{1}{S+2}\right]$

$A=1, \quad B=-1 \Rightarrow e^{-t} - e^{-2t}$

$$L^{-1}\left[\frac{1}{S+1}\right] = e^{-t} \; ; \; L^{-1}\left[\frac{1}{S+2}\right] = e^{-2t} \; ; \; L^{-1}[0] = 0$$

Put the above Laplace Transform values in $L^{-1}\left[(S\mathbb{I}-A)^{-1}\right]$ (Inverse)

$$\therefore L^{-1}\left[(S\mathbb{I}-A)^{-1}\right] = L^{-1}\begin{bmatrix} \frac{1}{S+1} & \frac{1}{(S+1)(S+2)} \\ 0 & \frac{1}{S+2} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} & e^{-t}-e^{-2t} \\ 0 & e^{-2t} \end{bmatrix}$$

PPQ

Problem:

A system is given by the state equation

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \; ; \; x(0) = x^0.$$

Sol: Using Laplace Transform technique, transform the state equations into a set of linear algebraic equations.

Sol: Compare Given state Model with the State equation.

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \quad ;$$

$$x(0) = \dot{x}^{\circ} = 1$$

$$\dot{x} = Ax + Bu$$

$$\therefore A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

When the given state Equation satisfy the linear algebraic equation then it is called "state Transition Matrix".

$$\phi(t) = L^{-1}\left[(sI - A)^{-1}\right] x(0) \qquad \left[\because x(0) = \dot{x}^{\circ} = 1\right]$$

$$\phi(t) = L^{-1}\left[(sI - A)^{-1}\right]$$

$$sI - A \Rightarrow s\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & -3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & -3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 0 & s+3 \end{bmatrix}$$

$$(SI-A)^{-1} = \frac{Adj(SI-A)}{|SI-A|}$$

$$|SI-A| = \begin{vmatrix} S & -1 & 0 \\ 0 & S & -1 \\ 1 & 0 & S+3 \end{vmatrix}$$

$$= S\left(S(S+3)+0\right) + 1\left(0(S+3)+1\right) + 0\left(0-S\right)$$

$$= S\left(S^2+3S\right) + 1 + 0$$

$$= S^3+3S^2+1$$

$$Adj(SI-A) = Adj\begin{bmatrix} S & -1 & 0 \\ 0 & S & -1 \\ 1 & 0 & S+3 \end{bmatrix} = Adj\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Row, column.

$$= \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Coefficient sign representation

Indicate as Row + column
⇒ even = +
odd = −
Ex: $A_{11} \Rightarrow A_{1+1} = A_2$
$= A_2$ is the

$$A_{12} = \begin{vmatrix} 0 & -1 \\ 1 & S+3 \end{vmatrix} \qquad A_{13} = \begin{vmatrix} 0 & S \\ 1 & 0 \end{vmatrix} \qquad \text{for } A_{11} = \begin{vmatrix} S & -1 \\ 0 & S+3 \end{vmatrix}$$

$= 0+1$ $\qquad = 0-S$ $\qquad = S(S+3)+0$

$\boxed{A_{12} = 1}$ $\qquad \boxed{A_{13} = -S}$ $\qquad \boxed{A_{11} = S^2+3S}$

$$A_{21} = \begin{vmatrix} -1 & 0 \\ 0 & S+3 \end{vmatrix} \qquad A_{22} = \begin{vmatrix} 0 & -1 \\ 1 & S+3 \end{vmatrix} \qquad A_{23} = \begin{vmatrix} 0 & S \\ 1 & 0 \end{vmatrix}$$

$= (-S-3) - 0$ $\qquad = 0+1$ $\qquad = 0-S$

$\boxed{A_{21} = -(S+3)}$ $\qquad \boxed{A_{22} = 1}$ $\qquad \boxed{A_{23} = -S}$

$$A_{31} = \begin{vmatrix} -1 & 0 \\ S & -1 \end{vmatrix} \qquad A_{32} = \begin{vmatrix} S & 0 \\ 0 & -1 \end{vmatrix} \qquad A_{33} = \begin{vmatrix} 0 & S \\ 1 & 0 \end{vmatrix}$$

$= 1 - 0$ $\qquad = -S - 0$ $\qquad = (0-S)$

$\boxed{A_{31} = 1}$ $\qquad \boxed{A_{32} = -S}$ $\qquad \boxed{A_{33} = -S}$

Cofactor of $(sI-A)$

~~Adj $(sI-A)$~~ $= \begin{bmatrix} +A_{11} & -A_{12} & +A_{13} \\ -A_{21} & +A_{22} & -A_{23} \\ +A_{31} & -A_{32} & +A_{33} \end{bmatrix}$ with sign representation

$$= \begin{bmatrix} s(s+3) & -1 & -s \\ s+3 & 1 & s \\ 1 & s & -s \end{bmatrix}$$

$\left[\because A_{12} = 1 \quad -A_{12} = -1\right]$

$\left[\because A_{21} = -(s+3) \quad -A_{21} = s+3\right]$

Cofactors of $(sI-A)^T$ Transpose $= \begin{bmatrix} s(s+3) & s+3 & 1 \\ -1 & 1 & s \\ -s & s & -s \end{bmatrix}$

$\left[\because A_{23} = -s \quad -A_{23} = s\right]$

$\left[\because A_{32} = -s \quad -A_{32} = s\right]$

$\left[\because \text{Cofactors of } (sI-A)^T \text{ is Called Adj } (sI-A)\right]$

$$\text{Adj}\,(sI-A) = \begin{bmatrix} s(s+3) & s+3 & 1 \\ -1 & 1 & s \\ -s & s & -s \end{bmatrix}$$

$$(sI-A)^{-1} = \frac{\text{Adj}(sI-A)}{|sI-A|}$$

$$= \begin{bmatrix} s(s+3) & s+3 & 1 \\ -1 & 1 & s \\ -s & s & -s \end{bmatrix} \frac{1}{s^3+3s^2+1}$$

$$= \begin{bmatrix} \frac{s(s+3)}{s^3+3s^2+1} & \frac{s+3}{s^3+3s^2+1} & \frac{1}{s^3+3s^2+1} \\ \frac{-1}{s^3+3s^2+1} & \frac{1}{s^3+3s^2+1} & \frac{s}{s^3+3s^2+1} \\ \frac{-s}{s^3+3s^2+1} & \frac{s}{s^3+3s^2+1} & \frac{-s}{s^3+3s^2+1} \end{bmatrix}$$

** PP9

Problem:

An LTI system is characterized by the homogeneous equation assuming the initial state vector $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$ Compute the solution of the homogenous equation.

Sol:

Compare with General state equation.

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u.$$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} ; \quad B \begin{bmatrix} 1 \\ 0 \end{bmatrix} ; \quad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\phi(t) = L^{-1}\left[(sI - A)^{-1}\right] x(0) = x(t)$ $\longrightarrow$ Homogeneous equation.

To get $\phi(t)$

$$sI - A = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} s-1 & 0 \\ -1 & s-1 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{Adj\,(sI - A)}{|sI - A|}$$

$$|sI - A| = \begin{vmatrix} s-1 & 0 \\ -1 & s-1 \end{vmatrix}$$

$$= (s-1)(s-1) + 0 \Rightarrow (s-1)(s-1)$$

$$= (s-1)^2.$$

$$\text{Adj}(sI-A) = \text{Adj}\begin{vmatrix} s-1 & 0 \\ -1 & s-1 \end{vmatrix} \qquad \left[\because \text{Adj } A = \begin{bmatrix} d & b \\ -c & a \end{bmatrix}\right.$$

$$= \begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix}$$

$$(sI-A)^{-1} = \begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix} \cdot \frac{1}{(s-1)^2}$$

$$= \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{s-1} \end{bmatrix}$$

$$L^{-1}(sI-A)^{-1} = L^{-1}\begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{(s-1)} \end{bmatrix}$$

$$= \begin{bmatrix} e^t & 0 \\ t\cdot e^t & e^t \end{bmatrix}$$

$$\phi(t) = L^{-1}(sI-A)^{-1} \cdot x(0)$$

$$\left[\because x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right.$$

$$= \begin{bmatrix} e^t & 0 \\ t\cdot e^t & e^t \end{bmatrix}_{2\times 2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{2\times 1}$$

$$= \begin{bmatrix} e^t \\ t\cdot e^t \end{bmatrix}_{2\times 1}$$

$$\therefore \phi(t) = L^{-1}(sI-A)^{-1}x(0) = x(t) = \begin{bmatrix} e^t \\ t\cdot e^t \end{bmatrix}$$

** PPQ

Problem: Obtain the state Transition Matrix for the State Model whose Matrix A is given by (Comput e^{At}).

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

Sol: Compare with the General state equation.

$$\dot{x}(t) = Ax + BU$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0 \cdot$$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$e^{At} = \phi(t) = \mathcal{L}^{-1}(sI - A)^{-1} \cdot x(0)$ = State Transition Matrix

$$(sI - A) = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & +1 \\ -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{\text{Adj}(sI - A)}{|(sI - A)|}$$

$$|sI - A| = \begin{vmatrix} s & -1 \\ 2 & s+3 \end{vmatrix}$$

$$= s(s+3) + 2$$

$$= s^2 + 3s + 2$$

$$\text{Adj}(sI - A) = \text{Adj}\begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$\text{Adj}(sI-A) = \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$[sI-A]^{-1} = \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \frac{1}{s^2+3s+2}$$

$$= \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \frac{1}{(s+2)(s+1)}$$

$$\mathcal{L}^{-1}[sI-A]^{-1} = \mathcal{L}^{-1}\begin{bmatrix} \frac{s+3}{(s+2)(s+1)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$\mathcal{L}^{-1}\left[\frac{s+3}{(s+2)(s+1)}\right] \Rightarrow \frac{s+3}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

$$\frac{s+3}{(s+1)(s+2)} = \frac{A(s+2)+B(s+1)}{(s+1)(s+2)}$$

$$s+3 = A(s+2)+B(s+1)$$

$s \Rightarrow -1$ $\quad -1+3 = A(-1+2)$

$$2 = A$$

$s \Rightarrow -2$ $\quad -2+3 = B(-2+1)$

$$1 = -B$$

$$B = -1$$

$$A = 2, \quad B = -1$$

$$\therefore \frac{s+3}{(s+1)(s+2)} = \frac{2}{(s+1)} - \frac{1}{(s+2)}$$

[$\because s^2+3s+2=0$

$s^2+2s+s+2=0$

$s(s+2)+1(s+2)=0$

$(s+2)(s+1)$

$$L^{-1}\left[\frac{s+3}{(s+1)(s+2)}\right] = L^{-1}\left[\frac{2}{s+1} - \frac{1}{s+2}\right]$$

$$= 2 \cdot e^{-t} - e^{-2t} \quad \text{— (1)}$$

$$L^{-1}\left[\frac{1}{(s+1)(s+2)}\right] \Rightarrow \frac{1}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

$$1 = A(s+2) + B(s+1)$$

$s \Rightarrow -1 \qquad 1 = A$

$s \Rightarrow -2 \qquad 1 = -B$

$\therefore A = 1, \; B = -1$

$$L^{-1}\left[\frac{1}{(s+1)(s+2)}\right] = L^{-1}\left[\frac{1}{(s+1)} - \frac{1}{(s+2)}\right]$$

$$= e^{-t} - e^{-2t} \quad \text{— (2)}$$

$$L^{-1}\left[\frac{-2}{(s+1)(s+2)}\right] \Rightarrow \frac{-2}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

$$-2 = A(s+2) + B(s+1)$$

$s \Rightarrow -2 \qquad -2 = -B \Rightarrow +B = +2$

$s \Rightarrow -1 \qquad -2 = A \Rightarrow A = -2$

$$L^{-1}\left[\frac{-2}{(s+1)(s+2)}\right] = L^{-1}\left[\frac{-2}{(s+1)} + \frac{2}{(s+2)}\right]$$

$$= -2 \cdot e^{-t} + 2 \cdot e^{-2t} \quad \text{— (3)}$$

$$L^{-1}\left[\frac{s}{(s+1)(s+2)}\right] \Rightarrow \frac{s}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

$$s = A(s+2) + B(s+1)$$

$S \Rightarrow -1 \quad \Rightarrow \quad S = A(S+2) + B(S+1)$

$-1 = A$

$S \Rightarrow -2 \quad \Rightarrow \quad -2 = -B$

$\therefore A = -1, \quad B = 2$

$$L^{-1}\left[\frac{S}{(S+1)(S+2)}\right] = L^{-1}\left[\frac{-1}{(S+1)} + \frac{2}{(S+2)}\right]$$

$$= -1e^{-t} + 2 \cdot e^{-2t}$$

$$= 2 \cdot e^{-2t} - e^{-t} \qquad \text{——— (4)}$$

By (1) (2) (3) & (4)

$$\phi(t) = L^{-1}\left[(SI - A)^{-1}\right] = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & 2e^{-2t} - e^{-t} \end{bmatrix}$$

Derivation of transfer function from the State Model:

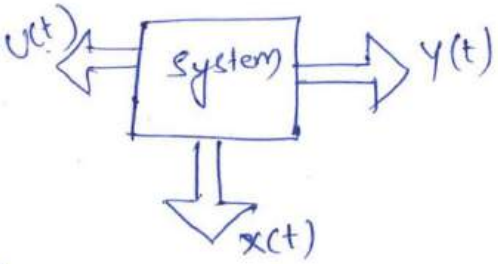
State Model = State equation + o/p equation.

$$\dot{x}(t) = Ax(t) + Bu(t) \quad \& \quad Y(t) = Cx(t) + Du(t)$$

——(1) ——(2)

Let Laplace Transform of eq(1)

$$L[\dot{x}(t)] = L\left[\frac{dx(t)}{dt}\right] = L[Ax(t) + Bu(t)]$$



$$SX(S) - X(0) = AX(S) + BU(S)$$

$$SX(S) - AX(S) = BU(S) + X(0)$$

$$X(S)[SI - A] = BU(S) + X(0)$$

$$X(S) = X(0)[SI - A]^{-1} + B(SI - A)^{-1}U(S)$$

——(3)

Laplace Transform of eq (2)

$$L[y(t)] = L[Cx(t) + DU(t)]$$

$$Y(s) = CX(s) + DU(s)$$

$$= C\left(\left[x(0)(sI-A)^{-1}\right] + \left(B(sI-A)^{-1}U(s)\right)\right]$$

$$+ DU(s)$$

T.F $x(0) = 0$ at $t_0 = 0$ $x(0) = 0$.

$$Y(s) = C[B(sI-A)^{-1}U(s)] + DU(s)$$

$$= U(s)[C(sI-A)^{-1}B + D]$$

$$\text{T.F} \Rightarrow \frac{Y(s)}{U(s)} = (sI-A)^{-1}BC + D$$ ✓

$$\phi(s) = (sI-A)^{-1}$$

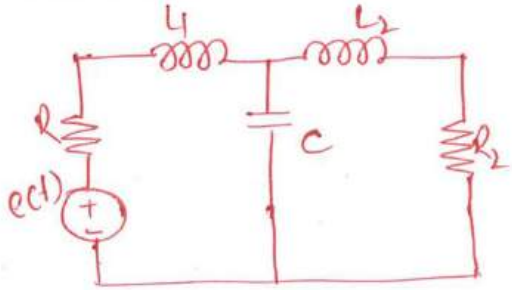
$$\phi(t) = L^{-1}(sI-A)^{-1}$$

$$\phi(t) = e^{At} \Rightarrow e^{at} \text{ is also state Transition.}$$

$\left[\because s\begin{bmatrix}1&0\\0&1\end{bmatrix} - A \quad \begin{bmatrix}s&0\\0&s\end{bmatrix} - A\right]$

Problem:

Obtain the State Model of the electrical network shown in fig. by using choosing minimal number of State variables.



Sol: Let us Choose the Current through the inductances i_1, i_2 and Voltage across the Capacitor V_c as State variables.

The assumed directions of currents and polarity of the voltage are shown in fig2.

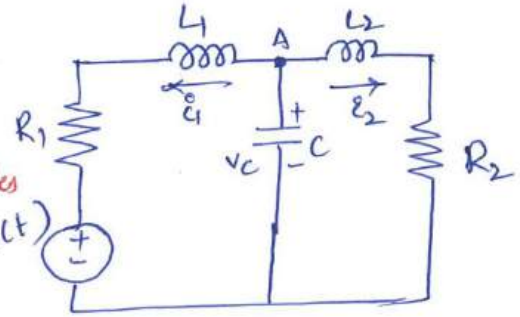


fig2.

Note:
The best choice of state variables in electrical network are currents and voltages in energy storage elements.

Let the three state variables x_1, x_2, x_3 be related to the physical quantities

$x_1 = i_1$ = Current through L_1

$x_2 = i_2$ = Current through L_2

$x_3 = v_c$ = Voltage across Capacitor.

At node A, by KCL

$$i_1 + i_2 + C\frac{dv_c}{dt} = 0. \quad \text{——— (1)}$$

On Sustituting the state variables for physical variables in eq (1). (i.e., $i_1 = x_1$, $i_2 = x_2$ and $\frac{dv_c}{dt} = \dot{x}_3$)

$$x_1 + x_2 + C\dot{x}_3 = 0$$

$$C\dot{x}_3 = -x_1 - x_2$$

$$\dot{x}_3 = -\frac{1}{C}x_1 - \frac{1}{C}x_2 \quad \text{——— (2)}$$

By KVL in the closed path shown in fig3.

$$e(t) = i_1 R_1 + L_1\frac{di_1}{dt} = v_c \quad \text{——— (3)}$$

On Substituting the state variables

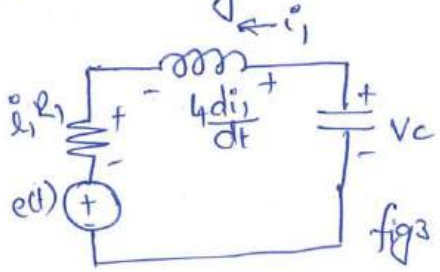


fig3.

for physical variables in eq (3)

(i.e., $i_1 = x_1$, $\frac{di_1}{dt} = \dot{x}_1$ and $v_c = x_3$)

$$e(t) + x_1 R_1 + L_1 \dot{x}_1 = x_3$$

Also, Let $u(t) = e(t) =$ input to the system.

$$\therefore u + x_1 R_1 + L_1 \dot{x}_1 = x_3$$

$$L_1 \dot{x}_1 = x_3 - \dot{x}_1 R_1 - u$$

$$\dot{x}_1 = -\frac{R_1}{L_1} x_1 + \frac{1}{L_1} x_3 - \frac{1}{L_1} u \quad \text{——— (4)}$$

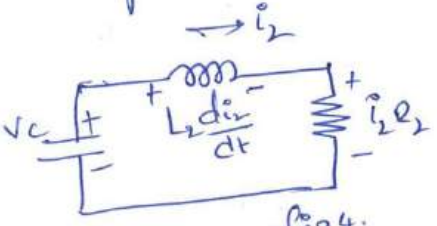


fig 4.

By KVL in the closed path shown in fig 4

$$v_c = L_2 \frac{di_2}{dt} + i_2 R_2 \quad \text{——— (5)}$$

On substituting the state variables for physical variable in eq (5)

i.e., $i_2 = x_2$, $\frac{di_2}{dt} = \dot{x}_2$ and $v_c = x_3$

$$x_3 = L_2 \dot{x}_2 + x_2 R_2$$

$$\therefore L_2 \dot{x}_2 = -x_2 R_2 + x_3$$

$$\dot{x}_2 = -\frac{R_2}{L_2} x_2 + \frac{1}{L_2} x_3 \quad \text{——— (6)}$$

The eq (2), (4) & (6) are the state equations of the system. Hence the state equations of the system are

$$\dot{x}_1 = -\frac{R_1}{L_1} x_1 + \frac{1}{L_1} x_3 - \frac{1}{L_1} u$$

$$\dot{x}_2 = -\frac{R_2}{L_2} x_2 + \frac{1}{L_2} x_3$$

$$\dot{x}_3 = -\frac{1}{C} x_1 - \frac{1}{C} x_2$$

On arranging the state equations in the matrix form we get,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & \frac{1}{L_1} \\ 0 & -\frac{R_2}{L_2} & \frac{1}{L_2} \\ -\frac{1}{C_1} & -\frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -\frac{1}{L_1} \\ 0 \\ 0 \end{bmatrix} [u] \quad \text{——— (7)}$$

Let us choose the voltage across the resistances as output variables and the output variables are denoted by y_1 and y_2.

$$\therefore y_1 = i_1 R_1 \quad \text{——— (8)}$$

$$y_2 = i_2 R_2 \quad \text{——— (9)}$$

On substituting the state variables in eq (8) and (9) (i.e, $i_1 = x_1$ and $i_2 = x_2$)

$$y_1 = x_1 R_1 \quad , \quad y_2 = x_2 R_2$$

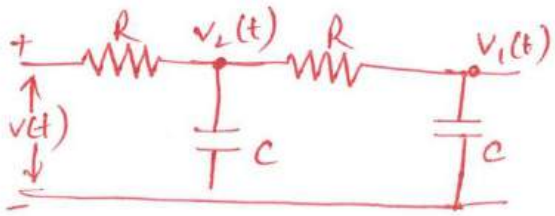
on arranging the above equations in the matrix form

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{——— (10)}$$

The state equation (eq (7)) and output equation (eq 10) together constitutes the "State Model".

Problem:

Obtain the state model of the electrical network shown in fig by choosing $v_1(t)$ and $v_2(t)$ as state variables.



Sol: Connect a voltage source at the i/p and convert to Current Source as shown in fig 1.

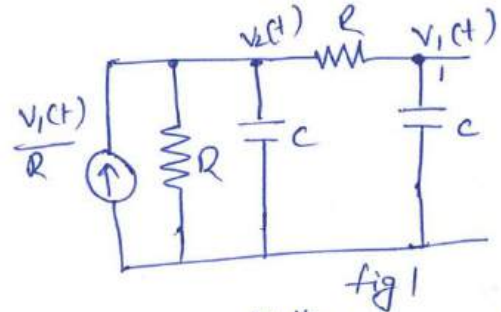


fig 1

At node 1 by KCL (fig 2)

$$\frac{V_1 - V_2}{R} + C\frac{dv_1}{dt} = 0 \quad \text{---(1)}$$

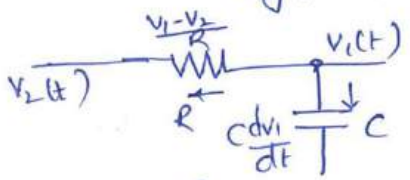


fig 2.

At node 2 by KCL (fig 3)

$$\frac{V_2 - V_1}{R} + \frac{V_2}{R} + C\frac{dv_2}{dt} = \frac{V(t)}{R} \quad \text{---(2)}$$

Let state variables be x_1 and x_2 and they are related physical variables as shown below.

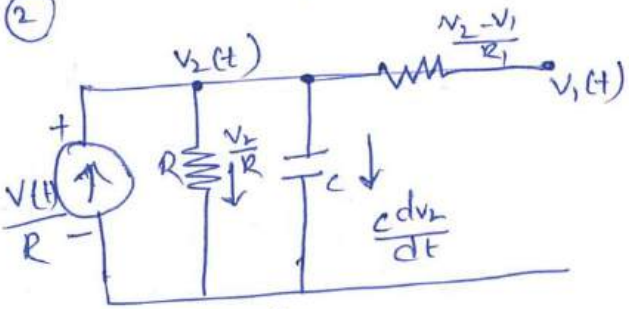


fig 3.

$V_1 = x_1$ and $V_2 = x_2$

Let $V(t) = v$ = input

On substituting the state variables in eq (1) & (2)

$$\frac{x_1 - x_2}{R} + C\frac{dx_1}{dt} = 0 \quad \text{---(3)}$$

$$\frac{x_2 - x_1}{R} + \frac{x_2}{R} + C\frac{dx_2}{dt} = \frac{v}{R} \quad \text{---(4)}$$

from eq (3) we get $\frac{x_1}{R} - \frac{x_2}{R} + C\dot{x}_1 = 0$

$$\therefore C\dot{x}_1 = -\frac{x_1}{R} + \frac{x_2}{R}$$

$$\dot{x}_1 = -\frac{1}{RC}x_1 + \frac{1}{RC}x_2 \quad \text{---(5)}$$

from eq (4) we get $\frac{x_2}{R} - \frac{x_1}{R} + \frac{x_2}{R} + C\dot{x}_2 = \frac{v}{R}$

$$\therefore C\dot{x}_2 = \frac{x_1}{R} - \frac{x_2}{R} - \frac{x_2}{R} + \frac{u}{R}$$

$$\dot{x}_2 = \frac{1}{RC}x_1 - \frac{2}{RC}x_2 + \frac{1}{RC}u \quad \text{——— (6)}$$

The eq (5) & (6) are state equations of the system.

Hence the state equations of the systems are

$$\dot{x}_1 = \frac{-1}{RC}x_1 + \frac{1}{RC}x_2$$

$$\dot{x}_2 = \frac{1}{RC}x_1 - \frac{2}{RC}x_2 + \frac{1}{RC}u$$

On arranging the state equations in the matrix form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}\frac{-1}{RC} & \frac{1}{RC}\\ \frac{1}{RC} & \frac{-2}{RC}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{RC}\end{bmatrix}\begin{bmatrix}u\end{bmatrix} \quad \text{——— (7)}$$

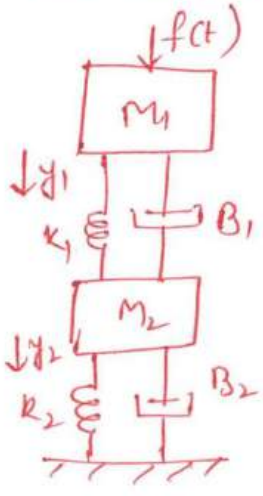
The output $y = V_1(t) = x_1$

$\therefore$ The output equation is $y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ ——— (8)

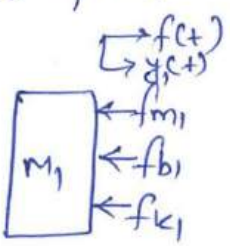
The State equation (7) and the output eq (8) together Constitute the state Model of the system.

Problem:

Construct the state Model of the Mechanical system shown in fig.



Sol: free body diagram of M_1 is shown in fig 1.



$$f_{m_1} = M_1\frac{d^2y_1}{dt^2}$$

$$f_{b_1} = B_1 \frac{d}{dt}(y_1 - y_2)$$

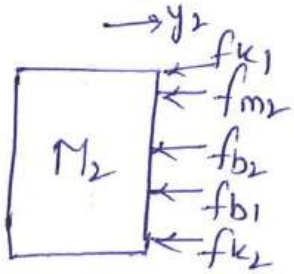
$$f_{k_1} = k_1 (y_1 - y_2)$$

By Newton's second law, the force balance equation at node M_1 is $-f(t) = -f_{m_1} + f_{b_1} + f_{k_1}$

$$f(t) = M_1 \frac{d^2 y_1}{dt^2} + B_1 \frac{d}{dt}(y_1 - y_2) + k_1 (y_1 - y_2)$$

$$= M_1 \frac{d^2 y_1}{dt^2} + B_1 \frac{dy_1}{dt} - B_1 \frac{dy_2}{dt} + k_1 y_1 - k_1 y_2 \quad \text{— (1)}$$

free body diagram of M_2 is shown in fig.



$$-f_{m_2} = M_2 \frac{d^2 y_2}{dt^2}$$

$$f_{b_2} = B_2 \frac{dy_2}{dt}$$

$$f_{b_1} = B_1 \frac{d}{dt}(y_2 - y_1)$$

$$f_{k_2} = k_2 y_2$$

$$f_{k_1} = k_1 (y_2 - y_1)$$

By Newton's second law, the force balance equation at node M_2 is,

$$f_{m_2} + f_{b_2} + f_{b_1} + f_{k_2} + f_{k_1} = 0.$$

$$\therefore M_2 \frac{d^2 y_2}{dt^2} + B_2 \frac{dy_2}{dt} + B_1 \frac{d}{dt}(y_2 - y_1) + k_2 y_2 + k_1 (y_2 - y_1) = 0.$$

$$M_2 \frac{d^2 y_2}{dt^2} + B_2 \frac{dy_2}{dt} + B_1 \frac{dy_2}{dt} - B_1 \frac{dy_1}{dt} + k_2 y_2 + k_1 y_2 - k_1 y_1 = 0$$

— (2)

Let us Choose the four state variables x_1, x_2, x_3 & x_4. Also, Let the input $f(t) = u$. The state variables are related to physical variables as follows.

$$x_1 = y_1, \quad x_2 = y_2, \quad x_3 = \frac{dy_1}{dt}, \quad x_4 = \frac{dy_2}{dt}, \quad \dot{x}_3 = \frac{d^2y_1}{dt^2}$$

$$\dot{x}_4 = \frac{d^2y_2}{dt^2}$$

On substituting $y_1 = x_1$; $y_2 = x_2$; $\frac{dy_1}{dt} = x_3$; $\frac{dy_2}{dt} = x_4$

$\frac{d^2y_1}{dt^2} = \dot{x}_3$ and $f(t) = u$ in eq (1).

$$u = M_1\dot{x}_3 + B_1x_3 - B_1x_4 + k_1x_1 - k_1x_2$$

$$M_1\dot{x}_3 = -B_1x_3 + B_1x_4 - k_1x_1 + k_1x_2 + u$$

$$\therefore \dot{x}_3 = -\frac{k_1}{M_1}x_1 + \frac{k_1}{M_1}x_2 - \frac{B_1}{M_1}x_3 + \frac{B_1}{M_1}x_4 + \frac{1}{M_1}u \quad \text{---(3)}$$

On substituting $y_1 = x_1$

$y_2 = x_2$ $\quad \frac{d^2y_2}{dt^2} = \dot{x}_4$ in eq (2)

$\frac{dy_1}{dt} = x_3$

$\frac{dy_2}{dt} = x_4$

$$M_2\dot{x}_4 + B_2x_4 + B_1x_4 - B_1x_3 + k_1x_2 + k_2x_2 - k_1x_1 = 0$$

$$\therefore M_2\dot{x}_4 = -B_2x_4 - B_1x_4 + B_1x_3 - k_2x_2 - k_1x_2 + k_1x_1$$

$$= -(B_2 + B_1)x_4 + B_1x_3 - (k_2 + k_1)x_2 + k_1x_1$$

$$\therefore \dot{x}_4 = \frac{k_1}{M_2}x_1 - \frac{(k_1 + k_2)}{M_2}x_2 + \frac{B_1}{M_2}x_3 - \frac{(B_1 + B_2)}{M_2}x_4. \quad \text{---(4)}$$

The state variable $x_1 = y_1$

On differentiating $x_1 = y_1$ with respect to t we get

$$\frac{dx_1}{dt} = \frac{dy_1}{dt}$$

Let $\frac{dx_1}{dt} = \dot{x}_1$ and $\frac{dy_1}{dt} = x_3$ — (5)

$\therefore \dot{x}_1 = x_3$

Let $\frac{dx_2}{dt} = \dot{x}_2$ and $\frac{dy_2}{dt} = x_4$ — (6)

$\therefore \dot{x}_2 = x_4$

The eq (3) (4) (5) (6) are state equations of the Mechanical system. Hence the state equations of the Mechanical system are:

$$\dot{x}_1 = x_3$$

$$\dot{x}_2 = x_4$$

$$\dot{x}_3 = -\frac{K_1}{M_1} x_1 + \frac{K_1}{M_1} x_2 - \frac{B_1}{M_1} x_3 + \frac{B_1}{M_1} x_4 + \frac{1}{M_1} u$$

$$\dot{x}_4 = \frac{K_1}{M_2} x_1 - \frac{(K_1+K_2)}{M_2} x_2 + \frac{B_1}{M_2} x_3 - \frac{(B_1+B_2)}{M_2} x_4$$

On arranging the state equations in the matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{K_1}{M_1} & \frac{K_1}{M_1} & -\frac{B_1}{M_1} & \frac{B_1}{M_1} \\ \frac{K_1}{M_2} & -\frac{(K_1+K_2)}{M_2} & \frac{B_1}{M_2} & \left(-\frac{B_1+B_2}{M_2}\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/M_1 \\ 0 \end{bmatrix} [u]$$

— (7)

Let the displacements y_1 and y_2 be the outputs of the system.

$\therefore y_1 = x_1$ and $y_2 = x_2$

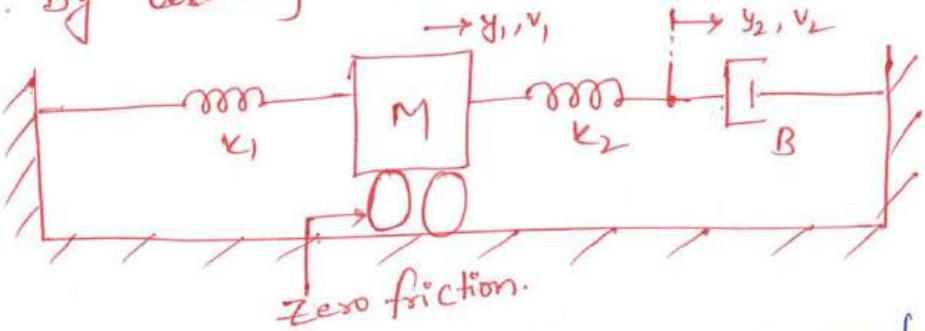
The output equation in matrix form is given by

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad \text{——— (8)}$$

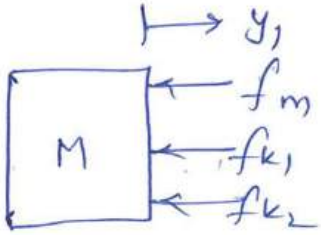
The state equation (eq (7)) and the output equation (eq (8)) together called state model of the system.

Problem:

Obtain the state model of the mechanical system shown in fig. by choosing a minimum of three state variables.



Sol: Let the three state variables be x_1, x_2 and x_3 as shown and they are related to physical variables as shown below.



$$x_1 = y_1$$

$$x_2 = y_2$$

$$x_3 = \frac{dy_1}{dt} = v_1$$

-from the free body diagram of mass M

$$f_m = M \frac{d^2 y_1}{dt^2}$$

$$f_{k_1} = k_1 y_1$$

$$f_{k_2} = k_2 (y_1 - y_2)$$

By Newton's Second Law, the force balance equation at node M is

$$f_m + f_{k_1} + f_{k_2} = 0$$

$$M \frac{d^2 y_1}{dt^2} + k_1 y_1 + k_2 (y_1 - y_2) = 0$$

$$M \frac{d^2 y_1}{dt^2} + k_1 y_1 + k_2 y_1 - k_2 y_2 = 0 \quad \text{——— (1)}$$

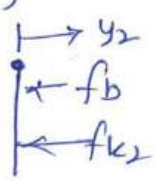
Put $\frac{d^2 y_1}{dt^2} = \dot{x}_3$, $y_1 = x_1$, $y_2 = x_2$ in eq (1)

$$M \dot{x}_3 + k_1 x_1 + k_2 x_1 - k_2 x_2 = 0$$

$$M \dot{x}_3 + (k_1 + k_2) x_1 - k_2 x_2 = 0$$

$$\dot{x}_3 = -\frac{(k_1 + k_2)}{M} x_1 + \frac{k_2}{M} x_2 \quad \text{——— (2)}$$

The free body diagram of node 2 (Meeting point of k_2 and B)



$$f_b = B \frac{dy_2}{dt}$$

$$f_{k_2} = k_2 (y_2 - y_1)$$

Writing force balance equation at the Meeting point of k_2 and B

$$f_b + f_{k_2} = 0$$

$$B\frac{dy_2}{dt} + k_2(y_2 - y_1) = 0$$

$$\therefore \frac{dy_2}{dt} + \frac{k_2}{B}y_1 - \frac{k_2}{B}y_2$$

put $\frac{dy_2}{dt} = \dot{x}_2$, $y_1 = x_1$ and $y_2 = x_2$

$$\therefore \dot{x}_2 = \frac{k_2}{B}x_1 - \frac{k_2}{B}x_2 \quad \text{——— (3)}$$

The state variable, $x_1 = y_1$, on differentiating this expression with respect to t we get

$$\frac{dx_1}{dt} = \frac{dy_1}{dt}$$

Let $\frac{dx_1}{dt} = \dot{x}_1$ and $\frac{dy_1}{dt} = x_3$ } ——— (4)

$$\therefore \dot{x}_1 = x_3$$

The state equations are eq (4), (3) & (2)

$$\dot{x}_1 = x$$

$$\dot{x}_2 = \frac{k_2}{B}x_1 - \frac{k_2}{B}x_2$$

$$\dot{x}_3 = -\frac{(k_1+k_2)}{M}x_1 + \frac{k_2}{M}x_2$$

On arranging the State ~~va~~ equations in the Matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{k_2}{B} & -\frac{k_2}{B} & 0 \\ -\frac{(k_1+k_2)}{M} & +\frac{k_2}{M} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{——— (5)}$$

If the desired outputs are y_1 and y_2 then

$y_1 = x_1$ & $y_2 = x_2$

O/p equation $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ ——— (6)

Statemodel = eq (5) & eq (6).

Problem :

Determine the state Model of armature Controlled Dc Motor.

Sol: The speed of Dc Motor is directly proportional to armature voltage and inversely proportional to flux.

The equivalent Circuit of Armature is shown in fig 2.

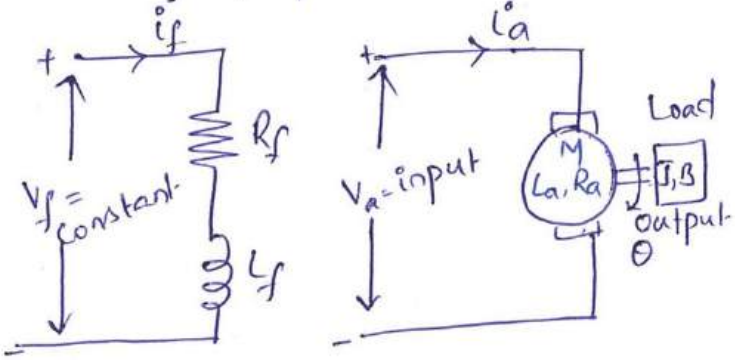


fig: Armature Controlled Dc Motor.

By KVL

$$i_a R_a + L_a \frac{di_a}{dt} + e_b = V_a \quad \text{——} (1)$$

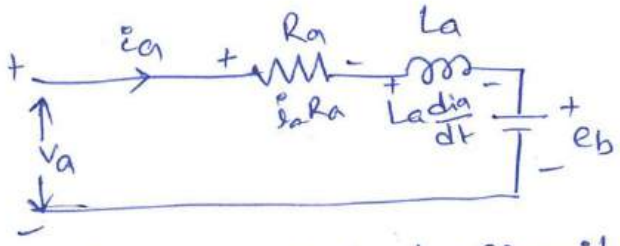


fig2: Equivalent Circuit of armature.

Torque of Dc motor is proportional to the product of flux and Current.

Since the flux is Constant in this system, the torque is proportional to i_a alone.

$$T \propto i_a$$

$$\therefore T = K_t i_a \quad \text{——} (2)$$

The Mechanical System of the motor is shown in fig 3. The differential equation governing the mechanical System of motor is given by

$$J \frac{d^2\theta}{dt^2} + B \frac{d\theta}{dt} = T \quad \text{——} (3)$$

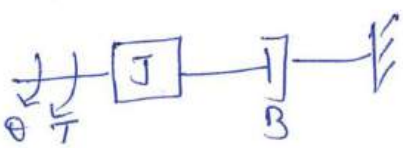


fig: 3

The back emf of Dc Machine is proportional to speed (angular velocity) of shaft.

$$\therefore e_b \propto \frac{d\theta}{dt}$$

$$e_b = k_b \frac{d\theta}{dt} \quad \text{——(4)}$$

from eq (1) & (4)

$$i_a R_a + L_a \frac{di_a}{dt} + k_b \frac{d\theta}{dt} = V_a \quad \text{——(5)}$$

from eq (2) & (3)

$$J \frac{d^2\theta}{dt^2} + B \frac{d\theta}{dt} = k_t i_a \quad \text{——(6)}$$

The eq (5) and (6) are the differential equations governing the armature Controlled DC motor.

Let us choose i_a, ω and θ as the state variables to model the armature Controlled dc motor. The physical variables i_a, ω and θ are related to the general notation of state variables x_1, x_2 and x_3.

$$x_1 = i_a$$

$$x_2 = \omega = \frac{d\theta}{dt}$$

$$x_3 = \theta$$

The input to the motor is the armature voltage V_a and let $V_a = u$

where u is the input variable.

On substituting state variables for the physical variables in eq (5)

$$x_1 R_1 + L_a \frac{dx_1}{dt} + k_b x_2 = u$$

Let $\frac{dx_1}{dt} = \dot{x}_1$

$$\therefore x_1 R_a + L_a \dot{x}_1 + k_b x_2 = u$$

$$\dot{x}_1 = -\frac{R_a}{L_a} x_1 - \frac{k_b}{L_a} x_2 + \frac{1}{L_a} u \quad \text{——(7)}$$

On substituting the state variables for physical variables in eq (6)

$$J\frac{d^2x_2}{dt^2} + B\frac{dx_3}{dt} = k_t x_1$$

Let $\frac{d^2x_3}{dt^2} = \dot{x}_2$

$$\frac{dx_3}{dt} = x_2$$

$$\therefore J\dot{x}_2 + Bx_2 = k_t x_1$$

$$\therefore \dot{x}_2 = \frac{k_1}{J}x_1 - \frac{B}{J}x_2 \quad \text{——— (8)}$$

The state variable $x_3 = \theta$.

$$\frac{dx_3}{dt} = \frac{d\theta}{dt}$$

Put $\frac{dx_3}{dt} = \dot{x}_3$ and $\frac{d\theta}{dt} = x_2$

$$\therefore \dot{x}_3 = x_2 \quad \text{——— (9)}$$

The eq (7) (8) & (9) are the state equations of the system

$$\dot{x}_1 = -\frac{R_a}{La}x_1 - \frac{kb}{La}x_2 + \frac{1}{La}u$$

$$\dot{x}_2 = \frac{k_1}{J}x_1 - \frac{B}{J}x_2$$

$$\dot{x}_3 = x_2$$

On arranging the state equations in the matrix form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-\frac{R_a}{La} & -\frac{kb}{La} & \frac{1}{La}\\ +\frac{k_1}{J} & \frac{-B}{J} & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}\frac{1}{La}\\ 0\\ 0\end{bmatrix}u$$

——— (10)

Let the desired output be i_a, ω and θ.

Let us equate the desired output quantities to standard notation y_1, y_2 and y_3 as

$$y_1 = i_a \; ; \; y_2 = \omega = \frac{d\theta}{dt} \; ; \; y_3 = \theta$$

On relating the outputs to state variables we get

$$y_1 = x_1$$
$$y_2 = x_2$$
$$y_3 = x_3$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The o/p equation in the matrix form as given in eq (11)

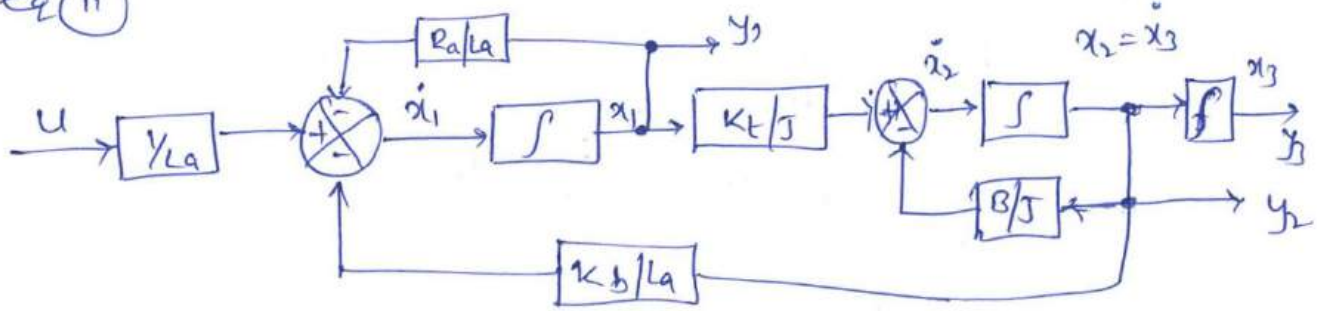


fig: Block diagram representation of the state model of armature Controlled dc motor.

eq (10) & (11) together Constitutes the state model of the armature Controlled dc motor.

Problem:

Determine the State model of field Controlled dc motor.

Sol: The speed of a Dc motor is directly proportional to armature voltage and inversly proportional to flux.

The equivalent circuit of field Controlled Dc motor is shown in fig.

By KVL

$$R_f i_f + L_f \frac{di_f}{dt} = V_f \quad \text{—(1)}$$

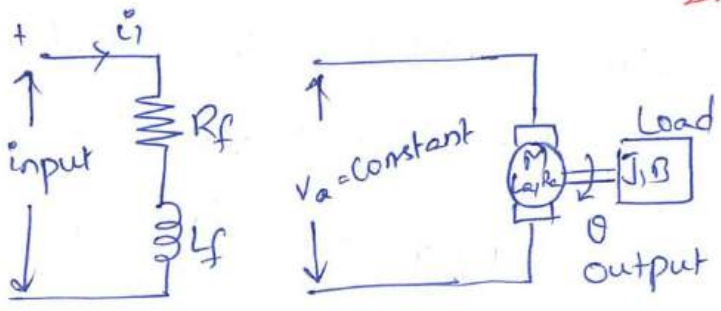


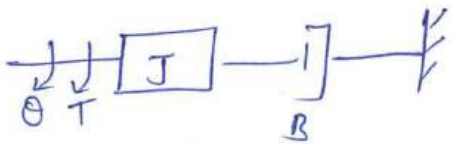
fig: field Controlled DC motor.

The torque of DC motor is proportional to product of flux and armature Current.

Since armature Current is Constant, The Torque ∝ to flux ~~in this~~ governing the mechanical system of the motor is given by

$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T \quad \text{—(3)} \quad , \quad T \propto i_f \; ; \; T = K_{tf}\, i_f \quad \text{—(2)}$$

from eq (2) & (3)



$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = K_{tf}\, i_f \quad \text{—(4)}$$

The eq (1) + (4) are the differential equations of the field Controlled DC motor.

Let us Choose i_t, ω and θ as state variable to model the field Controlled dc motor. The physical variables i_t, ω and θ are related to the state variables x_1, x_2, x_3

$$x_1 = i_f$$

$$x_2 = \omega = \frac{d\theta}{dt}$$

$$x_3 = \theta$$

The input to the system is the field voltage V_f

Let $V_f = u$ where u is the input.

On substituting the state variables and input variables for the physical variable in eq (1)

$$R_f x_1 + L_f \frac{dx_1}{dt} = u$$

Let $\frac{dx_1}{dt} = \dot{x}_1$

$\therefore R_f x_1 + L_f \dot{x}_1 = u$

$$\dot{x}_1 = -\frac{R_f}{L_f} x_1 + \frac{1}{L_1} u \quad \text{——— (5)}$$

On substituting the state variables for the physical variables in eq (4)

$$J \frac{d^2 x_3}{dt^2} + B \frac{dx_3}{dt} = k_{tf} x_1$$

Let $\frac{d^2 x_3}{dt^2} = \dot{x}_2$ and $\frac{dx_3}{dt} = x_2$

$\therefore J \dot{x}_2 + B x_2 = k_{tf} x_1$

$$\dot{x}_2 = \frac{k_{tf}}{J} x_1 - \frac{B}{J} x_2 \quad \text{——— (6)}$$

The state variable $x_3 = \theta$.

$$\frac{dx_3}{dt} = \frac{d\theta}{dt}$$

Put $\frac{dx_3}{dt} = \dot{x}_3$ and $\frac{d\theta}{dt} = x_2$

$\therefore \dot{x}_3 = x_2$

The eq (5) (6) & (7) are the state equations of the system

$$\dot{x}_1 = -\frac{R_f}{L_f} x_1 + \frac{1}{L_1} u$$

$$\dot{x}_2 = \frac{k_{tf}}{J} x_1 - \frac{B}{J} x_2$$

$$\dot{x}_3 = x_2$$

On arranging the state equations in the matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \frac{-R_f}{L_f} & 0 & 0 \\ \frac{K_f}{J} & -\frac{B}{J} & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{L_f} \\ 0 \\ 0 \end{bmatrix} u \quad \text{---------- (8)}$$

Let the desired output be ω and θ. Let us equate the desired output quantities to standard notation y_1 and y_2.

$$y_1 = \omega \; ; \; y_2 = \theta$$

On relating the outputs to state variable

$$y_1 = x_2 \; , \; y_2 = x_3$$

The output equation in the matrix form is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The state equation (eq 8) and the o/p equation (eq 9) together constitutes the state model of the system.

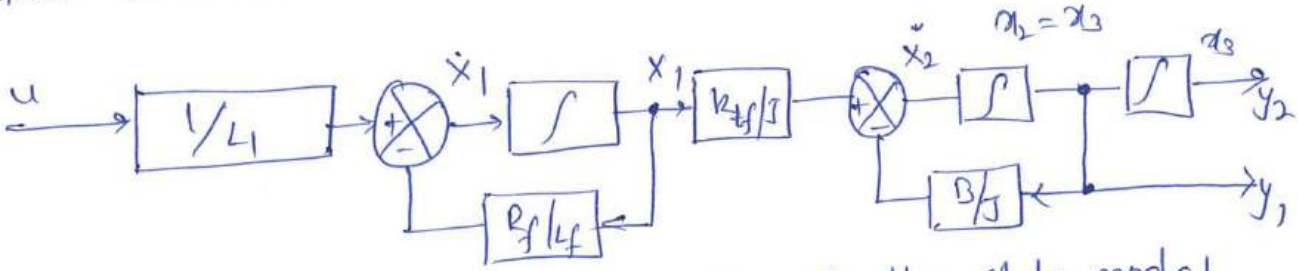


fig: Block diagram representation of the state model of field Controlled DC motor.

Problem:

Obtain the state model of the system whose transfer function is $\frac{Y(s)}{U(s)} = \frac{10}{s^3+4s^2+2s+1}$

Sol: Given $\frac{Y(s)}{U(s)} = \frac{10}{s^3+4s^2+2s+1}$ ——①

on cross multiplying the eq ① we get

$$Y(s)(s^3+4s^2+2s+1) = 10U(s)$$

$$s^3Y(s)+4s^2Y(s)+2sY(s)+Y(s) = 10U(s) \quad \text{——②}$$

On taking inverse Laplace Transform of eq ② we get

$$\dddot{y} + 4\ddot{y} + 2\dot{y} + y = 10u$$

Let us define state variables as

$$x_1 = y, \quad x_2 = \dot{y}, \quad x_3 = \ddot{y}$$

Put $\dddot{y} = \dot{x}_3$
$\ddot{y} = x_3$
$\dot{y} = x_2$
$y = x_1$
} in eq ③

$\therefore \dot{x}_3 + 4x_3 + 2x_2 + x_1 = 10u$

$$\dot{x}_3 = -x_1 - 2x_2 - 4x_3 + 10u$$

The state equations are $\dot{x}_1 = x_2$

$\dot{x}_2 = x_3$

$\dot{x}_3 = -x_1 - 2x_2 - 4x_3 + 10u$

The output equations ~~are~~ is $y = x_1$

The state Model in the Matrix form is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} [u] ;$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Problem:

Construct a state mode for a system Characterized by the differential equation

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y + u = 0.$$ Give the block diagram representation of the state model.

Sol: Let us Choose y and their derivatives as state variables. The system is governed by third order differential equation and so the number of state variables are three.

The state variables x_1, x_2 and x_3 are related to phase variables as follows.

$$x_1 = y$$

$$x_2 = \frac{dy}{dt} = \dot{x}_1$$

$$x_3 = \frac{d^2y}{dt^2} = \dot{x}_2$$

Put $y = x_1$

$$\frac{dy}{dt} = x_2$$

$$\frac{d^2y}{dt^2} = x$$

$$\frac{d^3y}{dt^3} = \dot{x}^3$$

$$\therefore \dot{x}_3 + 6x_3 + 11x_2 + 6x_1 + u = 0$$

$$\dot{x}_3 = -6x_1 - 11x_2 - 6x_3 - u$$

The state equations are

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -6x_1 - 11x_2 - 6x_3 - u$$

On arranging the state equations in the matrix form we get

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} [u]$$

Here y = output

but $y = x_1$

$\therefore$ The o/p equation is $y = [1 \ 0 \ 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

The state equation and output equation, constitutes the state Model of the systems.

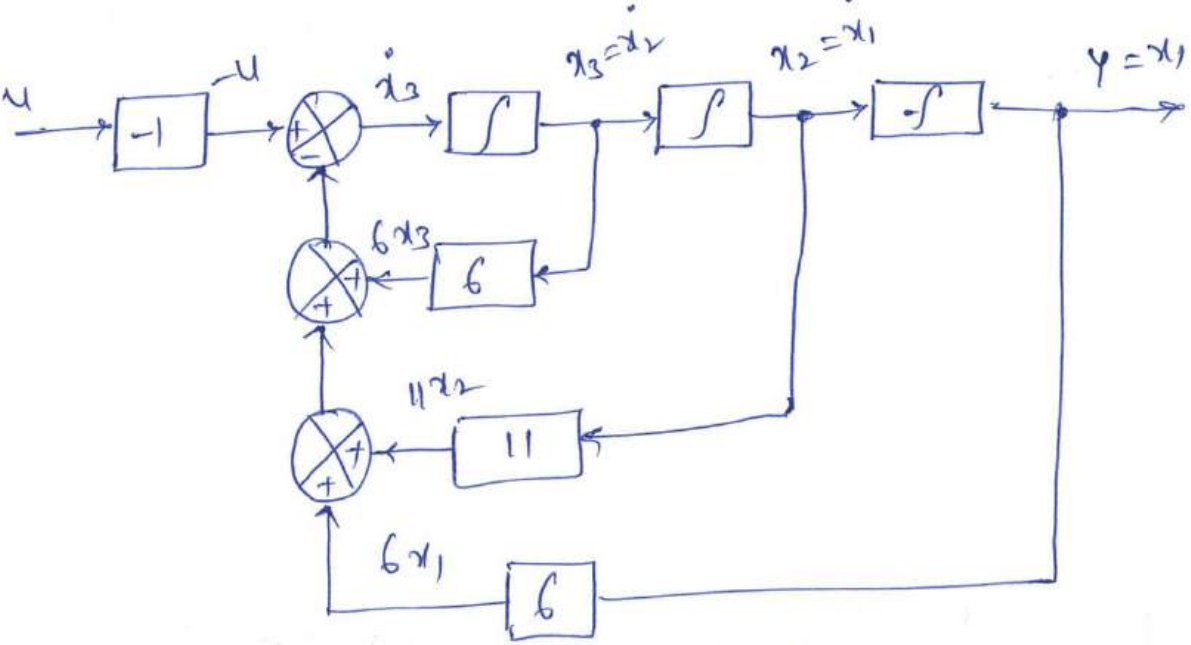


fig: Block diagram form of state diagram.

Concept of Controllability and observability

Controllability

The Controllability verifies the usefulness of state variable. In the Controllability test, we can find, whether the state variable can be Controlled to achieve the desired output. The choice of state variables is arbitary while forming the state model. After determing the state model, the Controllability of the state varieble is verified. If the state variable is not Controllable then we have to go for another Choice of state variable.

Defination:

A system is said to be Completely state Controllable if it possible to transfer the system state from any initial state $x(t_0)$ to any other desired state $x(t_d)$ in specified finite time by a Control vector $u(t)$.

Gilbert's method of testing Controllability.

Casi): When the system matrix has distinct Eigen values

In this case the system matrix can be diagonalized and the state model can be Converted to canonical form

Consider the state model of the system

$$\dot{X} = AX + BU$$
$$Y = CX + DU$$

The state model can be converted to canonical form by transformation $X = MZ$

Where M is the Modal Matrix

Z is the transformed state variable vector

The Transformed state model is given by

$$\dot{Z} = \tilde{A}Z + \tilde{B}U$$
$$Y = \tilde{C}Z + DU$$

where $\tilde{A} = M^{-1}AM$

$\tilde{B} = M^{-1}B$

$\tilde{C} = CM$

The necessary and sufficient condition for Complete Controllability is that, the matrix $\tilde{B}$ must have no rows will all zeros.

If any row of the matrix $\tilde{B}$ is zero then the corresponding state variable is uncontrollable.

Case ii) when the system matrix has repeated Eigen values.

In this case, the system matrix can not be diagonalized but can be transformed to Jordan Canonical form

Consider the state model of the system.

$$\dot{X} = AX + BU$$
$$Y = CX + DU$$

The state Model can be transformed to Jordon Canonical form by a transformation, $x = Mz$

where M is Modal Matrix

z is transformed state variable vector.

The transformed state Model is given by

$$\dot{z} = Jz + \tilde{B}U$$

$$Y = \tilde{C}z + DU$$

where $J = \tilde{M}^{-1}AM$

$$\tilde{B} = \tilde{M}^{-1}B$$

$$\tilde{C} = CM.$$

The system is Completely Controllable if the elements of any row of $\tilde{B}$ that Correspond to the last row of each Jordon block are not at all zero and the rows Corresponding to other state Variable must not have all zeros.

Observability:

A system is said to be Completely observable if every state $x(t)$ can be Completely identified by measurements of the output $Y(t)$ over a finite time interval.

Gilbert's Method of Testing observability.

Consider the state model of nth order system

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

The state model can be transformed to a canonical (or) Jordon canonical form by a transformation

$$X = MZ$$

where M is the Modal Matrix

Z is the transformed state variable vector

The Transformed state model is

$$\dot{Z} = AZ + \tilde{B}U \qquad \text{(or)} \qquad \dot{Z} = JZ + \tilde{B}U$$
$$Y = \tilde{C}Z + DU \qquad\qquad Y = \tilde{C}Z + DU$$

where $\Lambda = M^{-1}AM$; if Eigen values are distinct

$\tilde{B} = M^{-1}B$

$J = M^{-1}AM$; if Eigen values have Multiplicity $\tilde{C} = CM$.

The necessary and sufficient condition for complete observability is that none of the column of the Matrix $\tilde{C}$ be zero. If any of the column's of $\tilde{C}$ has all zeros then the corresponding state variable is not observable.

Effect of pole-zero Cancellation in Transfer function:

The Concepts of Controllability and observability are closely related to the properties of the transfer functi[illegible] Consider an nth order system with distinct eigen values.

The Transfer function of the system can be expressed as a ratio of two polynomial as shown in eq (1)

$$T(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s_1 + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} \; ; m \leq n \quad \text{—(1)}$$

$$= \frac{K(s+\beta_1)(s+\beta_2)\cdots(s+\beta_m)}{(s+\lambda_1)(s+\lambda_2)\cdots(s+\lambda_n)} \quad \text{---(2)}$$

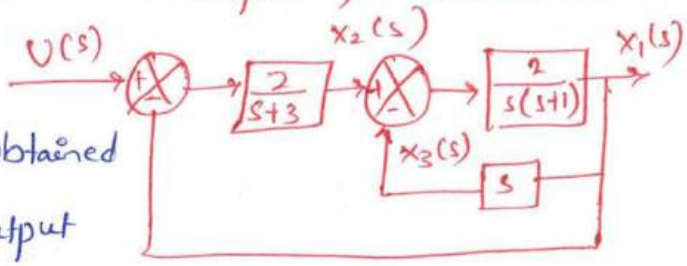
By partial fraction expansion technique eq(2) written as

$$\frac{Y(s)}{U(s)} = \frac{C_1}{s+\lambda_1} + \frac{C_2}{s+\lambda_2} + \cdots + \frac{C_i}{s+\lambda_i} + \cdots + \frac{C_n}{s+\lambda_n}$$

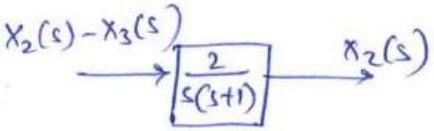
Where C_1, C_2, C_3 are residues.

If the Transfer function has idential pair of pole and zero at $\beta_1 = \lambda_i$, then $C_i = 0$. The effect of this Cancellation on Controllability and observability properties depends on the choice of state variables.

Problem: Write the state equations for the system shown in fig in which x_1, x_2 and x_3 Constitute the state vector. Determine wheather the system is Completely Controllable and observable.



Sol: The state equations are obtained by writing equations for the output of each block and then taking inverse laplace transform.



with reference to fig

$$X_1(s) = [X_2(s) - X_3(s)]\left[\frac{2}{s(s+1)}\right]$$

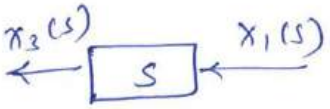
$$s(s+1)X_1(s) = 2X_2(s) - 2X_3(s)$$

$$s^2X_1(s) + sX_1(s) = 2X_2(s) - 2X_3(s)$$

on taking inverse laplace Transform

$$\ddot{x}_1 + \dot{x}_1 = 2x_2 - 2x_3 \quad \text{---(1)}$$

With reference to fig we can write

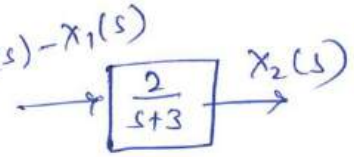


$$X_3(s) = s X_1(s) \quad \text{---(2)}$$

on taking inverse Laplace transform

$$x_3 = \dot{x}_1$$

With reference to fig we can write



$$X_2(s) = \left[U(s) - X_1(s)\right]\left[\frac{2}{s+3}\right]$$

$$X_2(s)(s+3) = 2U(s) - 2X_1(s)$$

$$s X_2(s) + 3X_2(s) = 2U(s) - 2X_1(s)$$

on taking inverse laplace transform

$$\dot{x}_2 + 3x_2 = 2u - 2x_1$$

$$\dot{x}_2 = -2x_1 - 3x_2 + 2u \quad \text{---(3)}$$

from eq (2) we get $\dot{x}_1 = x_3$;

$$\ddot{x}_1 = \dot{x}_3$$

put $\dot{x}_1 = x_3$ and $\ddot{x}_1 = \dot{x}_3$ in eq (1)

$$\therefore \dot{x}_3 + x_3 = 2x_2 - 2x_3$$

$$\dot{x}_3 = 2x_2 - 2x_3 - x_3$$

$$\dot{x}_3 = 2x_2 - 3x_3 \quad \text{---(4)}$$

The state equation are given by eq (2), (3) & (4)

$$\dot{x}_1 = x_3$$

$$\dot{x}_2 = -2x_1 - 3x_2 + 2u$$

$$\dot{x}_3 = 2x_2 - 3x_3$$

The o/p equation is $y = x_1$

The state model in the Matrix form is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} u$$

$$Y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\therefore \quad A = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

KALMAN'S TEST for Controllability:

$$A^2 = A \cdot A = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & -3 \\ 6 & 9 & -2 \\ -4 & -12 & 9 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -6 \\ 4 \end{bmatrix}$$

$$A^2 B = \begin{bmatrix} 0 & 2 & -3 \\ 6 & 9 & -2 \\ -4 & -12 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 18 \\ -24 \end{bmatrix}$$

The composite Matrix for Controllability

$$Q_c = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 4 \\ 2 & -6 & 18 \\ 0 & 4 & -24 \end{bmatrix}$$

Determinant of $Q_c = \begin{vmatrix} 0 & 0 & 4 \\ 2 & -6 & 18 \\ 0 & 4 & -24 \end{vmatrix}$

$= 0\left(-6 \times (-24)\right) - 0\left(\text{⊕} \times 8\right) + 4\left(2 \times 4\right) - 0)$

$= 4(8)$

$= 32$

Since $|Q_c| \neq 0$, the rank of $Q_c = 3$.

Hence the system is Completely state Controllable.

KALMAN'S TEST for Observability:

$$A^T = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix}^T = \begin{bmatrix} 0 & -2 & 0 \\ 0 & -3 & 2 \\ 1 & 0 & -3 \end{bmatrix}$$

$$C^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$(A^T)^2 = \begin{bmatrix} 0 & -2 & 0 \\ 0 & -3 & 2 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} 0 & -2 & 0 \\ 0 & -3 & 2 \\ 1 & 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 6 & -4 \\ 2 & 9 & -12 \\ -3 & -2 & 9 \end{bmatrix}$$

$$A^T C^T = \begin{bmatrix} 0 & -2 & 0 \\ 0 & -3 & 2 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$(A^T)^2 C^T = \begin{bmatrix} 0 & 6 & -4 \\ 2 & 9 & -12 \\ -3 & -2 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$$

The composite matrix for observability $Q_o = \begin{bmatrix} C^T & A^T C^T & (A^T)^2 C^T \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix}$$

Determinant of $Q_o = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & -3 \end{vmatrix}$

$$|Q_o| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & -3 \end{vmatrix}$$

$$= -2$$

$|Q_o| \neq 0$, the rank of $Q_o = 3$

Hence the system is Completely observable.

KALMAN's Method of Testing Controllability:

Consider a system with state equation $\dot{X} = AX + BU$.

For this system, a Composite matrix, Q_c can be formed such that

$$Q_c = [B \quad AB \quad A^2B \ldots A^{n-1}B]$$

where 'n' is the order of the system (n is also equal to number of state variables).

In this case the system is Completely state Controllable if the rank of the ~~Q_c~~ Composite matrix, Q_c is n

The rank of the matrix is n, if the determinant of $(n \times n)$ Composite Matrix Q_c is non-zero, i.e, if $|Q_c| \neq 0$, then rank of $Q_c = n$ and the system is Completely state Controllable.

KALMAN's Method of Testing observability:

Consider a system with state model

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

For this System, a Composite Matrix Q_0 Can be formed such that

$$\therefore Q_0 = \begin{bmatrix} C^T & A^TC^T & (A^T)^2C^T & (A^T)^3C^T & \cdots & (A^T)^{n-1}C^T \end{bmatrix}$$

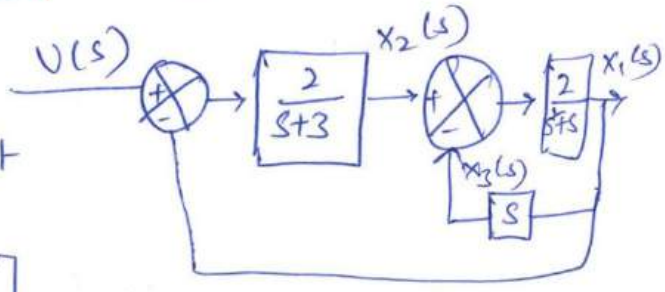
where 'n' is the order of the system.

In this Case the system is Completely observable if the rank of Composite matrix Q_0 is n.

The rank of the matrix is n, if the determinant of $n \times n$ Composite matrix Q_0 is non-zero.

* for the previous Controllability and observability problem, whether the system is Controllable or observable we can justify with "Gilbert's Test" also.

for the fig shown



We already get state model that

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} u$$

$$Y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

To find Eigen values for the Matrix A

$$A = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix}$$

The Characteristic equation is $|\lambda I - A| = 0$

$$[\lambda I - A] = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix}$$

$$[\lambda I - A] = \begin{bmatrix} \lambda & 0 & 1 \\ 2 & \lambda+3 & 0 \\ 0 & -2 & \lambda+3 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda & 0 & -1 \\ 2 & \lambda+3 & 0 \\ 0 & -2 & \lambda+3 \end{vmatrix} = \lambda(\lambda+3)(\lambda+3) - 0 - 1(-4+0)$$

$$= \lambda(\lambda+3)(\lambda+3) + 4$$

$$= (\lambda^2+3\lambda)(\lambda+3) + 4$$

$$= \lambda^3 + 3\lambda^2 + 3\lambda^2 + 9\lambda + 4$$

$$= \lambda^3 + 6\lambda^2 + 9\lambda + 4$$

$$= (\lambda+1)(\lambda^2+5\lambda+4)$$

$$= (\lambda+1)(\lambda+1)(\lambda+4)$$

$$= (\lambda+1)^2(\lambda+4)$$

The eigen values are

$\lambda_1 = -1$

$\lambda_2 = -1$

$\lambda_3 = -4$.

To find the Eign vectors

$$[\lambda_1 I - A] = \lambda_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & 0 & -1 \\ 2 & \lambda_1+3 & 0 \\ 0 & -2 & \lambda_1+3 \end{bmatrix}$$

Let C_{11}, C_{12} and C_{13} be cofactors along 1st row of the matrix $[\lambda_1 I - A]$

$$C_{11} = +1 \begin{vmatrix} \lambda_1+3 & 0 \\ -2 & \lambda_1+3 \end{vmatrix}$$

$$= (\lambda_1+3)^2 = \lambda_1^2 + 6\lambda_1 + 9$$

$$C_{12} = (-1) \begin{vmatrix} 2 & 0 \\ 0 & \lambda_1+3 \end{vmatrix}$$

$$= -2(\lambda_1+3) = -2\lambda_1 - 6$$

$$C_{13} = +1 \begin{vmatrix} 2 & \lambda_1+3 \\ 0 & -2 \end{vmatrix}$$

$$= -4$$

$$m_1 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} \lambda_1^2 + 6\lambda_1 + 9 \\ -2\lambda_1 - 6 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -4 \\ -4 \end{bmatrix}$$

$$m_2 = \begin{bmatrix} \frac{d}{d\lambda_1} C_{11} \\ \frac{d}{d\lambda_1} C_{12} \\ \frac{d}{d\lambda_1} C_{13} \end{bmatrix} = \begin{bmatrix} 2\lambda_1 + 6 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix}$$

$$[\lambda_3 I - A] = -4\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \begin{bmatrix}0&0&1\\-2&-3&0\\0&2&-3\end{bmatrix}$$

$$= \begin{bmatrix}-4&0&-1\\2&-1&0\\0&-2&-1\end{bmatrix}$$

Let C_{11} C_{12} and C_{13} be the cofactors along 1st row of the matrix $[\lambda_3 I - A]$

$$C_{11} = +1\begin{vmatrix}-1&0\\-2&-1\end{vmatrix} = 1$$

$$C_{12} = (-1)\begin{vmatrix}2&0\\0&-1\end{vmatrix} = 2$$

$$C_{13} = (+1)\begin{vmatrix}2&-1\\0&-2\end{vmatrix} = -4$$

$$\therefore M_3 = \begin{bmatrix}C_{11}\\C_{12}\\C_{13}\end{bmatrix} = \begin{bmatrix}1\\2\\-4\end{bmatrix}$$

To find Canonical form of state model

The model matrix, M is given by

$$M = [m_1, m_2, m_3]$$

$$= \begin{bmatrix}4&4&1\\-4&-2&2\\-4&0&-4\end{bmatrix}$$

$$M^{-1} = \frac{[\text{Cofactor of } M]^T}{\text{Determinant of } M} = \frac{M_{Cof}^T}{D_M}$$

$$\Delta_M = \begin{vmatrix}4&4&1\\-4&-2&2\\-4&0&-4\end{vmatrix} = -72$$

$$M_{Cof}^T = \begin{bmatrix}8&-24&-8\\16&-12&-16\\10&-12&8\end{bmatrix}^T = \begin{bmatrix}8&16&10\\-2&-12&-12\\-8&-16&8\end{bmatrix}$$

$$M^{-1} = \frac{1}{-72}\begin{bmatrix} 8 & 16 & 10 \\ -24 & -12 & -12 \\ -8 & -16 & 8 \end{bmatrix}$$

$$= \frac{1}{18}\begin{bmatrix} -2 & -4 & -2.5 \\ 6 & 3 & 3 \\ 2 & 4 & -2 \end{bmatrix}$$

$$J = M^{-1}AM = \frac{1}{18}\begin{bmatrix} -2 & -4 & -2.5 \\ 6 & 3 & 3 \\ 2 & 4 & -2 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix}\begin{bmatrix} 4 & 4 & 1 \\ -4 & -2 & 2 \\ -4 & 0 & -4 \end{bmatrix}$$

$$= \frac{1}{18}\begin{bmatrix} 8 & 7 & 5.5 \\ -6 & -3 & -3 \\ -8 & -16 & 8 \end{bmatrix}\begin{bmatrix} 4 & 4 & 1 \\ -4 & -2 & 2 \\ -4 & 0 & -4 \end{bmatrix}$$

$$= \frac{1}{18}\begin{bmatrix} -18 & 18 & 0 \\ 0 & -18 & 0 \\ 0 & 0 & -72 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

$$\tilde{B} = M^{-1}B = \frac{1}{18}\begin{bmatrix} -2 & -4 & -2.5 \\ 6 & 3 & 3 \\ 2 & 4 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -8/18 \\ 6/18 \\ 8/18 \end{bmatrix} = \begin{bmatrix} -4/9 \\ 3/9 \\ 4/9 \end{bmatrix}$$

$$\tilde{C} = CM = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 4 & 4 & 1 \\ -4 & -2 & 2 \\ -4 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 1 \end{bmatrix}$$

The Jordan Canonical form of state model is

$$\dot{z} = Jz + \tilde{B}u \;,\quad y = \tilde{C}z + Du$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} -4/9 \\ 3/9 \\ 4/9 \end{bmatrix}u \;;\quad y = \begin{bmatrix} 4 & 4 & 1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

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CS Notes (V unit).

7.12 SHORT QUESTIONS AND ANSWERS

Q7.1 *What are the advantages of state space analysis?*

1. The state space analysis is applicable to any type of systems. They can be used for modelling and analysis of linear & non-linear systems, time invariant & time variant systems and multiple input & multiple output systems.
2. The state space analysis can be performed with initial conditions.
3. The variables used to represent the system can be any variables in the system.
4. Using this analysis the internal states of the system at any time instant can be predicted.

Q7.2 *What are the drawbacks in transfer function model analysis?*

1. Transfer function is defined under zero initial conditions.
2. Transfer function is applicable to linear time invariant systems.
3. Transfer function analysis is restricted to single input and output systems.
4. Does not provides information regarding the internal state of the system.

Q7.3 *What is state and state variable?*

The state is the condition of a system at any time instant, t. A set of variable which describes the state of the system at any time instant are called state variables.

Q7.4 *What is a state vector?*

The state vector is a (n × 1) column matrix (or vector) whose elements are state variables of the system, (where n is the order of the system). It is denoted by **X(t)**.

Q7.5 *Write the state model of nth order system?*

The state model of a system consists of state equation and output equation. The state model of a nth order system with m-inputs and p-outputs are

$$\dot{X}(t) = AX(t) + BU(t) \quad \text{.....state equation.}$$
$$Y(t) = CX(t) + DU(t) \quad \text{.....output equation.}$$

where X(t) = State vector of order (n×1) ; U(t) = Input vector of order (m×1)

A = System matrix of order (n×n) ; B = Input matrix of order (n×m)

Y(t) = Output vector of order (p×1) ; C = Output matrix of order (p×n).

D = Transmission matrix of order (p×m).

Q7.6 *What is state space?*

The set of all possible values which the state vector **X(t)** can have (or assume) at time t forms the state space of the system.

Q7.7 *What is input and output space?*

The set of all possible values which the input vector **U(t)** can have (or assume) at time t forms the input space of the system.

The set of all possible values which the output vector **Y(t)** can have (or assume) at time t forms the output space.

Q7.8 *The state model of a linear time invariant system is given by*

$$\dot{X}(t) = A\,X(t) + BU(t)$$
$$Y(t) = C\,X(t) + DU(t)$$

Obtain the expression for transfer function of the system.

Solution : Given that $\dot{X}(t) = A\,X(t) + B\,U(t)$ .....(1)

and $Y(t) = C\,X(t) + D\,U(t)$ .....(2)

On taking Laplace transform of equation (1) with zero initial conditions we get,

$$s\,X(s) = A\,X(s) + B\,U(s)$$

$$s\,X(s) - A\,X(s) = B\,U(s)$$

$$(sI - A)\,X(s) = B\,U(s) \quad \text{.....(3)}$$

On premultiplying equation (3) by $(sI-A)^{-1}$ we get

$$X(s) = (sI - A)^{-1}\,B\,U(s) \quad \text{.....(4)}$$

On taking Laplace transform of equation (2) we get,

$$Y(s) = C\,X(s) + D\,U(s) \quad \text{.....(5)}$$

Substitute for X(s) from equation (4) in equation (5) we get,

$$Y(s) = C\,[(sI - A)^{-1}\,B\,U(s)] + D\,U(s)$$

$$= [C\,(sI - A)^{-1}\,B + D]\,U(s)$$

$$\therefore \frac{Y(s)}{U(s)} = C\,(sI - A)^{-1}\,B + D \quad \text{.....(6)}$$

The equation (6) is the transfer function of the system.

Q7.14

Q7.9 ***What is state diagram?***

The pictorial representation of the state model of the system is called state diagram. The state diagram of the system can be either in block diagram or in signal flow graph form.

Q7.10 ***Draw the block diagram representation of state model?***

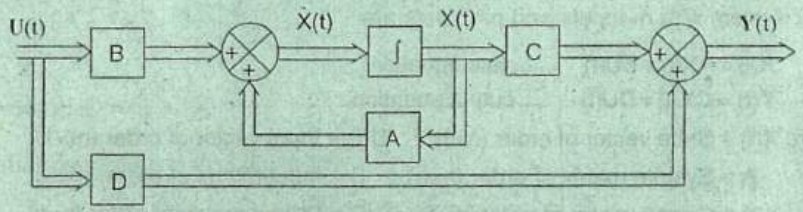


Fig Q7.10 : Block diagram of state model.

Q7.11 ***Draw the signal flow graph representation of state model?***

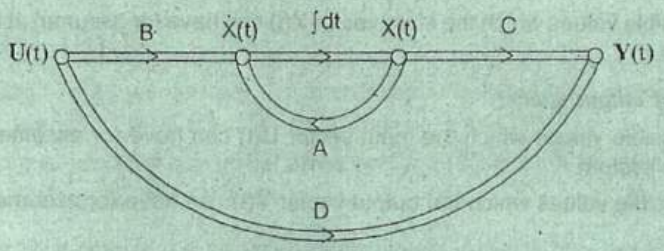


Fig Q7.11 : Signal flow graph of state model.

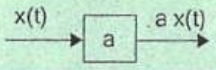
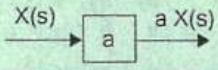
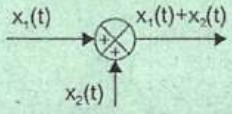
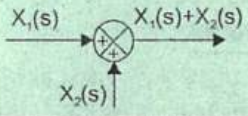
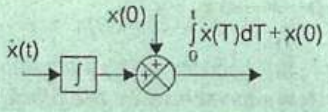
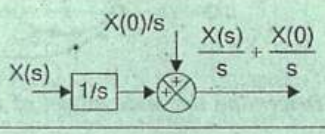
Q7.15

Q7.12 ***What are the basic elements used to construct the state diagram?***

The basic elements used to construct the state diagram are Scalar, Adder and Integrator.

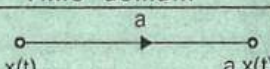
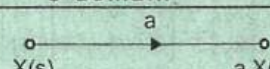
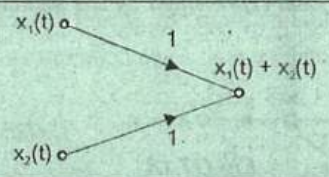
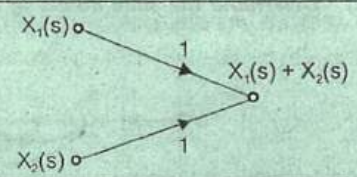
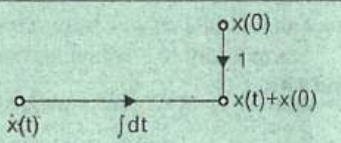
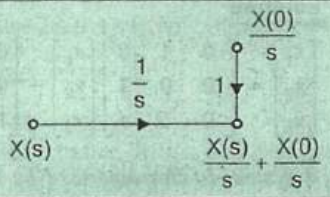
Q7.13 ***Sketch the basic elements used to construct the block diagram of a state model.***

The basic elements used to construct block diagram of a state model are shown in the following table.

Elements	Time domain	s-domain
Scalar		
Adder		
Integrator		

Q7.14 ***Sketch the basic elements used to construct the signal flow graph of a state model.***

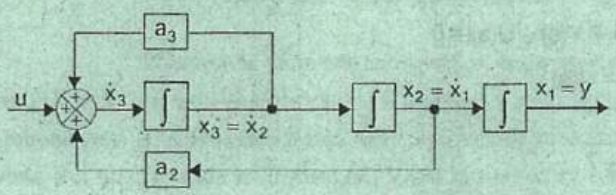
The basic elements used to construct the signal flow graph of a state model are shown in the following table.

Elements	Time domain	s-domain
Scalar		
Adder		
Integrator		

Q7.15 ***Draw the block diagram of the system described by the state model,***

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \quad \text{and} \quad y = x_1$$

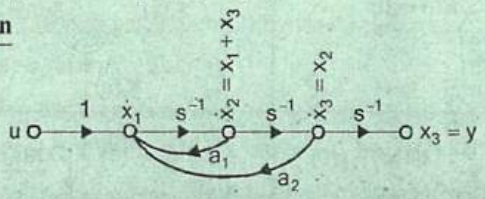
Solution



Q7.16 *Draw the signal flow graph of the system described by the state model.*

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}a_1 & a_2 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}1\\ 0\\ 0\end{bmatrix} u \quad \text{and } y = x_3$$

Solution



Q7.17 *Determine the state model of the system represented by the block diagram of fig Q7.17.*

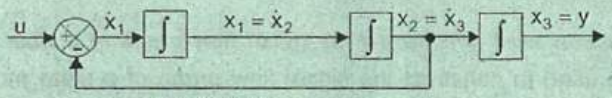


Fig Q7.17

Solution

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}1\\ 0\\ 0\end{bmatrix} u \text{ and } y = x_3$$

Q7.18 *Determine the state model of the system represented by the signal flow graph of fig Q7.18.*

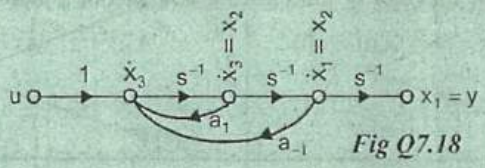


Fig Q7.18

Solution

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -2 & a_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix} u \text{ and } y = x_1$$

Q7.19 *A system is characterized by the differential equation,*

$$\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 7y - u = 0$$

Determine its transfer function.

Solution

Given that, $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 7y - u = 0$ (1)

On taking Laplace transform of equation (1) with zero initial conditions we get,

$$s^2 Y(s) + 10\,s\,Y(s) + 7\,Y(s) - U(s) = 0$$

$$(s^2 + 10\,s + 7)\,Y(s) = U(s)$$

$$\therefore \frac{Y(s)}{U(s)} = \frac{1}{s^2 + 10s + 7} \quad (2)$$

The equation (2) is the transfer function of the system.

Q7.20 *The transfer function of a system is given by* $\frac{Y(s)}{U(s)} = \frac{10}{4s^2 + 2s + 1}$. *Determine the differential equation governing the system.*

Solution

Given that, $\frac{Y(s)}{U(s)} = \frac{10}{4s^2 + 2s + 1}$

$\therefore [4s^2 + 2s + 1]\, Y(s) = 10\, U(s)$

$4s^2\, Y(s) + 2s\, Y(s) + Y(s) = 10\, U(s)$ .....(1)

On taking inverse Laplace transform of equation (1) we get,

$$4\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 10u$$

$$4\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y - 10u = 0 \quad \text{.....(2)}$$

The equation (2) is the differential equation governing the system.

Q7.21 *What are the advantages of state space modelling using physical variable?*

The advantages of choosing the physical variable are the following,

1. The state variable can be utilized for the purpose of feedback
2. The implementation of design with state variable feedback becomes straight forward.
3. The solution of state equation gives time variation of variables which have direct relevance to the physical system.

Q7.22 *What are phase variables?*

The phase variables are defined as those particular state variables which are obtained from one of the system variable and its derivatives. Usually the variable used is the system output and the remaining state variables are then derivatives of the output.

Q7.23 *What is bush form or companion form of state model?*

In bush form or companion form of state model, the system matrix, A has all1's in the upper off-diagonal and its last row is comprised of the negative of the coefficients of the original differential equation and all other elements are zero. The companion form of state model is shown below

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix} [u]$$

$$y = [1 \quad 0 \quad 0 \quad \cdots \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

Q7.24 *What are the advantages in choosing phase variables for state space modelling?*

1. Using phase variables the system state model can be written directly by inspection from the differential equation governing the system.
2. The phase variables provides a link between the transfer function design approach and time-domain design approach.

Q7.25 *What is the disadvantage in choosing phase variable for state-space modelling?*

The disadvantage in choosing phase variables is that the phase variables are not physical variables of the system and therefore are not available for measurement and control purposes.

Q7.26 *Write the canonical form of state model of nth order system.*

In canonical form of state model, the system matrix, A will be a diagonal matrix. The canonical form of state model in the matrix form is given below.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} [u]$$

$$y = [c_1 \;\; c_2 \;\; c_3 \;\; \cdots \;\; c_n] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + [b_o][u]$$

Q7.27 *What is the advantage and the disadvantage in canonical form of state model.*

The advantage of canonical form is that the state equations are independent of each other. The disadvantage is that the canonical variables are not physical variables and so they are not available for measurement and control.

Q7.28 *What is state transition matrix and how it is related to state of a system?*

The matrix exponential e^{At} is called state transition matrix. In the expanded form,

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \ldots\ldots + \frac{1}{1!}A^it^i + \ldots\ldots$$

The state transition matrix is used to find the state of the system, at any time instant t, from the knowledge of the state at time, t_0.

When the input is zero, $X(t) = e^{At}\, X(t_0)$

When the input vector is U(t), $X(t) = e^{A(t-t_0)}X(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}\, B\, U(\tau)\, dt$

where, X(t) = State vector at time, t and $X(t_0)$ = State vector at time, t_0.

Q7.29 *Write the properties of state transition matrix.*

The following are the properties of state transition matrix.

1. $\phi(0) = e^{A \times 0} = I$ (Unit matrix)
2. $\phi(t) = e^{At} = (e^{-At})^{-1} = [\phi(-t)]^{-1}$
3. $\phi(t_1 + t_2) = e^{A(t_1+t_2)} = e^{At_1}\, e^{At_2} = \phi(t_1)\, \phi(t_2) = \phi(t_2)\, \phi(t_1)$

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Q7.30 ***Write the solution of homogeneous state equations.***

The solution of homogeneous state equation is, $X(t) = e^{At} X_0$

where, $X(t)$ = State vector at time, t

e^{At} = State transition matrix

and X_0 = Initial condition vector at t = 0.

Q7.31 ***Write the solution of non-homogeneous state equations.***

The solution of non-homogeneous state equation is

$$X(t) = e^{A(t-t_0)} X(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} B U(\tau)\, d\tau$$

where, $X(t)$ = State vector of time, t ; $X(t_0)$ = Initial condition vector at $t = t_0$.

B = Input matrix and $U(t)$ = Input vector

If initial conditions are known at t = 0, then put $t_0 = 0$.

Q7.32 ***What is resolvant matrix?***

The Laplace transform of state transition matrix is called resolvant matrix.

Resolvant matrix, $\phi(s) = \mathcal{L}[\phi(t)] = \mathcal{L}[e^{At}]$

Also, $\phi(s) = [sI - A]^{-1}$

Q7.33 ***Define the characteristic equation of a matrix.***

The characteristic equation of a n x n matrix A is the nth degree polynomial of equation, $|\lambda I - A| = 0$, where I is the unit matrix.

Q7.34 ***What are eigenvalues and eigenvector?***

A non-zero column vector X is an eigenvector of a square matrix A, if there exists a scalar λ such that $AX = \lambda X$, then λ is an eigenvalue of A.

Q7.35 ***How the eigenvalues are calculated?***

The eigenvalues of a matrix A are the roots of characteristic equation of A. The characteristic equation is $|\lambda I - A| = 0$. Solving the characteristic equation for λ gives the eigenvalues of A.

Q7.36 ***Write any two properties of eigenvalues.***

1. A matrix and its transpose have the same eigenvalues.
2. The product of the eigenvalues (counting multiplicities) of the matrix equals the determinant of the matrix.

Q7.37 ***How the eigenvectors are calculated, when the eigenvalues are distinct?***

When the eigenvalues are distinct, then there exists only one independent eigenvector corresponding to each eigenvalue. The independent eigenvector corresponding to eigenvalue λ_i may be obtained by taking cofactors of matrix $[\lambda_i I - A]$ along any row.

If $C_{k1}, C_{k2}, \ldots C_{kn}$ are cofactors of matrix $[\lambda_i I - A]$ along k^{th} row then the eigenvector m_i corresponding to λ_i is given by,

$$m_i = \begin{bmatrix} C_{k1} \\ C_{k2} \\ \vdots \\ C_{kn} \end{bmatrix} \quad ; \quad k = 1 \text{ or } 2 \text{ or } \ldots\ldots n$$

Q7.38 ***What is similarity transformation?***

The process of transforming a square matrix A to another similar matrix B by a transformation $P^{-1}AP = B$ is called similarity transformation. The matrix P is called transformation matrix.

Q7.39 ***What is meant by diagonalization?***

The process of converting the system matrix A into a diagonal matrix by a similarity transformation using the modal matrix M is called diagonalization.

Q7.40 ***What is modal matrix?***

The modal matrix is a matrix used to diagonalize the system matrix. It is also called diagonalization matrix.

If A = System matrix

M = Modal matrix

and M^{-1} = Inverse of modal matrix, then $M^{-1}AM$ will be a diagonalized system matrix.

Note : Refer appendix for modal matrix

Q7.41 ***How the modal matrix is determined?***

The modal matrix M can be formed from eigenvectors. Let $m_1, m_2, m_3, \ldots m_n$ be the eigenvectors of a nth order system. Now the modal matrix M is obtained by arranging all the eigenvectors columnwise as shown below

Modal matrix, $M = [m_1 \quad m_2 \quad m_3 \quad \ldots\ldots\ldots \quad m_n]$

Q7.42 ***What is canonical form of state model?***

If the system matrix, A is in the form of diagonal matrix then the state model is called canonical form [In diagonal matrix, we have the eigenvalues on the main diagonal and all other elements are zero].

Q7.43 ***What is the transformation used to diagonalize a system matrix?***

The transformation used to diagonalize a system matrix is X = MZ

where X = Original state vector

Z = Transformed state vector ; M = Modal matrix.

Q7.44. ***Write the transformed canonical state model of a system.***

The transformed state model of a system is,

$$\dot{Z} = \Lambda Z + \tilde{B} U$$

$$Y = \tilde{C} Z + D U$$

where, Z = Transformed state vector ; $\tilde{B} = M^{-1}B$

M = Modal matrix ; $\tilde{C} = CM$

$\Lambda = M^{-1} AM$

Q7.45 ***When modal matrix is called vander monde matrix.***

When the system matrix A is in the companion or bush form then the modal matrix is given by a special matrix called vander monde matrix, V, shown below,

$$M = V = \begin{bmatrix} 1 & 1 & \cdots\cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots\cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots\cdots & \lambda_n^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots\cdots & \lambda_n^{n-1} \end{bmatrix}$$

Q7.46 ***What is Jordan canonical form?***

When the eigenvalues have multiplicity the system matrix cannot be diagonalized. But the transformation, X = MZ will transform the system matrix to a form called Jordan matrix, J, where J is M^{-1} AM. The transformed state model in this case is called Jordan canonical form.

Q7.47 ***What is Jordan matrix?***

The Jordan matrix is the transformed system matrix using the transformation, X = MZ. The Jordan matrix J, will have a Jordan block of size q x q correspond to a eigenvalue of λ_1 with multiplicity q. In the Jordan block the diagonal elements will be eigenvalues and elements just above the diagonal is one. A typical Jordan matrix is shown below,

$$J=\begin{bmatrix} \lambda_1 & 1 & 0 & \cdots & 0 \\ 0 & \lambda_1 & 1 & \cdots & 0 \\ 0 & 0 & \lambda_1 & \cdots & 0 \\ 0 & 0 & 0 & \lambda_4 \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \cdots & \lambda_n \end{bmatrix}$$

Q7.48 ***How the state transition matrix e^{At} is computed by canonical transformation.***

When the eigenvalues are distinct the following procedure can be used to compute e^{At}.

1. Compute the eigenvalues $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ and eigenvector $m_1, m_2, \ldots\ldots, m_n$ of the system matrix A.
2. Determine the modal matrix M and M^{-1}. Modal matrix, $M = [m_1\ m_2\ \ldots\ldots\ m_n]$.
3. The state transition matrix e^{At} is given by, $e^{At} = M\ e^{\Lambda t}\ M^{-1}$

where, $$e^{\Lambda t}=\begin{bmatrix} e^{\lambda_1 t} & 0 & \ldots\ldots & 0 \\ 0 & e^{\lambda_2 t} & \ldots\ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots\ldots & e^{\lambda_n t} \end{bmatrix}$$

Q7.49 ***Define controllability.***

A system is said to be completely state controllable if it is possible to transfer the system state from any initial state $X(t_0)$ at any other desired state X(t), in specified finite time by a control vector U(t).

Q7.50 ***What is the need for controllability test?***

The controllability test is necessary to find the usefulness of a state variable. If the state variables are controllable then by controlling (i.e., varying) the state variables the desired outputs of the system are achieved.

Q7.51 ***State the condition for controllability by Gilbert's method.***

Case(i) : When eigenvalues are distinct

Consider the canonical form of state model shown below which is obtained by using the transformation X = MZ.

$$\dot{Z} = \Lambda Z + \tilde{B} U$$
$$Y = \tilde{C} Z + D U$$

where, $\Lambda = M^{-1} AM$; $\tilde{C} = CM$

$\tilde{B} = M^{-1}B$ and M = Modal matrix

In this case the necessary and sufficient condition for complete controllability is that, the matrix $\tilde{B}$ must have no rows with all zeros. If any row of the matrix $\tilde{B}$ is zero then the corresponding state variable is uncontrollable.

Case (ii) : When eigenvalues have multiplicity

In this case the state model can be converted to Jordan canonical form shown below,

$$\dot{Z} = J Z + \tilde{B} U$$
$$Y = \tilde{C} Z + D U$$

; where $J = M^{-1} AM$

In this case the system is completely controllable, if the elements of any row of $\tilde{B}$ that correspond to the last row of each Jordan block are not all zero and all rows corresponding to other state variable must not have all zeros

Q7.52 ***State the condition for controllability by Kalman's method.***

For a nth order system described by state equation, $\dot{X} = AX + BU$, we can form a composite matrix, Q_c

where, $Q_c = [B \quad AB \quad A^2B \ldots\ldots A^{n-1}B]$

In this case the system is completely state controllable if the rank of the composite matrix Q_c is n.

Q7.53 ***What is the advantage and the disadvantage in Kalman's test for controllability?***

The advantage in Kalman's test is that the calculations are simpler. But the disadvantage in Kalman's test is that, we can't find the state variable which is uncontrollable. But in Gilbert's method we can find the uncontrollable state variable which is the state variable corresponding to the row of $\tilde{B}$ which has all zeros.

Q7.54 ***Define observability.***

A system is said to be completely observable if every state X(t) can be completely identified by measurements of the output Y(t) over a finite time interval.

Q7.55 ***What is the need for observability test?***

The observability test is necessary to find whether the state variables are measurable or not. If the state variables are measurable then the state of the system can be determined by practical measurements of the state variables.

Q7.56 ***State the condition for observability by Gilbert's method.***

Consider the transformed canonical or Jordan canonical form of the state model shown below which is obtained by using the transformation , $X = MZ$

$\dot{Z} = \Lambda Z + \tilde{B}U$ (or) $\dot{Z} = JZ + \tilde{B}U$

$Y = \tilde{C}Z + DU$ $\qquad\qquad$ $Y = \tilde{C}Z + DU$

where $\tilde{C} = CM$ and M = Modal matrix.

The necessary and sufficient condition for complete observability is that none of the columns of the matrix $\tilde{C}$ be zero. If any of the column is of $\tilde{C}$ has all zeros then the corresponding state variable is not observable.

Q7.57 ***State the condition for observability by Kalman's method?***

For a nth order system described by state model,

$\dot{X} = AX + BU$

$Y = CX + DU$

we can form a composite matrix Q_o, where

$Q_o = [C^T \quad A^TC^T \quad (A^T)^2C^T \quad (A^T)^3C^T \ldots\ldots (A^T)^{n-1}C^T]$

In this case the system is completely observable if the rank of composite matrix, Q_o is n.

Q7.58 ***What is the advantage and the disadvantage in Kalman's test for observability?***

The advantage in Kalman's test is that the calculations are simpler. The disadvantage in Kalman's test is that the nonobservable state variables cannot be determined.

Q7.59 ***State the duality between controllability and observability.***

The concepts of controllability and observability are dual concepts and it is proposed by Kalman as principle of duality.

The principle of duality states that a system is completely state controllable if and only if its dual system is completely observable or viceversa (i.e., if the system is observable then its dual is controllable).

Q7.60 ***What is the effect of pole-zero cancellation in transfer function.***

If cancellation of pole-zero occurs in the transfer function of a system, then the system will be either not state controllable or unobservable depending on how the state variables are defined (or chosen).

7.13 EXERCISES

E7.1 ***Obtain the state model of the electrical network shown below by choosing minimal number of state variables.***

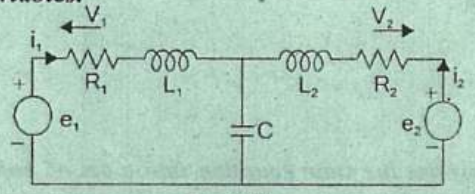

Fig E7.1a

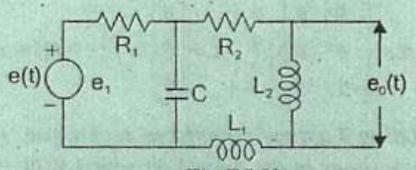

Fig E7.1b

E7.2 ***Construct the state model of mechanical systems shown in fig E7.2a & E7.2b.***

E7.3 ***Obtain the state model of the system whose transfer function is given as***

a) $\frac{1}{s^2+3s+2}$ b) $\frac{s+3}{s^2+3s+2}$

c) $\frac{s^2+4s+3}{s^3+9s+20}$ d) $\frac{s^2+6s+8}{(s+3)(s^2+2s+2)}$

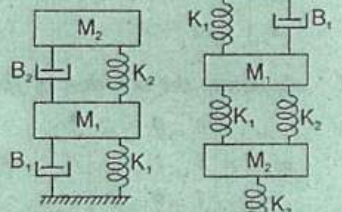

Fig E7.2a

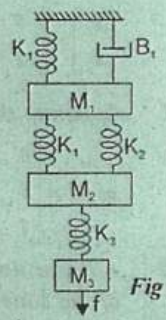

Fig E7.2b

E7.4 ***Construct a state model for the systems characterized by the following differential equations.***

a) $\dddot{y}+3\ddot{y}+2\dot{y}=\dot{u}+u$ b) $\dddot{y}+6\ddot{y}+11\dot{y}+6y=u$ c) $\ddot{y}+2\dot{y}+y=\dot{u}+u$

E7.5 ***In the state diagrams of systems shown in figE7.5a and figE7.5b. Assign state variables and obtain state model of the system.***

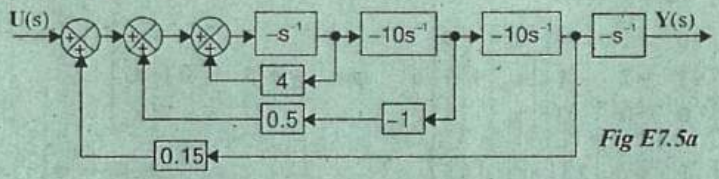

Fig E7.5a

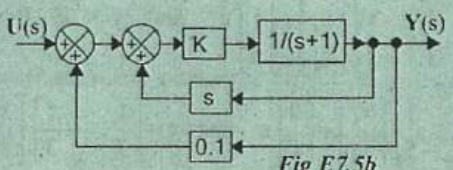

Fig E7.5b

E7.6 ***The state diagram of a linear system is shown in fig E7.6. Assign state variables and obtain the state model.***

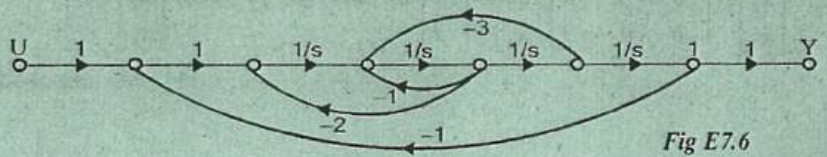

Fig E7.6

E7.7 *For the system matrices given below compute e^{At}*

a) $A=\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}$ b) $A=\begin{bmatrix}-2 & 1\\ 1 & -2\end{bmatrix}$ c) $A=\begin{bmatrix}-2 & 0\\ 0 & -1\end{bmatrix}$ d) $A=\begin{bmatrix}-1 & 0\\ 0 & -3\end{bmatrix}$

E7.8 *The following facts are known about the linear system,* $\dot{x}(t) = A\,x(t)$.

If $x(0)=\begin{bmatrix}2\\ -1\end{bmatrix}$ *then* $x(t)=\begin{bmatrix}e^{-2t}\\ -4e^{-2t}\end{bmatrix}$; *If* $\dot{x}(0)=\begin{bmatrix}1\\ -2\end{bmatrix}$ *then* $x(t)=\begin{bmatrix}e^{-t}\\ -2e^{-t}\end{bmatrix}$

Find e^{-At} *and hence A.*

E7.9 *A system is described by the state equation.*

$$\dot{X}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & 0 & -3\end{bmatrix}X+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u \quad ; \quad X(0)=x_0$$

Using Laplace transform technique, transform the state equation into a set of linear algebraic equations in the form $X(s) = \phi(s)\,X(0) + \phi(s)\,B\,U(s)$

E7.10 *Consider the state equation*

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

E7.11 *Compute the solution of following state equations.*

a) $\dot{X}=\begin{bmatrix}0 & 0 & -2\\ 0 & 1 & 0\\ 1 & 0 & 3\end{bmatrix}X \; ; \; X(0)=\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}$ b) $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ 1\end{bmatrix}u(t)$ *with* $X_0=\begin{bmatrix}1\\ 1\end{bmatrix}$

E7.12 *A linear time invariant system is described by the following state model. Obtain the canonical form of the state model.*

(a) $$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 0 & -20\\ 1 & 0 & -24\\ 0 & 1 & -9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}3\\ 1\\ 0\end{bmatrix}u$$

(b) $$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ -2 & -2 & -1\\ 0 & 0 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 1\\ 1\end{bmatrix}u \quad \text{and} \quad y=[3\;\;1\;\;0]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

(c) $$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}1 & 0 & 2\\ -1 & 1 & 1\\ 0 & 3 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}u \quad \text{and} \quad y=[1\;\;0\;\;1]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

(d) $$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -4 & -5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u \quad \text{and} \quad y=\begin{bmatrix}\frac{1}{3} & -\frac{1}{3}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

(e) $$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-1 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}1\\ -2\\ 1\end{bmatrix}u \quad \text{and} \quad y=[1\;\;1\;\;0]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

(f)

(g)

(h)

(i)

E7.13 Deter

a)

b)

c)

E7.14 Examin

a)

b)

E7.15 Convert th

a) $A=$

(f) $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ 1\end{bmatrix}[r]$ and $y=[-1 \quad -2]\begin{bmatrix}x_1\\ x_2\end{bmatrix}$

(g) $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -12 & -7\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}[r]$ and $y=[-10 \quad -4]\begin{bmatrix}x_1\\ x_2\end{bmatrix}+[u]$

(h) $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-2 & 0 & 1\\ 1 & -3 & 0\\ 1 & 1 & -1\end{bmatrix}\begin{bmatrix}x_2\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}u$ and $y=[2 \quad 1 \quad -1]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$

(i) $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ -2 & -2 & 0\\ 0 & 0 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 1\\ 1\end{bmatrix}u$ and $y=\begin{bmatrix}\frac{14}{5} & \frac{6}{5} & -\frac{1}{5}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$

f linear algebraic

7.13 *Determine the state controllability for the systems represented by the following state equations.*

a) $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}2 & 1\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ 0\end{bmatrix}u$

b) $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u$

$X_0=\begin{bmatrix}1\\ 1\end{bmatrix}$

c) $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ -1\end{bmatrix}u$

the canonical form

7.14 *Examine the observability of the systems given below using canonical form.*

a) $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}2 & -2 & 3\\ 1 & 1 & 1\\ 1 & 3 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}11\\ 1\\ -14\end{bmatrix}u$ and $y=[-3 \quad 5 \quad -2]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$

b) $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 1 & 1\\ 0 & -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u$ and $y=[3 \quad 4 \quad 1]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$

7.15 *Convert the following system matrix to canonical form.*

a) $A=\begin{bmatrix}-3 & 0 & 1\\ 1 & 1 & 0\\ 2 & -1 & 2\end{bmatrix}$ b) $A=\begin{bmatrix}1 & 1 & -1\\ 0 & 2 & 1\\ 1 & 2 & 1\end{bmatrix}$ c) $A=\begin{bmatrix}1 & 2 & 1\\ -1 & 0 & 2\\ 1 & 3 & -1\end{bmatrix}$

Answer for Exercise Problems

E7.1a $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-R_1/L_1 & 0 & -1/L_1\\ 0 & -R_2/L_2 & -1/L_2\\ 1/C & 1/C & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}1/L_1 & 0 & 0\\ 0 & 1/L_2 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}u_1(t)\\ u_2(t)\\ 0\end{bmatrix};\quad \begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}=\begin{bmatrix}R_1 & 0 & 0\\ 0 & R_2 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$

E7.1b $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & -1/C & 0\\ 1/L_1 & -R_2/L_1 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}1/CR_1 & 0 & 0\\ 0 & -1/L_1 & 0\\ 0 & -1/C_2 & 0\end{bmatrix}\begin{bmatrix}u_1(t)\\ u_2(t)\\ 0\end{bmatrix};\quad \begin{bmatrix}y_1\\ y_2\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & R_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$

E7.2a $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}=\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -K_2 & K_2 & -B_2 & B_2\\ 0 & -K_1 & 0 & -B_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix};\quad \begin{bmatrix}y_1\\ y_2\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$

E7.2b $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\\ \dot{x}_5\\ \dot{x}_6\end{bmatrix}=\begin{bmatrix}0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ -\frac{(K_1+K_2+K_3)}{M_1} & \frac{K_2+K_3}{M_1} & 0 & \left(\frac{-B_1}{M_1}\right) & 0 & 0\\ \left(\frac{K_2+K_3}{M_2}\right) & -\frac{(K_2+K_3+K_4)}{M_2} & \left(\frac{-K_4}{M_2}\right) & 0 & 0 & 0\\ 0 & \left(\frac{K_3}{M_3}\right) & \left(\frac{-K_3}{M_3}\right) & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 0\\ 0\\ 0\\ 1/M_3\end{bmatrix}u(t)$

$\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6\end{bmatrix}$

E7.3a $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-3 & 1\\ -2 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u\ ;\ Y=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$

E7.3b $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-2 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ 1\end{bmatrix}u\ ;\ Y=\begin{bmatrix}-1 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$

E7.3c $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -20 & -9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u\ ;\ Y=\begin{bmatrix}-17 & -5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+u$

E7.3d $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3$

E7.4a $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3$

E7.4c $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2$

E7.5a $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4$

E7.5b $\dot{x}_1 =$

E7.6 $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}$

E7.7a $e^{At} =$

E7.7b $e^{At} =$

E7.8 $e^{-At} =$

E7.9 $X(s) =$

E7.10 $e^{At} = \begin{bmatrix}t$

E7.3d $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-6 & 0 & 0\\ -8 & 0 & 0\\ -5 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}8\\ 6\\ 1\end{bmatrix} u \; ; \; Y = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$

E7.4a $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ 0\end{bmatrix}$

$Y = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$

E7.4b $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -6 & -11 & -6\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix} u \; ; \; Y = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$

E7.4c $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}; \; Y = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$

E7.5a $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & -10 & 0 & 0\\ 0 & 0 & -10 & 0\\ 0 & 0 & 0 & -10\\ 0 & -0.15 & 0.5 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 0\\ 1\end{bmatrix} u \; ; \quad Y = \begin{bmatrix}-1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$

E7.5b $\dot{x}_1 = \dfrac{1 - 0.1K}{1 - K} x_1 + \dfrac{K}{1 - K} u(s)$

E7.6 $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & -3 & 1 & 1\\ -1 & 0 & -2 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 0\\ 1\end{bmatrix} u \; ; \quad Y = \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$

E7.7a $e^{At} = \begin{bmatrix}0.2764\, e^{1.618t} + 0.7236\, e^{-0.618t} & 0.4472\, e^{1.618t} - 0.4472\, e^{-0.618t}\\ 0.4472\, e^{1.618t} - 0.4472\, e^{-0.618t} & 0.7236\, e^{1.618t} + 0.2764\, e^{-0.618t}\end{bmatrix}$

E7.7b $e^{At} = 0.5\begin{bmatrix}e^{-t} + e^{-3t} & e^{-t} - e^{-3t}\\ e^{-t} - e^{-3t} & e^{-t} + e^{-3t}\end{bmatrix}$

E7.7c $e^{At} = \begin{bmatrix}e^{-2t} & 0\\ 0 & e^{-t}\end{bmatrix}$

E7.7d $e^{At} = \begin{bmatrix}e^{-t} & 0\\ 0 & e^{-3t}\end{bmatrix}$

E7.8 $e^{-At} = \begin{bmatrix}2(e^{t} - e^{2t}) & \frac{1}{2}e^{t} - e^{2t}\\ e^{2t} - 4e^{t} & \frac{1}{2}e^{2t} - e^{t}\end{bmatrix}$

E7.9 $X(s) = \dfrac{1}{s^3 + 3s^2 + 1}\begin{bmatrix}s & -1 & 0\\ 0 & s & -1\\ 1 & 0 & s+3\end{bmatrix} x_0 + \dfrac{1}{s^3 + 3s^2 + 1}\begin{bmatrix}0\\ -1\\ s+3\end{bmatrix} u(s)$

E7.10 $e^{At} = \begin{bmatrix}te^{t} + e^{-t} & -te^{-t}\\ te^{-t} & -te^{-t} + e^{-t}\end{bmatrix}$

E7.11a $X(t) = \begin{bmatrix} 0 \\ e^t \\ 0 \end{bmatrix}$ E7.11b $X(t) = \begin{bmatrix} e^t + te^t \\ e^t \end{bmatrix} + \begin{bmatrix} e^t + te^t \\ e^t \end{bmatrix} u(t)$

E7.12

a) $\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0.22 \\ 0.33 \\ 0.22 \end{bmatrix} [u]$

b) $\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0.7321 & 0 \\ 0 & 0 & -2.732 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0.0528 \\ 0.1969 \end{bmatrix} [u] \; ; \; Y = \begin{bmatrix} -1 & 16.929 & 3.0722 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

c) $\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -1.777 & 0 & 0 \\ 0 & 1.3889 + j1.4174 & 0 \\ 0 & 0 & 1.3889 - j1.4174 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0.8943 + j0.1974 \\ -0.6047 + j0.1506 \\ -0.8481 + j0.6649 \end{bmatrix} [u]$

$Y = \begin{bmatrix} 1.2799 & (-0.36 - j1.3124) & (2.36 + j1.3124) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

d) $\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} [u] \; ; \; Y = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

e) $\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 2 \\ 0.5 \end{bmatrix} [u] \; ; \; Y = \begin{bmatrix} 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

f) $\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix} [u] \; ; \; Y = \begin{bmatrix} -2 & 4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

g) $\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & -11 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} [u] \; ; \; Y = \begin{bmatrix} 2 & 6 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + u$

h) $\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -0.5858 & 0 \\ 0 & 0 & -3.4142 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} -0.6667 \\ 0.3738 \\ -1.0405 \end{bmatrix} [u]$

$Y = \begin{bmatrix} -5 & -2.4145 & 0.4142 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

i) $\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1+j & 0 \\ 0 & 0 & -1-j \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0.2 \\ -(0.25 + j0.25) \\ -0.25 + j0.25 \end{bmatrix}$

$Y = \begin{bmatrix} -1 & \left(\frac{2}{5} + j\frac{14}{5}\right) & \left(\frac{2}{5} - j\frac{14}{5}\right) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

E7.13

a)

b)

c)

E7.14

a)

b)

E7.15

a)

b)

c)

E7.13

a) Non-controllable

b) Controllable

c) Controllable

E7.14

a) Non-observable

b) Observable

E7.15

a) $\begin{bmatrix} 1.1848 & 0 & 0 \\ 0 & 2.2267 & 0 \\ 0 & 0 & -3.4115 \end{bmatrix}$

b) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0.5+j1.6583 & 0 \\ 0 & 0 & 0.5-j1.6583 \end{bmatrix}$

c) $\begin{bmatrix} -1.943 & 0 & 0 \\ 0 & 1.9715+j1.3321 & 0 \\ 0 & 0 & 1.9715-j1.3321 \end{bmatrix}$