

Unit-1 - Signal Analysis

part-1

(1)

Introduction:

Signal: A signal is defined as a single-valued function of one or more independent variables that contains or conveys information about a physical quantity. voice, image etc are examples.

Example $x(t) = 0.7t$, where t is independent variable.
 $x(p, q) = 0.6p + 0.5q + 1.1z^2$

above examples are mathematical representation of signals

Analogy between Vectors and Signals:

A vector is a quantity that has both magnitude and direction and is represented either using the convention of a letter with a superscripted arrow such as $\vec{A}$, $\vec{B}$, etc, or with boldface type A, B , etc. The magnitude of a vector is represented as $|\vec{A}|$ or simply A .

A signal is broadly defined as a single-valued quantity that is a function of independent variables. This quantity has both magnitude

and direction, and therefore, all signals are vectors.

From the basic concepts of vector algebra we know that in a two-dimensional vector space the dot product of two vectors $\vec{A}$ and $\vec{B}$ is defined as the product of the magnitudes of $\vec{A}$ and $\vec{B}$ and the cosine of the angle between them. The dot product yields a scalar quantity.

$$\text{Thus, } \vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta = AB \cos \theta,$$

θ being the smaller of the angles between $\vec{A}$ and $\vec{B}$ and less than 180° .

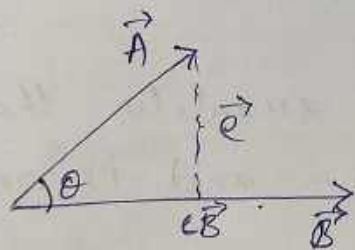


Fig. 1. projection of $\vec{A}$ along $\vec{B}$

If $c\vec{B}$ is a vector component of $\vec{A}$ along the direction of $\vec{B}$, where $c < 1$ is a scalar constant, $\vec{e}$ is another vector called the error vector, then the vector $\vec{A}$ can be expressed in terms of vector $\vec{B}$ as

$$\vec{A} = c\vec{B} + \vec{e} \quad \text{or}$$

$$\vec{e} = \vec{A} - c\vec{B}. \quad \text{--- (1)}$$

It may be observed that the length of the vector $\vec{e}$ will be minimum if vector $\vec{e}$ is normal to vector $\vec{B}$.

Hence, any vector $\vec{A}$ in the two-dimensional space can be expressed in terms of two orthogonal vectors, $\vec{B}$ and $\vec{e}$, (if the length of the vector $\vec{e}$ is a minimum). That is we have to minimise

$$e = |\vec{e}| = |\vec{A} - c\vec{B}|$$

If $c=0$, there is no component of $\vec{A}$ along $\vec{B}$ and the vectors $\vec{A}$ and $\vec{B}$ are said to be orthogonal to each other. Let us evaluate the value of c at the minimum of $\vec{e}$, From Fig 1.

$$\begin{aligned} CB &= |c\vec{B}| = A \cos \theta \\ CB^2 &= AB \cos \theta = \vec{A} \cdot \vec{B} \\ c\vec{B} \cdot \vec{B} &= \vec{A} \cdot \vec{B} \\ \therefore c &= \frac{\vec{A} \cdot \vec{B}}{\vec{B} \cdot \vec{B}} = \frac{1}{B^2} (\vec{A} \cdot \vec{B}) \quad \text{--- (2)} \end{aligned}$$

$\left[\because \cos \theta = \frac{CB}{A} \Rightarrow CB = A \cos \theta \right]$

Substituting the vector value of c from eqn (2) in eqn (1), the Error vector is

$$\begin{aligned} \vec{e} &= \vec{A} - c\vec{B} = \vec{A} - \frac{1}{B^2} (\vec{A} \cdot \vec{B}) \vec{B} \\ \vec{e} \cdot \vec{B} &= \vec{A} \cdot \vec{B} - c\vec{B} \cdot \vec{B} = \vec{A} \cdot \vec{B} - \frac{1}{B^2} (\vec{A} \cdot \vec{B}) B^2 \\ \therefore \boxed{\vec{e} \cdot \vec{B} &= \vec{A} \cdot \vec{B} - \vec{A} \cdot \vec{B} = 0} \quad \text{--- (3)} \end{aligned}$$

Thus, vector $\vec{e}$ is minimum if it is normal to vector $\vec{B}$. (4)

Note: If the two vectors $\vec{A}$ and $\vec{B}$ are orthogonal to each other, then $\vec{A} \cdot \vec{B} = 0$ and $c = 0$. That is, there is no component of $\vec{A}$ on $\vec{B}$.

Orthogonality in signals:

The concept of vector orthogonality can be extended in a similar manner to signals.

consider any two independent real signals $f(t)$ and $x(t)$ which are functions of time in the interval $t_1 \leq t \leq t_2$. The function $f(t)$ can be approximated in terms of the function $x(t)$ over the interval $t_1 \leq t \leq t_2$ as

$$f(t) \approx cx(t); \quad t_1 \leq t \leq t_2$$

The difference between $f(t)$ and $cx(t)$ is called the error signal $f_e(t)$, and it is given by

$$f_e(t) = f(t) - cx(t) \quad \text{--- (4)}$$

For the best approximation, the error should be a minimum, by taking mean square value (error may be either positive or negative)

That is,

(5)

$$e = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} f_e^2(t) dt = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} |f(t) - cx(t)|^2 dt \quad t_1 \leq t \leq t_2$$

where e is called the mean square error (MSE) or power of the error signal.

we know that the necessary condition to minimise the mean square error with respect to the constant

$$c \text{ is } \frac{de}{dc} = 0$$

$$\frac{de}{dc} = \frac{d}{dc} \left[\frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} f_e^2(t) dt \right] = \frac{d}{dc} \left[\frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} |f(t) - cx(t)|^2 dt \right] = 0$$

$$\Rightarrow \frac{d}{dc} \left(\int_{t_1}^{t_2} f^2(t) dt \right) + \frac{d}{dc} \int_{t_1}^{t_2} c^2 x^2(t) dt - \frac{d}{dc} \int_{t_1}^{t_2} 2cx(t)f(t) dt = 0$$

$$\Rightarrow 0 + 2c \int_{t_1}^{t_2} x^2(t) dt - 2 \int_{t_1}^{t_2} x(t)f(t) dt = 0$$

$$c = \frac{\int_{t_1}^{t_2} x(t)f(t) dt}{\int_{t_1}^{t_2} x^2(t) dt} \quad \text{--- (5)}$$

It is to be noted that the c obtained in eqn (5) is similar to the c obtained for vectors in eqn (2)

Substituting eqn (5) in eqn (4), the condition for the minimum of the error signal is

$$f_e(t) = f(t) - cx(t)$$

$$= f(t) - \frac{\int_{t_1}^{t_2} x(t)f(t) dt}{\int_{t_1}^{t_2} x^2(t) dt} x(t)$$

$$x(t) f_e(t) = x(t) f(t) - \left(\frac{\int_{t_1}^{t_2} x(t) f(t) dt}{\int_{t_1}^{t_2} x^2(t) dt} \right) x^2(t) \quad \text{--- constant} \quad (6)$$

Taking integration on both the sides w.r.t t over the interval $[t_1, t_2]$

$$\begin{aligned} \int_{t_1}^{t_2} x(t) f_e(t) dt &= \int_{t_1}^{t_2} x(t) f(t) dt - \left[\frac{\int_{t_1}^{t_2} x(t) f(t) dt}{\int_{t_1}^{t_2} x^2(t) dt} \right] \int_{t_1}^{t_2} x^2(t) dt \\ &= \int_{t_1}^{t_2} x(t) f(t) dt - \int_{t_1}^{t_2} x(t) f(t) dt \end{aligned}$$

$$\therefore \int_{t_1}^{t_2} x(t) f_e(t) dt = 0. \quad \text{--- (6)}$$

That is, the condition for the minimum of the error signal demands that the signals $f_e(t)$ and $x(t)$ be orthogonal to each other over the interval $t_1 \leq t \leq t_2$.

From eqn (3) and eqn (6), it is observed that signals have properties similar to vectors. The dot product or inner product of ~~any two the signals~~ ~~$f(t)$~~ ~~a~~ vectors gives the area common to the product of the vectors, which is true for signals also.

Note: If any two signals $f(t)$ and $x(t)$ are orthogonal to each other over an interval $t_1 \leq t \leq t_2$, then $c=0$

and there is no component of $f(t)$ on $x(t)$. That is, (7)
the condition for orthogonality is

$$\boxed{\int_{t_1}^{t_2} x(t) f(t) dt = 0} \quad \text{--- (7)}$$

Example: verify the orthogonality of the following functions: $S_1(t) = 1$ and $S_2(t) = c(1-2t)$ in the interval $[0, 1]$.

Solution:

$$\begin{aligned} \int_0^1 S_1(t) S_2(t) dt &= \int_0^1 1 \times c(1-2t) dt \\ &= c \left[t - \frac{2t^2}{2} \right]_0^1 \\ &= c[1-1] - c[0-0] \\ &= 0 \end{aligned}$$

Since $\int_0^1 S_1(t) S_2(t) dt = 0$, $S_1(t)$ and $S_2(t)$ are orthogonal over $[0, 1]$.

Signal Approximation using orthogonal Functions:

consider N real signals $x_1(t), x_2(t), x_3(t), \dots, x_N(t)$ as a set of mutually orthogonal signals in the N -dimensional signal space over an interval $t_1 \leq t \leq t_2$. The condition for orthogonality is

$$\int_{t_1}^{t_2} x_m(t) x_n(t) dt = \begin{cases} 0 & m \neq n \\ E_n & m = n \end{cases}$$

(8)

$$\text{where } E_n = \int_{t_1}^{t_2} x_n^2(t) dt$$

represents the energy of the signal for all n .

Now any function $f(t)$ can be approximated in terms of mutually orthogonal signals $x_1(t), x_2(t), x_3(t), \dots, x_N(t)$, called the basis functions, over the interval $t_1 \leq t \leq t_2$.

$$f(t) \approx c_1 x_1(t) + c_2 x_2(t) + c_3 x_3(t) + \dots + c_N x_N(t) = \sum_{n=1}^N c_n x_n(t),$$

(8)

where $c_1, c_2, c_3, \dots, c_N$ are constants.

If the error signal is $f_e(t)$,

$$\text{then } f(t) = \sum_{n=1}^N c_n x_n(t) + f_e(t)$$

$$\text{or } f_e(t) = f(t) - \sum_{n=1}^N c_n x_n(t).$$

For the best approximation, the error signal should be minimised. That is, minimise:

$$e = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} f_e^2(t) dt = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} \left| f(t) - \sum_{n=1}^N c_n x_n(t) \right|^2 dt, \quad t_1 \leq t \leq t_2$$

where e is called the mean square error or power of the error signals.

We know that the necessary condition to minimise the mean square error with respect to the constant c_j is

$$\frac{\partial e}{\partial c_j} = 0, \quad j = 1, 2, 3, \dots, N.$$

$$\frac{\partial e}{\partial c_j} = \frac{\partial}{\partial c_j} \left[\frac{1}{(t_2 - t_1)} \left(\int_{t_1}^{t_2} f_e^2(t) dt \right) \right] = \frac{\partial}{\partial c_j} \left[\frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} \left| f(t) - \sum_{n=1}^N c_n x_n(t) \right|^2 dt \right] = 0 \quad (9)$$

Due to orthogonality, all the terms other than the j th term (terms without c_j) are zero. Hence,

$$\frac{\partial}{\partial c_j} \left(\int_{t_1}^{t_2} f^2(t) dt \right) = 0, \quad \frac{\partial}{\partial c_j} \left(\int_{t_1}^{t_2} c_n^2 x_n^2(t) dt \right) = 0,$$

$$\frac{\partial}{\partial c_j} \left(\int_{t_1}^{t_2} c_n f(t) x_n(t) dt \right) = 0$$

the remaining two non-zero terms are:

$$\frac{\partial}{\partial c_j} \int_{t_1}^{t_2} [-2c_j f(t) x_j(t) + c_j^2 x_j^2(t)] dt = 0$$

changing the order of differentiation and integration,

$$\int_{t_1}^{t_2} \left[-2 \frac{\partial}{\partial c_j} (c_j f(t) x_j(t)) + \frac{\partial}{\partial c_j} (c_j^2 x_j^2(t)) \right] dt = 0$$

$$\int_{t_1}^{t_2} 2 f(t) x_j(t) dt = \int_{t_1}^{t_2} 2 c_j x_j^2(t) dt$$

$$\text{or } c_j = \frac{\int_{t_1}^{t_2} f(t) x_j(t) dt}{\int_{t_1}^{t_2} x_j^2(t) dt}$$

$$c_j = \frac{\int_{t_1}^{t_2} f(t) x_j(t) dt}{E_j} \quad \text{--- (9)}$$

Thus, for the best approximation of the function given in eqn (8), the constants should be chosen from eqn (9).

Example-2

(10)

prove that $\cos(n\omega_0 t)$ and $\cos(m\omega_0 t)$ are orthogonal over any interval $(t_0, t_0 + \frac{2\pi}{\omega_0})$ for integer values of n and m .

Solution:

Given that $f_1(t) = \cos(n\omega_0 t)$ and $f_2(t) = \cos(m\omega_0 t)$ over the interval $[t_0, t_0 + \frac{2\pi}{\omega_0}]$.

The condition for orthogonality between two signals $f_1(t)$ and $f_2(t)$ in this case is:

$$\int_{t_0}^{t_0 + \frac{2\pi}{\omega_0}} f_1(t) f_2(t) dt = 0$$

$$\int_{t_0}^{t_0 + \frac{2\pi}{\omega_0}} \cos(n\omega_0 t) \cdot \cos(m\omega_0 t) dt = \frac{1}{2} \int_{t_0}^{t_0 + \frac{2\pi}{\omega_0}} [\cos(n+m)\omega_0 t + \cos(n-m)\omega_0 t] dt$$

$$= \frac{1}{2} \left[\frac{\sin(n+m)\omega_0 t}{(n+m)\omega_0} + \frac{\sin(n-m)\omega_0 t}{(n-m)\omega_0} \right]_{t_0}^{t_0 + \frac{2\pi}{\omega_0}}$$

$$= \frac{1}{2} \left[\left(\frac{\sin[(n+m)(\omega_0 t_0 + 2\pi)]}{(n+m)\omega_0} + \frac{\sin[(n-m)(\omega_0 t_0 + 2\pi)]}{(n-m)\omega_0} \right) \right.$$

$$\left. - \left(\frac{\sin(n+m)\omega_0 t_0}{(n+m)\omega_0} + \frac{\sin(n-m)\omega_0 t_0}{(n-m)\omega_0} \right) \right]$$

$$= 0$$

$$[\because \sin(2\pi N + \theta) = \sin \theta]$$

Therefore, $\cos(n\omega_0 t)$ and $\cos(m\omega_0 t)$ are orthogonal over any interval $(t_0, t_0 + \frac{2\pi}{\omega_0})$ for all integer values of n and m .

Mean Square Error :

(11)

The mean square error (MSE) in signal approximation can be evaluated from the orthogonal functions when the optimum values of $c_1, c_2, c_3, \dots, c_N$ are obtained from below expression

$$c_j = \frac{1}{E_j} \int_{t_1}^{t_2} f(t) x_j(t) dt \quad \text{--- (1)}$$

Now the mean square error

$$e = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} \left| f(t) - \sum_{n=1}^N c_n x_n(t) \right|^2 dt$$

$$e = \frac{1}{(t_2 - t_1)} \left[\int_{t_1}^{t_2} f^2(t) dt + \sum_{n=1}^N c_n^2 \int_{t_1}^{t_2} x_n^2(t) dt - 2 \sum_{n=1}^N c_n \int_{t_1}^{t_2} f(t) x_n(t) dt \right]$$

From the eqn (1)

$$c_n E_n = \int_{t_1}^{t_2} f(t) x_n(t) dt$$

Therefore the MSE is

$$e = \frac{1}{(t_2 - t_1)} \left[\int_{t_1}^{t_2} f^2(t) dt + \sum_{n=1}^N c_n^2 E_n - 2 \sum_{n=1}^N c_n c_n E_n \right]$$

$$= \frac{1}{(t_2 - t_1)} \left[\int_{t_1}^{t_2} f^2(t) dt - \sum_{n=1}^N c_n^2 E_n \right]$$

$$[\because E_n = \int_{t_1}^{t_2} x_n^2(t) dt]$$

$$e = \frac{1}{t_2 - t_1} \left[\int_{t_1}^{t_2} f^2(t) dt - (c_1^2 E_1 + c_2^2 E_2 + c_3^2 E_3 + \dots + c_N^2 E_N) \right]$$

closed or complete set of mutually orthogonal functions:

The mean square error for N-dimensional orthogonal space is given by (12)

$$e = \frac{1}{(t_2 - t_1)} \left[\int_{t_1}^{t_2} f^2(t) dt - (C_1^2 E_1 + C_2^2 E_2 + C_3^2 E_3 + \dots + C_N^2 E_N) \right]$$

It is observed that as $N \rightarrow \infty$ the approximation becomes more accurate with large number of orthogonal functions and the error $e \rightarrow 0$ and the orthogonal set becomes a complete set.

When the error becomes zero,

$$\int_{t_1}^{t_2} f^2(t) dt = \sum_{n=1}^{\infty} C_n^2 E_n$$

The energy of $f(t)$ is equal to the sum of the energies of the orthogonal components. This equation is called Parseval's Theorem.

~~And~~

And the signal $f(t)$ can be written as

$$f(t) = \sum_{n=1}^{\infty} C_n x_n(t), \quad t_1 \leq t \leq t_2$$

Hence the infinite series with weighted sum of orthogonal functions converges to the function $f(t)$ such that the mean square error is zero

A set of functions $x_1(t), x_2(t), x_3(t) \dots x_N(t)$ mutually orthogonal over the interval $t_1 \leq t \leq t_2$ is

said to be closed or a complete set if there exists no function $f(t)$ for which it is true that

$$\int_{t_1}^{t_2} f(t) x_n(t) dt = 0, \text{ for } n=1, 2, 3, \dots$$

That is, there is no function $f(t)$ existing other than the set of functions $x_1(t), x_2(t), x_3(t), \dots, x_n(t)$. Obviously, this means that the function $f(t)$ should be equal to any one of the basis function in the complete orthogonal set.

Orthogonality in complex signals:

consider that the set of signals $x_1(t), x_2(t), x_3(t), \dots, x_n(t)$ have complex. then they are mutually orthogonal if.

$$\int_{t_1}^{t_2} x_m(t) x_n^*(t) dt = \int_{t_1}^{t_2} x_m^*(t) x_n(t) dt = \begin{cases} 0 & \text{for } m \neq n \\ E_n & \text{for } m = n \end{cases}$$

Then $f(t)$ can be expressed as,

$$f(t) = \sum_{n=1}^{\infty} c_n x_n(t)$$

where c_n are given by

$$c_n = \frac{1}{E_n} \int_{t_1}^{t_2} f(t) x_n^*(t) dt$$

where E_n is given for complex signals

$$E_n = \int_{t_1}^{t_2} x_n(t) \cdot x_n^*(t) dt$$

(14)

Example-3:

prove that $\sin(n\omega_0 t)$ and $\cos(m\omega_0 t)$ are orthogonal over any interval $(t_0, t_0 + \frac{2\pi}{\omega_0})$ for any values of n and m .

Sol:

Given $f_1(t) = \sin(n\omega_0 t)$ and $f_2(t) = \cos(m\omega_0 t)$, over the interval $(t_0, t_0 + \frac{2\pi}{\omega_0})$ The condition for orthogonality is

$$\int_{t_0}^{t_0 + \frac{2\pi}{\omega_0}} f_1(t) f_2(t) dt = 0$$

$$\int_{t_0}^{t_0 + \frac{2\pi}{\omega_0}} \sin(n\omega_0 t) \cdot \cos(m\omega_0 t) dt = \frac{1}{2} \int_{t_0}^{t_0 + \frac{2\pi}{\omega_0}} [\sin(n+m)\omega_0 t + \sin(n-m)\omega_0 t] dt$$

$$= -\frac{1}{2} \left[\frac{\cos(n+m)\omega_0 t}{(n+m)\omega_0} + \frac{\cos(n-m)\omega_0 t}{(n-m)\omega_0} \right]_{t_0}^{t_0 + \frac{2\pi}{\omega_0}}$$

$$= -\frac{1}{2} \left[\frac{\cos[(n+m)(\omega_0 t + 2\pi)]}{(n+m)\omega_0} + \frac{\cos[(n-m)(\omega_0 t + 2\pi)]}{(n-m)\omega_0} - \left(\frac{\cos(n+m)\omega_0 t_0}{(n+m)\omega_0} + \frac{\cos(n-m)\omega_0 t_0}{(n-m)\omega_0} \right) \right]$$

$$= 0$$

$$[\because \cos(2N\pi + \theta) = \cos \theta]$$

Therefore $\sin(n\omega_0 t)$ and $\cos(m\omega_0 t)$ are orthogonal over any interval $(t_0, t_0 + \frac{2\pi}{\omega_0})$ for all integer values of n & m .

Ex-4 prove that the set of exponentials
 $1, e^{\pm j\omega_0 t}, e^{\pm 2j\omega_0 t}, e^{\pm 3j\omega_0 t}, \dots$ is orthogonal over
 any interval T_0 .

Sol: Here we have to check orthogonality of complex
 functions. It is given by

$$\int_{t_1}^{t_2} x_m(t) x_n^*(t) dt = \begin{cases} 0 & \text{for } m \neq n \\ E_n & \text{for } m = n \end{cases}$$

for $x_m(t) = e^{jm\omega_0 t}$ and

$x_n(t) = e^{jn\omega_0 t}$ we can write

$$\begin{aligned} \int_{t_0}^{t_0+T_0} e^{jm\omega_0 t} \cdot (e^{jn\omega_0 t})^* dt &= \int_{t_0}^{t_0+T_0} e^{jm\omega_0 t} \cdot e^{-jn\omega_0 t} dt \\ &= \int_{t_0}^{t_0+T_0} e^{j(m-n)\omega_0 t} dt = \frac{1}{j(m-n)\omega_0} \left[e^{j(m-n)\omega_0 t} \right]_{t_0}^{t_0+T_0} \\ &= \frac{1}{j(m-n)\omega_0} \left[e^{j(m-n)\omega_0(t_0+T_0)} - e^{j(m-n)\omega_0 t_0} \right] \\ &= \frac{1}{j(m-n)\omega_0} \left[e^{j(m-n)\omega_0 t_0} \cdot e^{j(m-n)\omega_0 T_0} - e^{j(m-n)\omega_0 t_0} \right] \\ &= \frac{1}{j(m-n)\omega_0} e^{j(m-n)\omega_0 t_0} \left[e^{j(m-n)\omega_0 T_0} - 1 \right] \end{aligned}$$

Here $\omega_0 = \frac{2\pi}{T_0}$, Hence $\omega_0 T_0 = 2\pi$ Therefore $e^{j(m-n)\omega_0 T_0} = e^{j(m-n)2\pi} = 1$ always, Hence

above equation becomes,

(16)

$$\int_{t_0}^{t_0+T_0} e^{jm\omega t} [e^{jn\omega t}]^* dt = \frac{1}{j(m-n)\omega} e^{j(m-n)\omega t} [1-1]$$

$$= 0$$

Thus the complex exponentials are orthogonal over any time period T_0 .

Now consider ~~again~~ the case, where $m=n$,

$$\int_{t_0}^{t_0+T_0} [e^{jm\omega t} (e^{jn\omega t})^*] dt = \int_{t_0}^{t_0+T_0} (e^{jm\omega t} e^{-jn\omega t}) dt$$

$$= \int_{t_0}^{t_0+T_0} e^{j(m-n)\omega t} dt$$

For $m=n$, above equation will be

$$\int_{t_0}^{t_0+T_0} e^{j(m-n)\omega t} dt = \int_{t_0}^{t_0+T_0} 1 dt = t \Big|_{t_0}^{t_0+T_0} = t_0+T_0 - t_0 = T_0$$

Hence

$$\int_{t_0}^{t_0+T_0} e^{jm\omega t} e^{-jn\omega t} dt = \begin{cases} 0 & \text{for } m \neq n \\ T_0 & \text{for } m = n \end{cases}$$

Ex-5: A rectangular function is defined as

$$f(t) = \begin{cases} A, & 0 \leq t \leq \pi/2 \\ -A, & \pi/2 \leq t \leq 3\pi/2 \\ A, & 3\pi/2 \leq t \leq 2\pi \end{cases}$$

Approximate the above function by $A \cos(t)$ between the interval $(0, 2\pi)$ such that the mean square error is minimum. (17)

Sol:

Given

$$f(t) = \begin{cases} A, & 0 \leq t \leq \pi/2 \\ -A, & \pi/2 \leq t \leq 3\pi/2 \\ A, & 3\pi/2 \leq t \leq 2\pi \end{cases} \quad \text{and}$$

$$x(t) = A \cos t$$

For the best approximation, with a given function $x(t) = A \cos t$ in the interval $(0, 2\pi)$.

$$f(t) = c x(t) = c A \cos t, \quad 0 \leq t \leq 2\pi \quad \text{--- (1)}$$

$$\text{Where } c = \frac{\int_0^{2\pi} f(t) x(t) dt}{\int_0^{2\pi} x^2(t) dt} \quad \left[\because c = \frac{\int_{t_1}^{t_2} f(t) x(t) dt}{\int_{t_1}^{t_2} x^2(t) dt} \right]$$

$$c = \frac{\int_0^{\pi/2} A \cdot A \cos t dt + \int_{\pi/2}^{3\pi/2} (-A) A \cos t dt + \int_{3\pi/2}^{2\pi} A \cdot A \cos t dt}{\int_0^{2\pi} (A \cos t)^2 dt}$$

$$= \frac{A^2 \sin(t) \Big|_0^{\pi/2} - A^2 \sin t \Big|_{\pi/2}^{3\pi/2} + A^2 \sin(t) \Big|_{3\pi/2}^{2\pi}}{\frac{A^2}{2} \left[t \Big|_0^{2\pi} + \frac{\sin 2t}{2} \Big|_0^{2\pi} \right]}$$

$$= \frac{A^2(1-0) - A^2(-1-1) + A^2(0-(-1))}{\frac{A^2}{2} [2 \times 2\pi + \frac{1}{2}(0-0)]} = \frac{A^2 + 2A^2 + A^2}{A^2 \frac{4\pi}{2}} = \frac{4A^2}{2A^2 \pi} = \frac{2}{\pi}$$

$$= \frac{4A^2}{\pi A^2} = \frac{4}{\pi}$$

(18)

substituting c in eqn (1)

$$\text{eqn (1)} \Rightarrow \boxed{f(t) = \frac{4}{\pi} \cos t}, \quad 0 < t < 2\pi$$

_____ x _____ x _____ x _____

→ If the number of mutually orthogonal signals increases for approximation of a signal, then mean square error will decrease and approximation will be more accurate.

Unit-I , part-2

(1)

Signal: A signal is defined as a single-valued function of one or more independent variables that contains or conveys information about a physical quantity.

Examples: speech, image, voltage, electric current pressure signal • temperature etc.

$x_1(t) = 0.7t$ and $x_2(t) = 1.8t^2$ are examples of ~~one~~ function of one independent variables.

And $x(p, z) = 0.6p + 0.5z + 1.1z^2$ is a signal, which is function of two independent variable.

Amplitude: The value of a signal at any specified value of the independent variable is called its amplitude.

Waveform: The sketch or plot of the amplitude of a signal as a function of independent variable is called ~~its~~ its waveform.

The signals can be classified into two parts

depending upon Independent variable (time).

(2)

A) continuous Time (CT) signals

B) Discrete Time (DT) signals

A) continuous-time signal: A continuous time

signal $f(t)$ is a function of a continuous time variable t . It is also called analog signal and analog signal is continuous in both time and amplitude.

Examples: Most of the signals available in the nature are analog signals, like speech, human voice. etc.

$f(t) = 0.7t$, where t is continuous time indep. variable

B) Discrete Time signals:

A discrete-time signal is one that can be represented by a sequence of numbers.

(or)

which is defined only at specific or regular time instants, which is denoted by $f[n]$. Digital

signal is a special case of discrete time signal.

Digital signal is discrete in both time and amplitude.

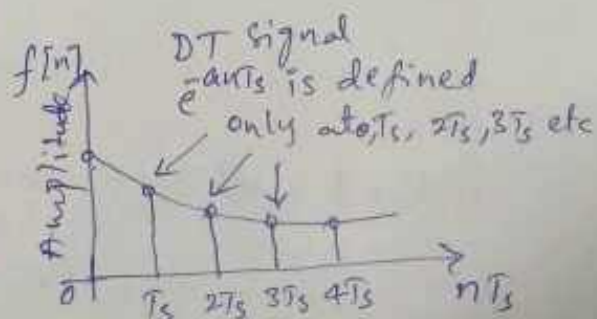
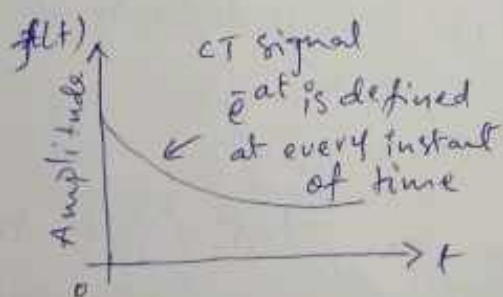


Fig 1. CT & DT signals

→ $f(t) = e^{at}$ is defined at every instant of time. (3)

→ $f[n] = e^{anT_s}$ is defined only at $0, T_s, 2T_s, 3T_s, \dots$
... It is not defined over continuous time.

Special cases of CT and DT:

i) Analog Signal: when amplitude of CT signal varies continuously, it is called analog signal. In other words amplitude and time both are continuous for analog signal.

ii) Digital signals: when amplitude of DT signal takes only finite values, it is called digital signal. In other words amplitude and time both are discrete for digital signal.

The continuous time and discrete time signals are classified into the following parts.

1. Deterministic and Nondeterministic (Random) Signals.
2. periodic and non-periodic signals
3. Symmetric (Even) and Antisymmetric (odd)
4. Energy and power signals.

5. Causal and Noncausal Signals.

(4)

1. Deterministic and Nondeterministic (Random) Signals:

The signal that can be completely specified by a mathematical equation is called a deterministic signal.

ex: sinusoidal, exponential etc.

$$f_1(t) = A \cos(\omega t), f_2(t) = At, f_3(t) = Ae^{-at}$$

The signal whose characteristics are random in nature is called a nondeterministic or random signal.

ex: The noise signals from various sources like electronic amplifiers, oscillators, radio receivers etc.

2. periodic and nonperiodic (aperiodic) signals:

The signal which satisfies the condition $f(t+T_0) = f(t)$ is called a periodic signal. Where T_0 is fundamental period of the signal. A periodic signal will have a definite pattern that repeats again and again over a certain period of time.

A signal which does not satisfy the condition $f(t+T_0) = f(t)$ is called an aperiodic or nonperiodic signal.

Where T_0 is fundamental time period of the signal.

Inverse of T_0 is called fundamental frequency f_0 in cycles per second or Hz, and $2\pi f_0 = \omega_0$ is called the fundamental angular frequency in rad/sec. (5)

The sinusoidal signals and complex exponential signals are always periodic signals with a periodicity of T_0 .

proof:

(a) consider cosine signal.

$$\text{Let } f(t) = A \cos(\omega_0 t)$$

$$\begin{aligned}\therefore f(t+T) &= A \cos[\omega_0(t+T)] \\ &= A \cos[\omega_0 t + \omega_0 T] \\ &= A \cos\left[\omega_0 t + \omega_0 \times \frac{2\pi}{\omega_0}\right] = A \cos[2\pi + \omega_0 t] \\ &= A \cos(\omega_0 t) \\ &= f(t). \quad \left[\because \cos(2\pi + \theta) = \cos \theta\right]\end{aligned}$$

$$\therefore f(t+T) = f(t).$$

Hence periodic signal.

(b) consider sine signal.

$$\text{Let } f(t) = A \sin(\omega_0 t)$$

$$\begin{aligned}\therefore f(t+T) &= A \sin[\omega_0(t+T)] = A \sin[\omega_0 t + \omega_0 T] \\ &= A \sin\left[\omega_0 t + \omega_0 \times \frac{2\pi}{\omega_0}\right] \\ &= A \sin \omega_0 t \quad \left[\because \sin(2\pi + \theta) = \sin \theta\right] \\ &= f(t)\end{aligned}$$

$$\therefore f(t+T) = f(t)$$

Hence periodic signal.

① consider complex exponential signal.

⑥

$$\text{Let } f(t) = A e^{j\omega_0 t}$$

$$\therefore f(t+T_0) = A e^{j\omega_0(t+T_0)} = A e^{j\omega_0 t + j\omega_0 T_0}$$

$$= A e^{j\omega_0 t} \cdot e^{j\omega_0 T_0}$$

$$= A e^{j\omega_0 t} \cdot e^{j2\pi}$$

$$= A e^{j\omega_0 t} (\cos(2\pi) + j\sin(2\pi))$$

$$= A e^{j\omega_0 t} (1 + 0)$$

$$= A e^{j\omega_0 t} = f(t)$$

$$\therefore f(t+T_0) = f(t)$$

Hence periodic signal.

Ex-1. verify whether the following continuous time signals are periodic. If periodic, find the fundamental period.

(a) $f(t) = 2\cos t/4$ b) $f(t) = 3\cos(5t + \pi/6)$

c) $f(t) = e^{-j\frac{2\pi}{7}t}$ d) $f(t) = e^{\alpha t}; \alpha > 1$.

Sol:

(a) Given that $f(t) = 2\cos t/4$

The given function is cosine signal, which is always periodic.

comparing given signal $f(t) = 2\cos t/4$ with standard form $A\cos(\omega_0 t)$, we get

$$\omega_0 = \frac{1}{4}$$

$$= 2\pi f_0 = \frac{1}{4}$$

$$\frac{2\pi}{T_0} = \frac{1}{4}$$

$$\text{period, } T_0 = 8\pi$$

Now let us verify with obtained period, whether given signal is periodic or not.

$$f(t) = 2\cos(t/4)$$

condition for periodicity is $f(t+T_0) = f(t)$.

$$\begin{aligned} f(t+T_0) &= f(t+8\pi) = 2\cos\left(\frac{t+8\pi}{4}\right) \\ &= 2\cos\left(\frac{t}{4} + 2\pi\right) \\ &= 2\cos t/4 \end{aligned}$$

$$\therefore f(t+T_0) = f(t) = f(t).$$

Hence given signal is periodic and its time period is 8π .

b) Given that, $f(t) = 3\cos(5t + \pi/6)$

Given signal is cosine signal, which is always periodic.

$$f(t+T_0) = 3\cos(5(t+T_0) + \pi/6) = 3\cos(5t + 5T_0 + \pi/6)$$

$$= 3\cos[(5t + \pi/6) + 5T_0]$$

$$\text{Let } 5T_0 = 2\pi \quad [\because \text{given function is periodic}]$$

$$T_0 = 2\pi/5$$

$$\therefore f(t) f(t+T_0) = f(t+2\pi/5) = 3\cos[(5t + \pi/6) + 5 \times 2\pi/5]$$

(8)

$$= 3 \cos[(5t + \pi/6) + 2\pi] = 3 \cos(5t + \pi/6) = f(t)$$

$\therefore f(t+T_0) = f(t)$, condition is satisfied.

Hence given signal is periodic and its period is $T_0 = 2\pi/5$.

(c) Given that, $f(t) = e^{-j\frac{2\pi t}{7}}$

The given signal is a complex exponential signal. which is always periodic, comparing $f(t) = e^{-j\frac{2\pi t}{7}}$ with standard form $e^{-j\omega_0 t}$, we get,

$$\omega_0 = \frac{2\pi}{7}$$

$$2\pi f_0 = 2\pi/7$$

$$\frac{1}{T_0} = \frac{1}{7} \Rightarrow T_0 = 7$$

$\therefore$ period, $T_0 = 7$

$\therefore f(t)$ is periodic signal with period, $T = 7$.

(d) Given that, $f(t) = e^{\alpha t}$; $\alpha > 1$

condition for periodicity is

$$f(t+T_0) = f(t)$$

$$\therefore f(t+T) = e^{\alpha(t+T_0)} = e^{\alpha t} \cdot e^{\alpha T_0}$$

For any value of α , $e^{\alpha T_0} \neq 1$. and so $f(t+T) \neq f(t)$

Since $f(t+T) \neq f(t)$, the signal $f(t)$ is non-periodic.

Note: Individual signals like $e^{j\omega t}$, $\cos(\omega t)$, $\sin(\omega t)$ are periodic.
Step to follow for finding time period. (9)

1. Identify the individual time periods, T_1, T_2, T_3 etc
2. Calculate $\frac{T_1}{T_2}, \frac{T_1}{T_3}, \frac{T_1}{T_4} \dots$ (you may take ~~any~~ T_2 or T_3 etc)
3. If ratios of 2nd step are rational numbers then the overall signal is periodic.
4. Calculate LCM of denominators of $\frac{T_1}{T_2}, \frac{T_1}{T_3}, \frac{T_1}{T_4}$ etc
[not T_2, T_3, T_4 etc but after simplification of $\frac{T_1}{T_2}, \frac{T_1}{T_3}, \frac{T_1}{T_4}$ etc then consider denominators]
5. Time period $T_0 = (\text{LCM}) \times T_1$

Example - 2.

Determine the periodicity of the following continuous time signals.

(a) $f(t) = 2\cos\frac{2\pi t}{3} + 3\cos(\frac{2\pi t}{7})$, (b) $f(t) = 2\cos 8t + 3\sin 7t$.

Sol: Given signal $f(t) = 2\cos(\frac{2\pi t}{3}) + 3\cos(\frac{2\pi t}{7})$

compare given signal with standard expression

i.e., $A\cos(\omega t)$

$$\omega_1 = \frac{2\pi}{3}$$

$$\omega_2 = \frac{2\pi}{7}$$

~~Ans~~

$$2\pi f_1 = \frac{2\pi}{3}$$

$$\frac{1}{T_1} = \frac{1}{3}$$

$$T_1 = 3,$$

$$2\pi f_2 = \frac{2\pi}{7}$$

$$\frac{1}{T_2} = \frac{1}{7}$$

$$T_2 = 7.$$

(10)

Take $\frac{T_1}{T_2} = \frac{3}{7}$, and $\frac{3}{7}$ is a rational number

So, given signal is periodic.

period of the signal (T_0) is given by

$$T_0 = (\text{LCM of the ratio } \frac{T_1}{T_2} \text{ denominator of the ratio } \frac{T_1}{T_2}) \times T_1$$

$$= 7 \times 3 = 21$$

$$\therefore \text{Fundamental Timeperiod } (T_0) = 21.$$

(b) Given signal is $f(t) = 2\cos(3t) + 3\sin(7t)$

Sol: Given signal is $f(t) = 2\cos(3t) + 3\sin(7t)$, compare given signal with the standard signal i.e.,

$$A_1 \cos \omega_1 t + A_2 \sin \omega_2 t.$$

where $\omega_1 = 3,$

$$2\pi f_1 = 3$$

$$f_1 = \frac{3}{2\pi}$$

$$T_1 = \frac{2\pi}{3}$$

$$\omega_2 = 7$$

$$2\pi f_2 = 7$$

$$f_2 = \frac{7}{2\pi}$$

$$T_2 = \frac{2\pi}{7}$$

Take $\frac{T_2}{T_1}$ or $\frac{T_1}{T_2}$

consider $\frac{T_1}{T_2} = \frac{\frac{2\pi}{3}}{\frac{2\pi}{7}} = \frac{2\pi}{3} \times \frac{7}{2\pi} = \frac{7}{3}$, and $\frac{7}{3}$ is a

rational number. So given signal is periodic.

The Fundamental Time period of the signal (T_0) is given by

$$T_0 = (\text{LCM of the denominator of the ration } \frac{T_1}{T_2}) \times T_1$$
$$= 3 \times \frac{2\pi}{3} = 2\pi$$

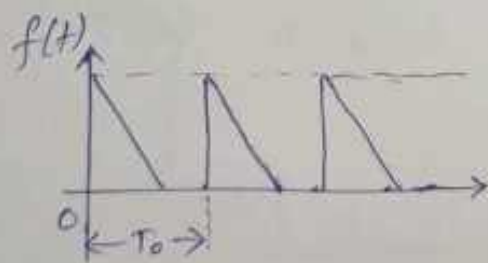
$$\therefore \text{Fundamental Time period } (T_0) = 2\pi$$

For discrete-time signals, $f[n]$ is said to be periodic signal if it satisfies the condition $f[n] = f[n \pm N]$ or $f[n \pm N] = f[n]$, where N is a positive integer

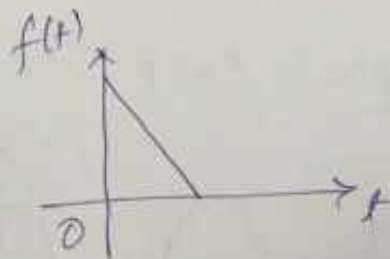
This condition is true if $N = \frac{2\pi k}{\omega_0}$ is integer for any integer k or $\frac{N}{k} = \frac{2\pi}{\omega_0} = \text{rational number}$.

The smallest integer N is called fundamental period.

$\omega_0 = \frac{2\pi}{N}$ rad/sec. is the fundamental angular frequency.



(a) continuous time periodic signal.

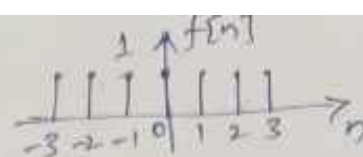


(b) Non-periodic signal.

Fig. 1(a)



(a) Discrete time periodic



(b) Non-periodic signal

Fig 2 Signal.

Note: 1. periodic signals are infinite-duration signals that repeat the same shape continuously at every finite period. Finite duration signals or time-limited signals can never be periodic.

2. The sum of continuous time periodic signals may not be periodic, whereas sum of discrete time periodic signals is always periodic.

3) Symmetric (Even) and Antisymmetric (odd) Signals:

Even signal: When a signal exhibits symmetry with respect to $t=0$ then it is called an even signal.

Therefore, the even signal satisfies the condition,

$$\boxed{f(-t) = f(t)}$$

example: $\cos t$ is even signal because $\cos(-t) = \cos t$.

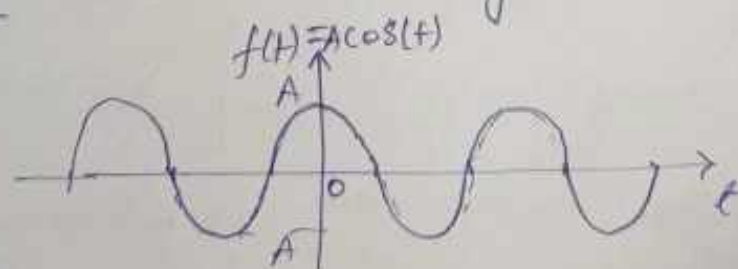


Fig 1. Symmetric or even signal.

odd signal: when a signal exhibits antisymmetry with respect to $t=0$, then it is called an odd signal. ~~there~~ Therefore, the odd signal satisfies the condition,

$$\boxed{f(-t) = -f(t)}$$

Example: $\sin t$ is an odd signal, because $\sin(-t) = -\sin t$

Sine signals are odd signals.

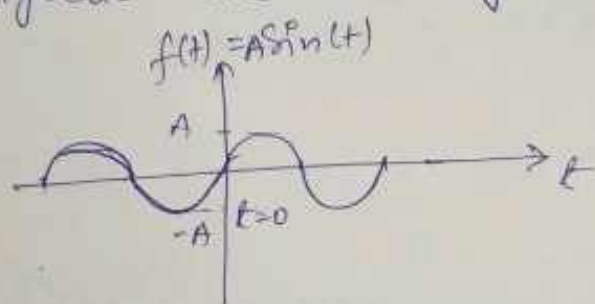


Fig 2. Antisymmetric or odd signals.

→ Any signal $f(t)$ can be expressed as a sum of even and odd signal. That is $\boxed{f(t) = f_e(t) + f_o(t)}$

where $f_e(t)$ is even part of the signal $f(t)$ and

$f_o(t)$ is odd part of the signal $f(t)$.

Also

$$\boxed{\begin{aligned} f_e(t) &= \frac{1}{2}[f(t) + f(-t)] \\ f_o(t) &= \frac{1}{2}[f(t) - f(-t)] \end{aligned}}$$

for discrete time signals

$$\boxed{f[n] = f_e[n] + f_o[n]}$$

$$\boxed{\begin{aligned} f_e[n] &= \frac{1}{2}[f[n] + f[-n]] \\ f_o[n] &= \frac{1}{2}[f[n] - f[-n]] \end{aligned}}$$

Note: 1. $f_e(t) + f_e(t) \rightarrow$ even signal

$f_o(t) + f_o(t) \rightarrow$ odd signal

$f_e(t) + f_o(t) \rightarrow$ has no symmetry

2. $f_e(t) \times f_e(t) \rightarrow$ even symmetry

$f_o(t) \times f_o(t) \rightarrow$ even symmetry

$f_e(t) \times f_o(t) \rightarrow$ odd symmetry.

Example-3

Determine the even and odd parts of the following continuous time signals.

(a) $f(t) = e^t$

Sol: $f(t) = e^t$
 $f(-t) = e^{-t}$

Even part $f_e(t) = \frac{1}{2}[f(t) + f(-t)] = \frac{1}{2}[e^t + e^{-t}]$

odd part $f_o(t) = \frac{1}{2}[f(t) - f(-t)] = \frac{1}{2}[e^t - e^{-t}]$

(b) $f(t) = (1 + \sin t)^2$

Sol: $f(t) = (1 + \sin t)^2 = 1 + \sin^2 t + 2\sin t$

$f(t) = 1 + \sin^2 t - 2\sin t$

Even part $f_e(t) = \frac{1}{2}[f(t) + f(-t)] = \frac{1}{2}[1 + \sin^2 t + 2\sin t + 1 + \sin^2 t - 2\sin t]$

$f_e(t) = 1 + \sin^2 t$

Odd part $f_o(t) = \frac{1}{2}[f(t) - f(-t)] = \frac{1}{2}[1 + \sin^2 t + 2\sin t - 1 + \sin^2 t - 2\sin t]$

$f_o(t) = 2\sin t$

(or)

$f_o(t) = f(t) - f_e(t) = (1 + \sin t)^2 - 1 - \sin^2 t$

$= 1 + \sin^2 t + 2\sin t - 1 - \sin^2 t$

$f_o(t) = 2\sin t$

4) Energy and power signal:

(15)

Energy signal: A signal is said to be an energy signal if and only if it has finite energy and zero ~~power~~ average power ($0 < E < \infty$, $P=0$).

All time-limited signals of finite amplitude are energy signals. Deterministic and aperiodic signals are examples of energy signal.

Total energy is given by (for continuous time signals)

$$E = \lim_{T \rightarrow \infty} \int_{-T}^T |f(t)|^2 dt$$

(or)

$$E = \int_{-\infty}^{\infty} |f(t)|^2 dt$$

Similarly, for discrete-time signal $f[n]$, the total energy or normalized energy content E of $f[n]$ is

$$E = \sum_{n=-\infty}^{\infty} |f[n]|^2$$

Power signal: A signal is said to be power signal if and only if it has finite average

power and infinite energy ($0 < p < \infty, E \rightarrow \infty$).

most of the periodic signals are power signals.

Average power of continuous signal is given

by

$$P = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(t)|^2 dt$$

or

$$P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{+T/2} |f(t)|^2 dt \quad \text{or} \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |f(t)|^2 dt$$

Average power of Discrete time signal is

given by

$$P = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |f(n)|^2$$

Note: If the given signal is periodic, then ~~periodic~~
(T and N) have finite value. Hence there is no need
to ~~put~~ take limits $\left[\lim_{T \rightarrow \infty} \right]$

→ Steps to be followed while calculating ~~either~~
energy or power of the signal

1. observe the signal carefully. If it is ~~infinite~~
periodic and infinite duration, then it can be

power signal. Hence calculate its power directly.

2. If the signal is periodic but of finite duration, then it can be energy signal. Hence calculate its energy directly.

3. If the signal is not periodic, then it can be energy signal. Hence calculate its energy.

Ex-1: Find the average power and energy

for the signal $f(t) = 3 \sin(2\pi t)$, $-\infty < t < \infty$

Sol: Given $f(t) = 3 \sin(2\pi t)$, $-\infty < t < \infty$

Since $f(t)$ is periodic with period $T = 1 \text{ sec}$
the average power of the signal is

$$P = \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 dt$$

$$P = \frac{1}{1} \int_{-1/2}^{1/2} |3 \sin(2\pi t)|^2 dt = 9 \int_{-1/2}^{1/2} \frac{1 - \cos(4\pi t)}{2} dt$$

$$= 9 \int_{-1/2}^{1/2} \frac{1}{2} dt - \frac{9}{2} \int_{-1/2}^{1/2} \cos 4\pi t dt = \frac{9}{2} (t)_{-1/2}^{1/2} - \frac{9}{2} \frac{\sin(4\pi t)}{4\pi} \Big|_{-1/2}^{1/2}$$

$$= \frac{9}{2} \left(\frac{1}{2} + \frac{1}{2} \right) - \frac{9}{8\pi} \left[\sin\left(\frac{2\pi \times 1}{2}\right) + \sin\left(\frac{2\pi \times 1}{2}\right) \right]$$

$$= \frac{9}{2} - \frac{9}{8\pi} [\sin(2\pi) + \sin(2\pi)] = \frac{9}{2} - \frac{9}{8\pi} (0 + 0) = \frac{9}{2} \text{ watts}$$

The total energy of the signal is

(18)

$$E = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |3 \sin(200t)|^2 dt$$

$$= 9 \int_{-\infty}^{\infty} \frac{1}{2} [1 - \cos(400t)] dt = \frac{9}{2} \left[\int_{-\infty}^{\infty} 1 dt - \int_{-\infty}^{\infty} \cos(400t) dt \right]$$

$$= \infty$$

So the energy of the signal is infinite and its average power is finite. Therefore signal is a ~~finite~~ power signal.

Example-2: prove the following

- i) The power of the energy signal is zero over infinite time.
- ii) The energy of the power signal is infinite over infinite time.

Sol: i) power of the energy signal

Let $f(t)$ be an energy signal and the power of the signal is given by

$$P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{T} \left[\lim_{T \rightarrow \infty} \int_{-T/2}^{T/2} |f(t)|^2 dt \right]$$

$$= \lim_{T \rightarrow \infty} \frac{1}{T} \left[\int_{-\infty}^{\infty} |f(t)|^2 dt \right]$$

(19)

Here limits are changed to $-\infty, \infty$
as $T \rightarrow \infty$

$$= \lim_{T \rightarrow \infty} \frac{1}{T} E \quad \left[\because \text{Since the quantity inside bracket is } E \right]$$

$$= 0 \times E$$

$$= 0$$

Thus, power of energy signal is zero over infinite time.

ii) Energy of the power signal

Solution: Let $f(t)$ be a power signal and energy of the signal $f(t)$ is given by

$$E = \lim_{T \rightarrow \infty} \int_{-T/2}^{T/2} |f(t)|^2 dt = \lim_{T \rightarrow \infty} \left[T \cdot \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 dt \right]$$

$$E = \lim_{T \rightarrow \infty} T \left[\frac{1}{T} \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 dt \right]$$

$$= \lim_{T \rightarrow \infty} T \cdot P \quad \left[\because \text{quantity inside brackets is power } P. \right]$$

$$= \infty \cdot P = \infty$$

$$E = \infty.$$

Thus energy of the power signal is infinite over infinite time.

5) Causal and Non-causal signals:

Causal signal: A signal is said to be causal, if it is defined for $t \geq 0$. That is, if $f(t)$ is a causal signal, then $f(t) = 0$, for $t < 0$.

Non-causal signal: A signal is said to be non-causal, if it is defined for ~~either~~ both $t \leq 0$ and $t > 0$. That is, if $f(t)$ is a noncausal signal then $f(t) \neq 0$ for all values of t .

Anticausal signal: A signal is said to be anticausal if it is defined for $t \leq 0$. That is if $f(t)$ is an anticausal signal, then $f(t) = 0$, for $t > 0$.

Step signal $f(t) = A; t \geq 0$
 unit step signal $f(t) = 1; t \geq 0$
 Exponential signal, $f(t) = A e^{bt} u(t)$ } causal signals.

Exponential signal, $f(t) = A e^{bt}$, for all t . $\rightarrow$ noncausal

complex exponential signal, $f(t) = A e^{jbt}$, for all t . $\rightarrow$ noncausal

Exponential Signal:

Exponential signals are extremely important in the study of signals and in the analysis of systems. They are used as a driving function for most of the applications of analysing linear systems. Some of their unique properties are:

- i) Exponential signals are invariant under the action of linear time-invariant systems. That is the response of an LTI system to an exponential signal is also an exponential signal.
- ii) Any arbitrary signal can be expressed as a sum of various exponential functions.
- iii) Transform tools like Fourier, Laplace and Z-transform use complex exponential signals for analysis.

A continuous time complex exponential signal in its most general form, is written as

$$f(t) = c e^{at} \quad , \quad -\infty < t < \infty$$

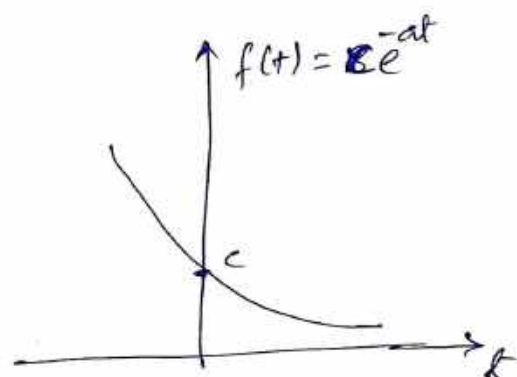
Where c and a are complex parameters. Depend upon the parameters, this complex exponential signal can be divided into two categories: a) under

i) Real exponential signal: A real exponential signal is defined as

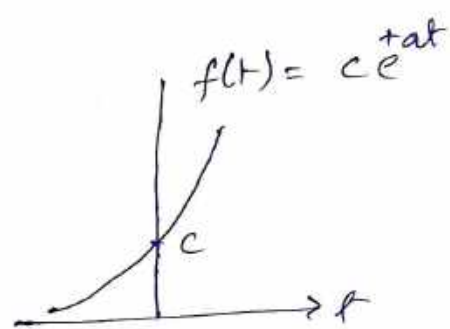
$$f(t) = ce^{at}$$

Where c and a are real parameters. And c is amplitude of the exponential signal. It can be divided into two types depending upon a :

- Decaying exponential, $a < 0$
- Growing exponential, $a > 0$



(a) Decaying for $a < 0$



(b) Growing for $a > 0$.

Fig. continuous time real exponential $f(t) = ce^{at}$.

ii) Imaginary Exponential signal: An imaginary exponential signal is defined as

$$f(t) = e^{j\omega_0 t}, \quad -\infty < t < \infty$$

where $\omega_0 = \frac{2\pi}{T}$ is the fundamental frequency

From Euler's relation it can be expressed as

$$f(t) = e^{j\omega_0 t} = \cos \omega_0 t + j \sin \omega_0 t, \quad \text{for } t \geq 0$$

$$f(t) = e^{-j\omega_0 t} = \cos \omega_0 t - j \sin \omega_0 t, \quad \text{for } t < 0$$

Sinusoidal signal:

A continuous^{time} sinusoidal signal, in general, can be expressed as

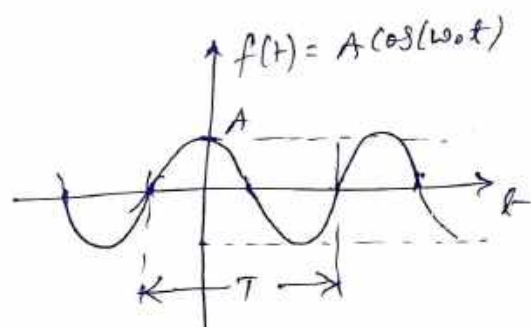
$$f(t) = A \cos(\omega_0 t + \phi).$$

Where A is the amplitude,

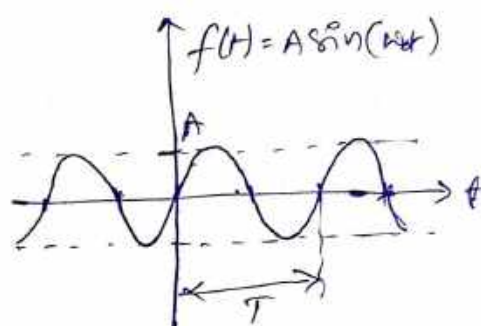
ω_0 is the fundamental frequency $\omega_0 = \frac{2\pi}{T}$,

T is fundamental time period $[\omega_0 = 2\pi f_0 = \frac{2\pi}{T}]$

ϕ is phase shift.



a) cosine function,
 $\phi = 0$



b) sine function,
 $\phi = 0$

Fig. 2: Graphical representation of ~~cosine~~ and sine signals in continuous time.

→ An important property of the sinusoidal signal is that it is periodic with single fundamental frequency f_0 and time period T . Since $f(t)$ is periodic with period T ,

$$f(t) = f(t+T) \text{ for all } t$$

$$\text{i.e., } f(t+T) = A \cos(\omega_0(t+T) + \phi)$$

⊛ If $\omega_0 \neq 0$, then the fundamental period $T = \frac{2\pi}{\omega_0}$ Sec,
or $\omega_0 T = 2\pi$. Then

$$f(t+T) = A \cos(\omega_0(t+T) + \phi) = A \cos(\omega_0 t + 2\pi + \phi)$$

$$= A \cos(\omega_0 t + \phi)$$

$$= f(t)$$

$$\therefore f(t+T) = f(t).$$

If $\omega_0 = 0$, then $f(t) = A \cos \phi$, which is constant for any value of T .

Sinusoidal signals in terms of complex Exponential:

The imaginary exponential signal $f(t) = e^{j\omega_0 t}$ can be written in terms of sinusoidal signals with the same fundamental period by using Euler's relation.

$$e^{j\omega_0 t} = \cos \omega_0 t + j \sin \omega_0 t$$

Similarly, a sinusoidal signals can be written in terms of periodic complex exponentials, with the same fundamental period.

$$\cos(\omega_0 t) = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t}) \quad \text{and}$$

$$\sin(\omega_0 t) = \frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t})$$

with phase angle

$$\left. \begin{aligned} \cos(\omega_0 t + \phi) &= \frac{1}{2}(e^{j\omega_0 t} e^{j\phi} + e^{-j\omega_0 t} e^{-j\phi}) \\ \sin(\omega_0 t + \phi) &= \frac{1}{2j}(e^{j\omega_0 t} e^{j\phi} - e^{-j\omega_0 t} e^{-j\phi}) \end{aligned} \right\} \text{--- ①}$$

Sinusoidal signals in terms of complex exponential

$$\cos(\omega_0 t + \phi) = \operatorname{Re} \{ e^{j(\omega_0 t + \phi)} \}$$

$$\sin(\omega_0 t + \phi) = \text{Im} \{ e^{j(\omega_0 t + \phi)} \}$$

(5)

with $\phi = 0$.

$$\cos(\omega_0 t) = \text{Re} \{ e^{j\omega_0 t} \} \text{ and}$$

$$\sin(\omega_0 t) = \text{Im} \{ e^{j\omega_0 t} \}$$

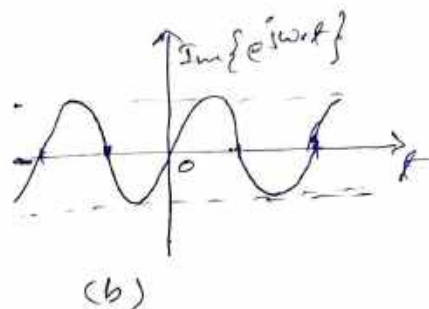
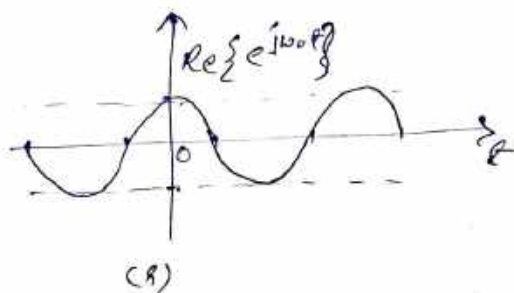


Fig 3. (a) Real part & (b) Imaginary part of the complex exponential

Damped Sinusoidal signal: Sinusoidal signals multiplied

by either growing or decaying exponentials are commonly referred to as damped sinusoids

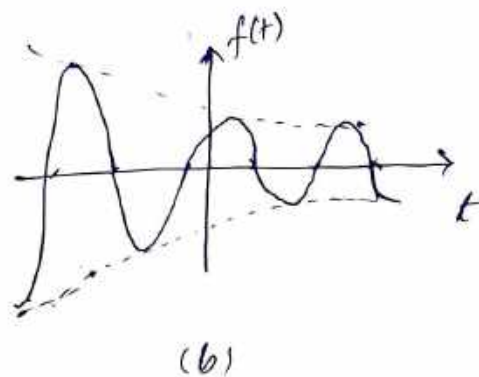
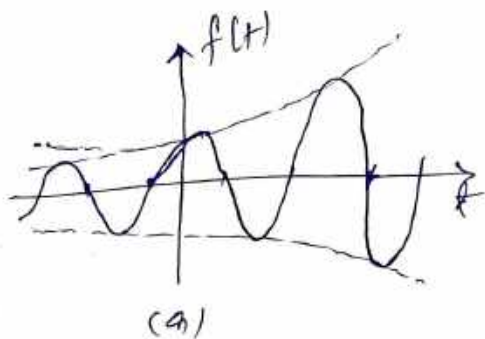


Fig 4. (a) A growing sinusoidal signal $f(t) = Ce^{at} \cos(\omega_0 t + \phi)$, $a > 0$
 (b) A decaying sinusoidal $f(t) = Ce^{at} \cos(\omega_0 t + \phi)$, $a < 0$

⑥ Ex: determine whether or not each of the following signals is periodic. If periodic, determine the fundamental period.

(a) ~~f(t)~~ $f(t) = e^{j2t}$

Steps to follow for finding time period.

1. Identify the individual time periods, T_1, T_2, T_3 etc.
2. calculate $\frac{T_1}{T_2}, \frac{T_1}{T_3}, \frac{T_1}{T_4} \dots$
3. If ratios of 2nd step are rational number then the overall signal is periodic
4. calculate LCM of denominators of ratios of 2nd step [not T_2, T_3, T_4 , but ratios of $\frac{T_1}{T_2}, \frac{T_1}{T_3}, \frac{T_1}{T_4}$]
5. Time period $T = (\text{LCM}) \times T_1$

Note: Individual signals like $e^{j\omega t}, \cos(\omega t), \sin(\omega t)$ are periodic signals.

Sol: Given $f(t) = e^{j2t} = \cos 2t - j\sin 2t$

$$\omega_1 = 2$$

$$\omega_2 = 2$$

$$2\pi f_1 = 2$$

$$2\pi f_2 = 2$$

$$f_1 = \frac{1}{\pi}$$

$$f_2 = \frac{1}{\pi}$$

$$T_1 = \pi$$

$$T_2 = \pi$$

$$\frac{T_1}{T_2} = \frac{\pi}{\pi} = 1 \text{ rational number, so given signal is periodic}$$

$$\text{period of given signal} = \text{LCM of } T_1 \times T_2$$

$$= 1 \times \pi = \pi \text{ sec.}$$

$$f(t) = \cos 50\pi t + \sin 30\pi t$$

x=2 $f(t) = \cos 50\pi t + \sin 60\pi t$

Sol: compare above signal with standard expression

$$f(t) = A_1 \cos \omega_1 t + A_2 \sin \omega_2 t$$

⑦

concept of Impulse function (Dirac delta):

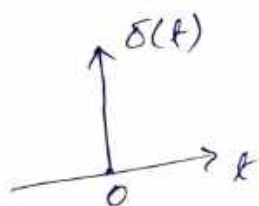
The continuous time unit impulse function is defined as

$$\delta(t) = \begin{cases} \infty, & t=0 \\ 0, & \text{otherwise.} \end{cases}$$

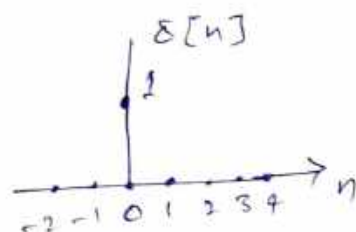
For discrete signals unit impulse function is defined as

$$\delta[n] = \begin{cases} 1, & n=0 \\ 0, & \text{otherwise } (n \neq 0) \end{cases}$$

→ Discrete time impulse function is also called as Kronecker delta function.



(a) continuous time unit impulse



(b) Discrete time unit impulse

Fig.5. unit impulse function.

From graphical representation the impulse function is equal to zero everywhere except at $t=0$.

Since at the time instant $t=0$ it is not possible to define the function, the impulse function may be defined for an infinitesimal time period Δ as shown in fig.6.

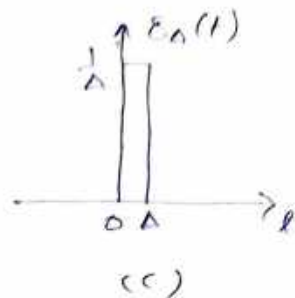
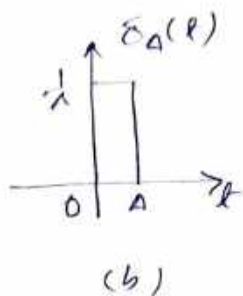
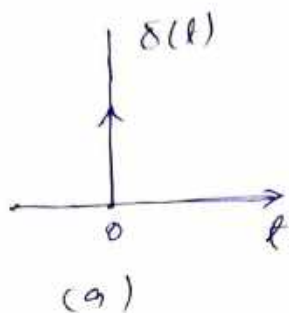


Fig 6(a) unit impulse function and
(b) and (c) unit impulse function for the period Δ

At the limit, the continuous time unit impulse function can be defined as

$$\delta(t) = \lim_{\Delta \rightarrow 0} \delta_A(t) = \begin{cases} \infty, & t=0 \\ 0, & \text{otherwise} \end{cases}$$

properties of the unit impulse function:

1. The area of an impulse function is always unity.

$$\int_{-\infty}^{\infty} \delta(t) dt = \int_{0^-}^{0^+} \delta(t) dt = 1 \text{ (area or strength)}$$

2. Impulse is an even function of time [even symmetry]

$$\delta(-t) = \delta(t)$$

3. Scaling property

$$\delta(at) = \frac{1}{|a|} \delta(t)$$

4. product property (or) sampling property.

$$f(t) \cdot \delta(t-t_0) = f(t_0) \cdot \delta(t-t_0)$$

5. convolution property.

$$f(t) * \delta(t-t_0) = f(t-t_0)$$

6. Sampling (or) Shifting: which states that the multiplication of a continuous function $f(t)$ with an impulse function $\delta(t-t_0)$ yields the sampled value of $f(t)$ at $t=t_0$.

$$(9) \int_{t_1}^{t_2} f(t) \delta(t-t_0) dt = \begin{cases} f(t_0) & ; t_1 \leq t_0 \leq t_2 \\ 0 & ; \text{otherwise} \end{cases}$$

and

Note: $\int_{-\infty}^{\infty} f(t) \delta(t) dt = f(0)$

Ex: $\int_{-1}^2 (t+t^2) \delta(t-5) dt$

= 0, since 5 is not within the given limits.

unit step function:

For continuous time.

The unit step function $u(t)$ is defined as

$$u(t) = \begin{cases} 1 & ; t > 0 \\ 0 & ; t < 0 \end{cases}$$

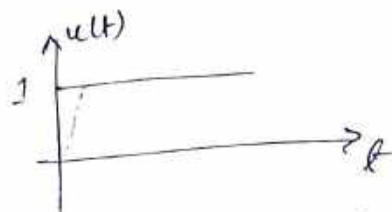


Fig. 7. unit step function [continuous time]

Since the function is ~~defined~~ discontinuous at $t=0$, the value at $t=0$ is undefined

Delayed unit step function is defined as

$$u(t-t_0) = \begin{cases} 1 & ; t > t_0 \\ 0 & ; t < t_0 \end{cases}$$

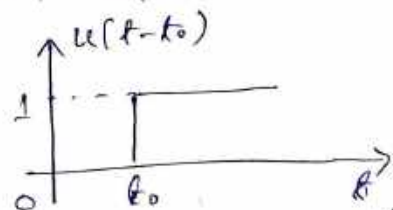


Fig. 8. Delayed unit step

Note: $\frac{d u(t)}{dt} = \delta(t)$ | $\frac{d^2 u(t)}{dt^2} = \frac{d \delta(t)}{dt} = \delta'(t)$ (which is called doublet.
 $\int_{-\infty}^t \delta(\tau) d\tau = u(t)$ $\rightarrow$ area under doublet becomes zero [it is odd function]

Signum function:

(10)

$$\text{sgn}(t) = \begin{cases} -1; & t < 0 \\ +1; & t > 0 \end{cases}$$

$$\text{sgn}(t) = 2u(t) - 1$$

$$\begin{array}{l} \delta(t) \xrightarrow{\text{diff}} u(t) \xrightarrow{\text{diff}} \delta(t) \\ \delta(t) \xrightarrow{\text{integral}} u(t) \xrightarrow{\text{integral}} \delta(t) \end{array}$$

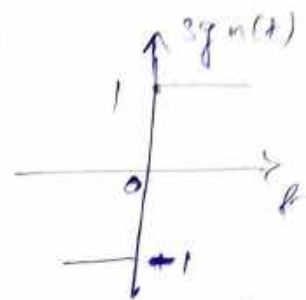


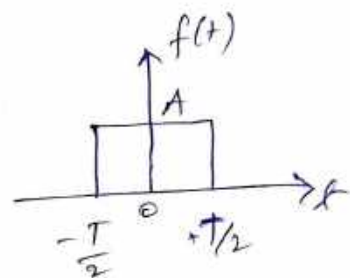
Fig. Signum function.

Additional signals:

1) Rectangular or Gate function.

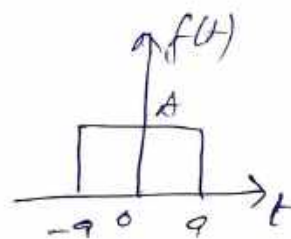
$$f(t) = A \text{rect}(t/T) \text{ (or) } A\pi(t/T)$$

$$f(t) = A \text{rect}(t/T) = \begin{cases} A; & -T/2 < t < T/2 \\ 0; & \text{otherwise} \end{cases}$$



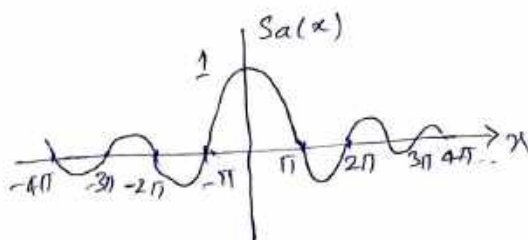
-or-

$$f(t) = A \text{rect}\left(\frac{t}{2a}\right) = \begin{cases} A; & -a < t < a \\ 0; & \text{otherwise} \end{cases}$$

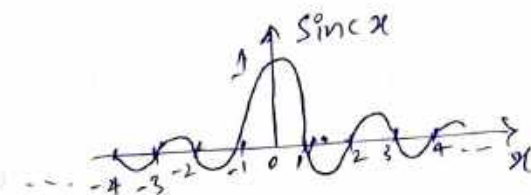


2) Sampling function:

$$\text{Sa}(x) = \frac{\sin x}{x}$$



$$\text{3) } \text{sinc } x = \frac{\sin \pi x}{\pi x}$$

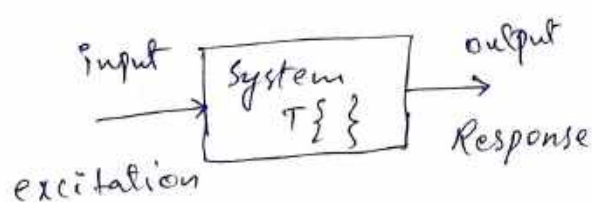


(11)

System: A system is defined as a group of objects that can carry one or more signals and combined ~~them~~ in a manner intended to achieve some desired response.

or

It is an operator which maps relation between input and output.



where the symbol $T\{\}$ is transformation.

There are two types of systems based on time.

1. continuous time system
2. Discrete time system.

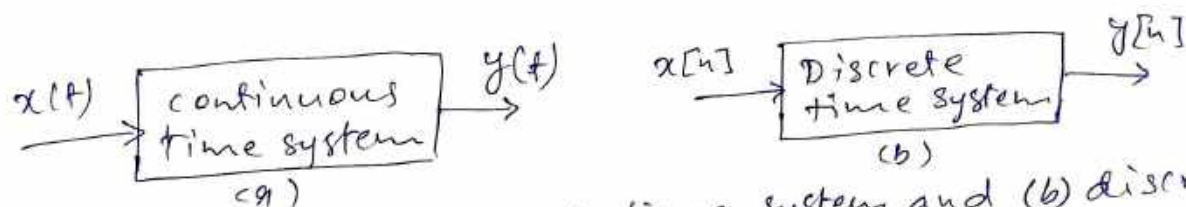


Fig. (a) continuous time system and (b) discrete time system.

classification of systems: [Based on their properties]

1. Linear & non linear
2. Time-Invariant and variant
3. Causal & Non-causal
4. Static & Dynamic
5. stable & unstable
6. Invertible & non invertible

1. Linear and non-linear:

(12) A system is said to be linear if it satisfies the scaling property (homogeneity) and principle of ~~superposition~~ ~~additivity~~ additivity. If a system violates any one of these properties, it is called a nonlinear system.

or

If a system, which follows principle of superposition is called linear system.

Additivity:

Let $f_1(t)$ & $f_2(t)$ be any input signals to a system. Then output be $g_1(t)$ & $g_2(t)$ respectively.

$$g_1(t) = T\{f_1(t)\}$$

$$g_2(t) = T\{f_2(t)\}$$

$$g(t) = g_1(t) + g_2(t) = T\{f_1(t) + f_2(t)\}$$

$$\downarrow$$

Sum of two outputs

$$T\{f_1(t) + f_2(t)\} = T\{f_1(t)\} + T\{f_2(t)\}$$

Homogeneity (scaling property):

$$a_1 g_1(t) = T\{a_1 f_1(t)\}$$

$$a_2 g_2(t) = T\{a_2 f_2(t)\}$$

$$a_1 g_1(t) + a_2 g_2(t) = T\{a_1 f_1(t) + a_2 f_2(t)\}$$

2. Time-invariant and Time variant:

A system is said to be time invariant, if the in

and output characteristics do not change with time. That is, the shape of the response depends only on the shape of the input, not on the time when it is applied. Otherwise it is time varying system.

$$f(t) \rightarrow g(t) \quad \text{or} \quad g(t) = T\{f(t)\}$$

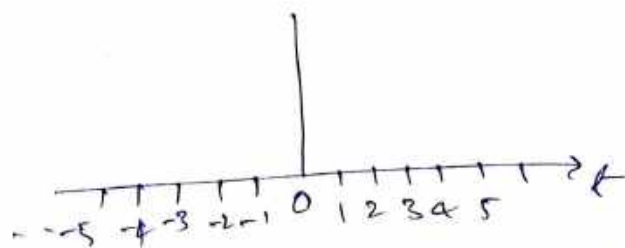
$$f(t-t_0) \rightarrow g(t-t_0) \quad \text{or} \quad g(t-t_0) = T\{f(t-t_0)\}$$

3. Causal & non-causal:

A system is said to be causal system if the present output depends on present input & past values of the input ^{past outputs} but not on future values. It is also called as non-anticipating system.

If present output of a system depends on present input and future inputs or past inputs & future inputs is called non-causal.

If present output of a system depends only on future inputs is called anti-causal system.



If 0 is considered to be present, then 1, 2, 3, ... and -1, -2, -3, ... are future and past values respectively.

Ex: $g(t) = f(t) \times (2t+5)$

(1x)

$$g(0) = f(0) \times 5$$

↑ present

∴ causal

$$g(1) = 7f(1)$$

↑ present value

$$g(-1) = 3f(-1)$$

↑ c/p depends on present

Ex: $g(t) = f(5-t)$

$$g(0) = f(5-0)$$

↑ future

$$g(1) = f(5-1)$$

$$g(1) = f(4)$$

↑ future value

$$g(6) = f(5-6)$$

$$g(6) = f(-1)$$

↑ past value

∴ non causal system.

4. Static (memoryless) or Dynamic (Memory):

A system is said to be static or memoryless or instantaneous system if and only if its present output depends on only the present input. Otherwise it is dynamic system.

Ex: $g(t) = f(t) + 2f(t)$ $f(t) + 2f(t)$
 → All static systems are called Causal system

Example of dynamic system is $g(t) = f(t+1)$

5. Stable and unstable systems.

A system is said to be Bounded Input, Bounded output ~~stable~~ (BIBO) stable if and only if for any bounded input the output is bounded.

The condition for BIBO stability for any bounded input signal $f(t)$ is,

(15)

$$|f(t)| \leq M_f < \infty ; \quad -\infty < t < \infty$$

$$|g(t)| \leq M_g < \infty ; \quad -\infty < t < \infty$$

where $g(t)$ is output. M_f & M_g are amplitudes

→ A bounded signal has an amplitude remains finite

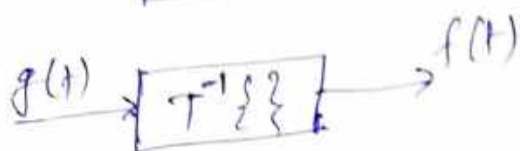
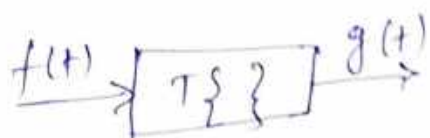
Ex: $g(t) = T\{f(t)\} = f(t) e^{at} \sin \omega t$, for $a < 0$

→ If the output of a system is not bounded for any bounded input is called unstable system.

Ex: $g(t) = T\{f(t)\} = f(t) e^{at} \sin \omega t$, unstable for $a > 0$.

(6) Invertible and non-Invertible.

A system is said to be invertible if the input of the system can be recovered from the output. otherwise it is non-invertible.



Ex: Test the following systems for linearity.

① $y(t) = tx(t)$.

Sol: $x(t)$ be the input to produce the output $y(t)$

$$y(t) = T\{x(t)\} = tx(t)$$

consider two signals $x_1(t)$ & $x_2(t)$, let $y_1(t)$ & $y_2(t)$ be responses of a system T for inputs $x_1(t)$ & $x_2(t)$ respectively

$$\therefore y_1(t) = T\{x_1(t)\} = tx_1(t) \quad \text{--- ①}$$

$$y_2(t) = T\{x_2(t)\} = tx_2(t) \quad \text{--- ②}$$

Let $x_3(t) = a_1x_1(t) + a_2x_2(t)$

Let $y_3(t)$ be response of the system T for input $x_3(t)$.
[then $y_3(t) = a_1y_1(t) + a_2y_2(t)$]

$$y_3(t) = T\{x_3(t)\} = T\{a_1x_1(t) + a_2x_2(t)\} = T\{a_1x_1(t)\} + T\{a_2x_2(t)\}$$

$$= a_1T\{x_1(t)\} + a_2T\{x_2(t)\}$$

$$= a_1tx_1(t) + a_2tx_2(t) \quad [\because \text{from eq ① \& ②}]$$

$$y_3(t) = a_1y_1(t) + a_2y_2(t)$$

Since, $y_3(t) = a_1y_1(t) + a_2y_2(t)$, the system is linear.

② $y(t) = x(t^2)$

Sol: Let $x(t)$ be the input to produce the output $y(t)$

$$y(t) = T\{x(t)\} = x(t^2)$$

$y_1(t)$ & $y_2(t)$

consider two inputs $x_1(t)$ & $x_2(t)$ and let $y_1(t)$ & $y_2(t)$ be corresponding outputs of a system.

(17)

$$y_1(t) = T\{x_1(t)\} = x_1(t) \quad \text{--- (1)}$$

$$y_2(t) = T\{x_2(t)\} = x_2(t) \quad \text{--- (2)}$$

$$\text{Let } x_3(t) = a_1 x_1(t) + a_2 x_2(t)$$

Let $y_3(t)$ be the response of the system for input

$$x_3(t), \text{ and } y_3(t) = a_1 y_1(t) + a_2 y_2(t). \quad \text{--- (3)}$$

$$\text{Now } y_3(t) = T\{x_3(t)\} = T\{a_1 x_1(t) + a_2 x_2(t)\}$$

$$= a_1 T\{x_1(t)\} + a_2 T\{x_2(t)\}$$

* from (1) & (2)

$$y_3(t) = a_1 x_1(t) + a_2 x_2(t)$$

$$y_3(t) = a_1 y_1(t) + a_2 y_2(t) \quad \text{--- (4)}$$

from eq (3) & (4) given system is linear.

- 3) consider the system output $y(t) = T\{x(t)\} = x(t) \cos \omega_c t$.
 Determine whether it is i) Linear ii) time invariant
 iii) causal iv) memoryless or v) stable.

Sol:

i) Given the system output is $y(t) = T\{x(t)\} = x(t) \cos \omega_c t$

Let $x_1(t)$ & $x_2(t)$ be any two inputs and $y_1(t)$ & $y_2(t)$ be corresponding outputs. then

$$y_1(t) = T\{x_1(t)\} = x_1(t) \cos \omega_c t \quad \text{--- (1)}$$

$$y_2(t) = T\{x_2(t)\} = x_2(t) \cos \omega_c t \quad \text{--- (2)}$$

$$\text{say } x_3(t) = a_1 x_1(t) + a_2 x_2(t)$$

(18)

Let $y_2(t)$ be output of the system for input $x_2(t)$

$$\text{and } y_3(t) = a_1 y_1(t) + a_2 y_2(t) \quad \text{--- (2)}$$

$$\text{Now take } y_3(t) = T \{ x_3(t) \}$$

$$\begin{aligned} &= T \{ a_1 x_1(t) + a_2 x_2(t) \} \\ &= (a_1 x_1(t) + a_2 x_2(t)) \cos \omega_c t \\ &= a_1 x_1(t) \cos \omega_c t + a_2 x_2(t) \cos \omega_c t \end{aligned}$$

$$y_3(t) = a_1 y_1(t) + a_2 y_2(t) \quad \text{--- (4)}$$

from eq (3) & (4) system is linear

ii) To check for time invariance, find the response for delayed input i.e., $y_1(t) = y_2(t) = x(t-t_0) \cos \omega_c t$ and then find delayed response for ~~the delayed~~ input.

$$y_2(t) = y_1(t-t_0) = x(t-t_0) \cos \omega_c (t-t_0).$$

Here, $y_1(t) \neq y_2(t)$, Hence the system is time-variant.

iii) given output of the system is $y(t) = x(t) \cos \omega_c t$

put $t=0$

$$y(0) = x(0) \cos \omega_c (0)$$

↑ ↑
present value present value

put $t=-1$

$$y(-1) = x(-1) \cos \omega_c (-1)$$

↑ ↑
present value present value

$$\text{put } t=1; \quad y(1) = x(1) \cos \omega_c (1)$$

Since, the output does not depend on future values of input, the system is causal.

iv) Since the output depends only on present values, the system is static i.e., memoryless.

v) To check for stability

$$|x(t)| \leq |x(t)| |\cos \omega_c t| < \infty$$

$$|y(t)| \leq |x(t)| |\cos \omega_c t| < \infty$$

Since $|\cos \omega_c t| \leq 1$.

Thus, if the input is bounded, the output is bounded and the system is BIBO stable.

④ State whether the given system is time invariant or not.

$$i) y(t) = 2tx(t)$$

Sol: Test-1: Response for delayed input.

$$y_1(t) = 2tx(t-t_0) \quad \text{--- (1)}$$

Test-2: Delayed response

$$y_2(t) = 2(t-t_0)x(t-t_0) \quad \text{--- (2)}$$

$\therefore$ Here, $y_1(t) \neq y_2(t)$, therefore the system is time vari

⑤ Test whether given system is linear or not
 $y(t) = x^2(t)$.

Sol: Given system response is $y(t) = x^2(t)$.

Let $y_1(t)$ & $y_2(t)$ be the response of the system for inputs $x_1(t)$ & $x_2(t)$ respectively.

(20)

$$\therefore y_1(t) = T\{x_1(t)\} = x_1^2(t)$$

$$y_2(t) = T\{x_2(t)\} = x_2^2(t)$$

$$\text{Let } x_3(t) = a_1 x_1(t) + a_2 x_2(t)$$

Let $y_3(t)$ be the response of the system for input $x_3(t)$, and * system may be linear, then

$$y_3(t) = a_1 y_1(t) + a_2 y_2(t) \quad \text{--- (3)}$$

$$\begin{aligned} \text{Now } y_3(t) &= T\{x_3(t)\} = T\{(a_1 x_1(t) + a_2 x_2(t))^2\} \\ &= (a_1 x_1(t) + a_2 x_2(t))^2 \\ &= \{a_1^2 x_1^2(t) + a_2^2 x_2^2(t) + 2a_1 a_2 x_1(t) x_2(t)\} \end{aligned}$$

$$= \underbrace{\{a_1^2 x_1^2(t)\}}_{y_1(t)} + \underbrace{\{a_2^2 x_2^2(t)\}}_{y_2(t)} + 2a_1 a_2 x_1(t) x_2(t)$$

$$y_3(t) = a_1^2 y_1(t) + a_2^2 y_2(t) + 2a_1 a_2 x_1(t) x_2(t)$$

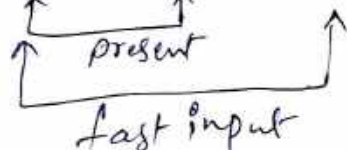
$\therefore$ Here $y_3(t) \neq a_1 y_1(t) + a_2 y_2(t)$. Hence the given system is non linear.

⑥ Test the causality of following system.

$$i) \quad y(t) = x(t) - x(t-1)$$

Sol: put $t=0$

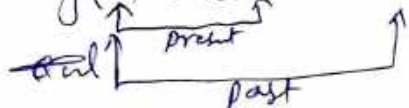
$$y(0) = x(0) - x(-1)$$



$\Rightarrow$ the response at $t=0$, $y(0)$ depends on the present input $x(0)$ and past input $x(-1)$.

put $t=1$

$$y(1) = x(1) - x(0)$$

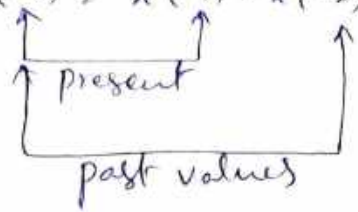


$\Rightarrow$ the response at $t=1$, present output $y(1)$ depends on present input $x(1)$ and past input $x(0)$

(21)

put $t = -1$

$$y(-1) = x(-1) - 2(-2)$$



$\Rightarrow$ the present response $y(-1)$ depends on the present input $x(-1)$ and past input $x(-2)$.

$\therefore$ From the above analysis we can say that for any value of input t , the system present response (output) depends on present and past inputs so given system is causal system.

b) Try this $y(t) = x(t) + 2x(3-t) \rightarrow$ non-causal.

Additional problems on systems:

I. Test the following system for linearity

i) $y(t) = x(t^2)$ (Linear)
ii) $y(t) = x^2(t)$ (N.L.)
iii) $y(t) = Ax(t) + B$ (N.L.)

II. State whether the following systems are time invariant or not

i) $y(t) = 2tx(t)$ T.V.
ii) $y(t) = 3x(t^2)$ T.V.
iii) $y(t) = x(t) + c$ T.In
iv) $y(t) = 2e^{x(t)}$ T.In
v) $y(t) = x(t) + \frac{d}{dt}x(t)$ T.In

III Test the causality of the following systems.

i) $y(t) = x(t) - x(t-1)$ causal
ii) $y(t) = x(t) + 2x(3-t)$ N.C
iii) $y(t) = tx(t)$ causal
iv) $y(t) = x(t^2)$ N.C

(22)

Determine the power and energy for the following continuous time signals.

i) $x(t) = e^{-2t} u(t)$, ii) $x(t) = 8 \cos(500\pi t)$

$E = \frac{1}{4}$ Joules, $P = 0$

$E = \infty$, $P = 4.5$ watts.

Unit-II

Fourier Series and Fourier Transform

①

23 Function: A function is defined as a set of rules which substitutes one number for another.

The terms 'Signal' and 'function' are interchangeably used as there is no difference between the two. And therefore, "signal" may be defined as a set of rules to assign a number $f(t)$ for every number t (if time t is independent variable).

periodic signal: If $f(t+T) = f(t)$ then $f(t)$ is a periodic signal. Where T is fundamental period of the signal $f(t)$.

Fourier Series: In, mathematics, a Fourier series decomposes periodic functions or periodic signals into the sum of a set of simple oscillating functions namely sines and cosines or complex exponentials.

A generalized Fourier series is the representation of a function $f(t)$ by the linear combination of the elements of a complete set of infinite mutually orthogonal functions.

Dirichlet's conditions:

There are sufficient conditions that must be satisfied by a function $f(t)$ for its Fourier series representation. These conditions are known as Dirichlet's conditions.

1. The function $f(t)$ is a single valued function of the variable t within the interval (t_1, t_2) .
2. The function $f(t)$ has a finite number of discontinuities in the interval (t_1, t_2) .
3. The function $f(t)$ has a finite number of maxima and minima in the interval (t_1, t_2) .
4. The function $f(t)$ is absolutely integrable, i.e.,

$$\int_{t_1}^{t_2} |f(t)| dt < \infty$$

Representation of Fourier Series:

There are two important representations of Fourier series.

- They are
1. Trigonometric Fourier series
 2. Exponential Fourier Series.

Advantages of Fourier Series Representation.

- i) It allows a simple, consistent, and unique method to find out the coefficients (constant weights) for each mean component and.

2000
Four
It has only one fundamental frequency. (3)

Trigonometric Fourier Series:

In trigonometric Fourier series, a periodic function $f(t)$ is represented in terms of a linear weighted sum of harmonic signals of sine and cosine terms over any interval $(t_0, t_0 + \frac{2\pi}{\omega_0})$ with fundamental freq. ω_0 .

$$f(t) = a_0 + a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t + a_3 \cos 3\omega_0 t + \dots + a_n \cos n\omega_0 t \\ + b_1 \sin \omega_0 t + b_2 \sin 2\omega_0 t + b_3 \sin 3\omega_0 t + \dots + b_n \sin n\omega_0 t + \dots$$

or

$$f(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos n\omega_0 t + b_n \sin n\omega_0 t], \quad (t_0 < t < t_0 + T)$$

Where n is an integer,

T is period of $f(t)$.

$\omega_0 = \frac{2\pi}{T}$ is called fundamental frequency.

and $n\omega_0$ is the n th harmonic of the sinusoid freq. ω_0 .

Then Fourier coefficients are:

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) dt,$$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos n\omega_0 t dt,$$

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin n\omega_0 t dt$$

Note: It is observed that a_0 is the average (DC) value of $f(t)$ over the interval $(t_0 < t < t_0 + T)$

compact form of Fourier Trigonometric Fourier

(28) The Fourier Series can also be represented in polar form, which is more compact and easier to manipulate. We know that

$$a_n \cos n\omega t + b_n \sin n\omega t = A_n \cos(n\omega t + \phi_n)$$

or

$$A_n \angle \phi_n = a_n - j b_n$$

where amplitude $A_n = \sqrt{a_n^2 + b_n^2}$ and

$$\text{phase angle } \phi_n = -\tan^{-1}\left(\frac{b_n}{a_n}\right).$$

Then the trigonometric Fourier series can be written as

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega t + \phi_n)$$

Thus the compact form of the trigonometric Fourier series is also called cosine representation or the polar form of a periodic function $f(t)$.

Evaluation of Trigonometric Fourier coefficients:

Evaluation of a_0 :

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

$$\int_{t_0}^{t_0+T} f(t) dt = \int_{t_0}^{t_0+T} a_0 dt + \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T} \cos n\omega t dt + \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T} \sin n\omega t dt$$

$$= a_0 \int_{t_0}^{t_0+T} dt + \sum_{n=1}^{\infty} a_n(0) + \sum_{n=1}^{\infty} b_n(0)$$

$$= a_0 T \int_{t_0}^{t_0+T} + 0 = a_0 T.$$

(5)

Therefore

$$\int_{t_0}^{t_0+T} f(t) dt = a_0 T$$

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) dt$$

Evaluation of a_n :

Take the series $\int_{t_0}^{t_0+T} f(t) dt = \int_{t_0}^{t_0+T} \left[a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + b_n \sin n\omega t \right] dt$

then $\int_{t_0}^{t_0+T} f(t) \cos n\omega t dt = \int_{t_0}^{t_0+T} a_0 \cos n\omega t dt + \int_{t_0}^{t_0+T} \sum_{n=1}^{\infty} a_n \cos n\omega t \cdot \cos n\omega t dt$
 $+ \int_{t_0}^{t_0+T} \sum_{n=1}^{\infty} b_n \sin n\omega t \cdot \cos n\omega t dt.$

$$\Rightarrow \int_{t_0}^{t_0+T} f(t) \cos n\omega t dt = a_0(0) + a_n \frac{T}{2} + \sum_{n=1}^{\infty} b_n(0)$$

$$= a_n \frac{T}{2}.$$

$[\cos^2 \theta = \frac{1 + \cos 2\theta}{2}]$

Therefore, $a_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos n\omega t dt$

Evaluation of b_n :

Take the series $\int_{t_0}^{t_0+T} f(t) dt = \int_{t_0}^{t_0+T} \left[a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t) \right] dt$

then $\int_{t_0}^{t_0+T} f(t) \sin n\omega t dt = \int_{t_0}^{t_0+T} a_0 \sin n\omega t dt + \int_{t_0}^{t_0+T} [a_n \cos n\omega t + b_n \sin n\omega t] \sin n\omega t dt$
 $= \int_{t_0}^{t_0+T} a_0 \sin n\omega t dt + \sum_{n=1}^{\infty} \int_{t_0}^{t_0+T} \cos n\omega t \sin n\omega t dt + \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T} \sin n\omega t \sin n\omega t dt$

(28)

$$\int_{t_0}^{t_0+T} f(t) \sin n\omega_0 t dt = a_0(0) + a_n(0) + b_n \frac{T}{2}$$

$$= b_n \frac{T}{2}$$

Therefore, $b_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin n\omega_0 t dt$

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin n\omega_0 t dt$$

$$[\sin^2 \theta = 1]$$

Ex-1:

Evaluate the trigonometric Fourier Series for the waveform shown in below fig.

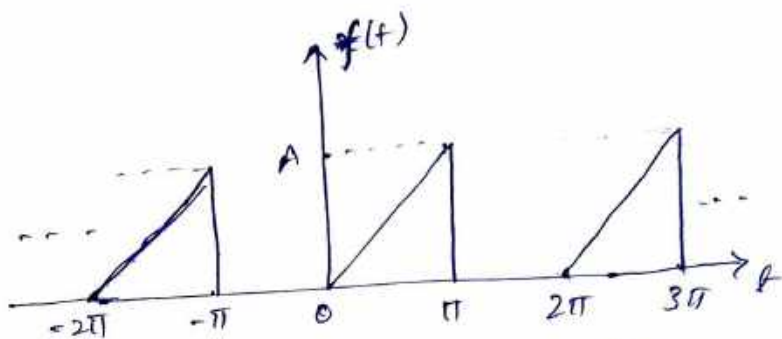


Fig. Triangular waveform

Solution:

From the given Fig., the triangular function can be expressed as

$$f(t) = \begin{cases} \frac{At}{\pi}, & 0 \leq t \leq \pi \\ 0, & \pi \leq t \leq 2\pi \end{cases}$$

Time period $T = 2\pi$ Sec, fundamental frequency $\omega_0 = 1$ rad/sec.

The trigonometric Fourier series is

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

(7)

$$\text{where } a_0 = \frac{1}{T} \int_0^T f(t) dt = \frac{1}{2\pi} \left[\int_0^{\pi} \frac{At}{\pi} dt \right] + 0$$

$$= \frac{1}{2\pi^2} A \left[\frac{t^2}{2} \right]_0^{\pi} = \frac{1}{2\pi^2} A \left(\frac{\pi^2}{2} - 0 \right) = \frac{A}{4}$$

$$\boxed{a_0 = \frac{A}{4}}$$

$$a_n = \frac{2}{T} \int_0^T f(t) \cos n\omega_0 t dt = \frac{2}{2\pi} \int_0^{\pi} \frac{At}{\pi} \cos nt dt + 0$$

$$= \frac{2A}{\pi^2} \left[t \int_0^{\pi} \cos nt dt - \int_0^{\pi} \left[\int \cos nt dt \cdot \frac{d}{dt} t \right] \right]$$

$$[\because \int u v dx = u \int v dx - \int \left[\int v dx \frac{du}{dx} \right]]$$

$$= \frac{A}{\pi^2} \left[t \frac{\sin nt}{n} + \frac{\cos nt}{n^2} \right]_0^{\pi}$$

$$= \frac{A}{\pi^2} \left[\left(\pi \frac{\sin n\pi}{n} + \frac{\cos n\pi}{n^2} \right) - \left(0 \frac{\sin 0}{n} + \frac{\cos 0}{n^2} \right) \right]$$

$$= \frac{A}{\pi^2} \left[0 + \frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2} \right] = \frac{A}{n^2 \pi^2} [\cos n\pi - 1]$$

$$\therefore a_n = \begin{cases} -\frac{2A}{n^2 \pi^2} & \text{for } n \text{ is odd} \\ 0 & \text{for } n \text{ is even} \end{cases}$$

$$b_n = \frac{2}{T} \int_0^T f(t) \sin n\omega_0 t dt = \frac{2}{2\pi} \int_0^{\pi} \frac{At}{\pi} \sin nt dt + 0$$

$$= \frac{A}{\pi^2} \left[t \left(-\frac{\cos nt}{n} \right) + \frac{\sin nt}{n^2} \right]_0^{\pi} \quad [\because \int u v dx = u \int v dx - \int \left[\int v dx \frac{du}{dx} \right]]$$

$$= \frac{A}{\pi^2} \left[\left(\pi \times -\frac{\cos n\pi}{n} + \frac{\sin n\pi}{n^2} \right) - \left(0 \times -\frac{\cos 0}{n} + \frac{\sin 0}{n^2} \right) \right]$$

$$= \frac{A}{\pi^2} \times -\frac{\cos n\pi}{n} = -\frac{A \cos n\pi}{n\pi^2} = \frac{A}{n\pi^2} (-1)^{n+1}$$

(20)

The trigonometric Fourier Series is

$$f(t) = \frac{A}{A} - \frac{2A}{\pi^2} \sum_{n(\text{odd})}^{\infty} \frac{\cos n\pi t}{n^2} + \frac{A}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin n\pi t}{n}$$

or

$$f(t) = \frac{A}{A} - \frac{2A}{\pi^2} \left(\cos t + \frac{1}{9} \cos 3t + \frac{1}{25} \cos 5t + \dots \right) + \frac{A}{\pi} \left(\sin t - \frac{1}{2} \sin 2t + \frac{1}{3} \sin 3t - \frac{1}{4} \sin 4t + \dots \right)$$

Ex-2 Expand a function $f(t)$ shown in Fig by trigonometric Fourier series over the interval $(0,1)$.

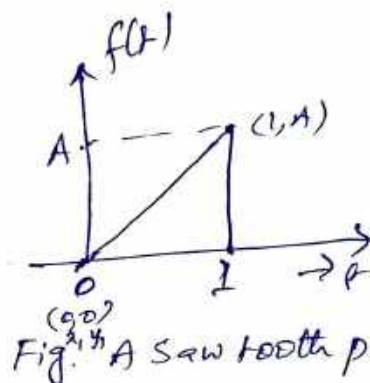
Sol: It is obvious from Fig. that

$$f(t) = At, \quad (0 \leq t \leq 1)$$

compare with limits, (t_0, t_0+T)

$$t_0 = 0, \quad t_0 + T = 0 + T = 1 \\ \Rightarrow T = 1.$$

$$\text{and } \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{1}, \text{ as } T=1$$



$$\begin{aligned} y - y_1 &= \frac{y_2 - y_1}{x_2 - x_1} (x - x_1) \\ f(t) - 0 &= \frac{A - 0}{1 - 0} (t - 0) \\ f(t) &= At. \end{aligned}$$

the trigonometric Fourier Series is given by

$$\begin{aligned} f(t) &= a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \\ &= a_0 + \sum_{n=1}^{\infty} (a_n \cos n 2\pi t + b_n \sin n 2\pi t) \end{aligned}$$

where the coefficients are evaluated as follows

$$a_0 = \frac{1}{T} \int_0^T f(t) dt = \frac{1}{1} \int_0^1 At dt = \frac{A}{2}$$

(9)

$$\begin{aligned}
 a_n &= \frac{2}{T} \int_0^T f(t) \cos n\omega_0 t \, dt = \frac{2}{1} \int_0^1 A t \cos n 2\pi t \, dt \\
 &= 2A \left[t \frac{\sin 2\pi n t}{2\pi n} - \int_0^1 \frac{\sin 2\pi n t}{2\pi n} \, dt \right] \\
 &= \frac{2A}{2\pi n} \left[t \sin 2\pi n t + \frac{\cos 2\pi n t}{-2\pi n} \right]_0^1 = \frac{A}{\pi n} \left[1 \sin 2\pi n + \frac{\cos 2\pi n}{2\pi n} \right. \\
 &\quad \left. - 0 - \frac{\cos 2\pi}{2\pi n} \right] \\
 &= \frac{A}{\pi n} [0 + 1 - 0 - 1] = 0
 \end{aligned}$$

$$a_n = 0$$

$$\begin{aligned}
 b_n &= \frac{2}{T} \int_0^T f(t) \sin n\omega_0 t \, dt = 2 \int_0^1 A t \sin 2\pi t \, dt \\
 &= 2A \left[-t \frac{\cos 2\pi n}{2\pi n} + \int_0^1 \frac{\cos 2\pi n}{2\pi n} \, dt \right] = \frac{2A}{2\pi n} \left[-t \cos 2\pi n t + \frac{\sin 2\pi n t}{2\pi n} \right]_0^1 \\
 &= \frac{A}{\pi n} \left[1 \cos 2\pi n + \frac{\sin 2\pi n}{2\pi n} + 0 \cos 2\pi n \times 0 - \frac{\sin 2\pi n \times 0}{2\pi n} \right] \\
 &= \frac{A}{\pi n} [-1 + 0 + 0 - 0] = -\frac{A}{\pi n}
 \end{aligned}$$

$$b_n = -\frac{A}{\pi n}$$

The trigonometric series will be as follows

$$f(t) = \frac{A}{2} - \frac{A}{\pi} \sin 2\pi t - \frac{A}{2\pi} \sin 4\pi t - \frac{A}{3\pi} \sin 6\pi t - \dots$$

$$f(t) = \frac{A}{2} - \frac{A}{\pi} \sum_{n=1}^{\infty} \frac{\sin 2\pi n t}{n} \quad (0 \leq t \leq 1)$$

Exponential Fourier Series:

Any real valued periodic function $f(t)$ can be expressed as a linear combination of complex exponential functions over any interval $(t_0, t_0 + \frac{2\pi}{\omega_0})$.

(32)

The expression is called the exponential Fourier series of $f(t)$ and is expressed as

$$f(t) = c_0 + c_1 e^{j\omega_0 t} + c_2 e^{j2\omega_0 t} + c_3 e^{j3\omega_0 t} + \dots + c_n e^{jn\omega_0 t} + c_{-1} e^{-j\omega_0 t} + c_{-2} e^{-j2\omega_0 t} + c_{-3} e^{-j3\omega_0 t} + \dots + c_{-n} e^{-jn\omega_0 t}$$

(or)

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

Evaluation of c_n :

The exponential Fourier series is given by

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

Taking integration and multiplying with $e^{-jn\omega_0 t}$, over the limits (t_0, t_0+T)

$$\int_{t_0}^{t_0+T} f(t) e^{-jn\omega_0 t} dt = \int_{t_0}^{t_0+T} \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \cdot e^{-jn\omega_0 t} dt$$

$$= \sum_{n=-\infty}^{\infty} c_n \int_{t_0}^{t_0+T} e^{jn\omega_0 t - jn\omega_0 t} dt$$

$$= \sum_{n=-\infty}^{\infty} c_n \int_{t_0}^{t_0+T} 1 dt = \sum_{n=-\infty}^{\infty} c_n \left. t \right|_{t_0}^{t_0+T}$$

$$= \sum_{n=-\infty}^{\infty} c_n T \quad \text{or} \quad c_n T$$

$$\frac{1}{T} \int_{t_0}^{t_0+T} f(t) e^{-jn\omega_0 t} dt = c_n \quad (\text{or})$$

$$c_n = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) e^{-jn\omega_0 t} dt$$

Noted:
It is

the average value of $f(t)$ over one period T

Note 1:

It is observed that for $n=0$, $c_0 = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) dt$ is the average value (DC) of $f(t)$ over the interval (t_0, t_0+T)

Note 2: Fourier Theorem:

→ Any periodic function $f(t)$ with period T can be expressed in terms of a trigonometric or exponential Fourier series over the entire interval $-\infty < t < \infty$. This statement is called Fourier theorem.

Relation between the Trigonometric and Exponential Fourier Series:

Let us consider the exponential Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$f(t) = c_0 + c_1 e^{+j\omega_0 t} + c_2 e^{+j2\omega_0 t} + \dots + c_n e^{+jn\omega_0 t} + \dots \\ + c_{-1} e^{-j\omega_0 t} + c_{-2} e^{-j2\omega_0 t} + \dots + c_{-n} e^{-jn\omega_0 t} + \dots$$

replacing the exponential functions by their trigonometric identities, the series becomes

$$f(t) = c_0 + c_1 (\cos \omega_0 t + j \sin \omega_0 t) + c_2 (\cos 2\omega_0 t + j \sin 2\omega_0 t) + \dots \\ + c_n (\cos n\omega_0 t + j \sin n\omega_0 t) + \dots \\ + c_{-1} (\cos \omega_0 t - j \sin \omega_0 t) + c_{-2} (\cos 2\omega_0 t - j \sin 2\omega_0 t) + \dots \\ + c_{-n} (\cos n\omega_0 t - j \sin n\omega_0 t) + \dots$$

(34)

rearranging the right-hand side, we have

$$f(t) = f_0 + (C_1 + C_{-1}) \cos \omega_0 t + \dots + (C_n + C_{-n}) \cos n \omega_0 t + \dots \\ + j(C_1 - C_{-1}) \sin \omega_0 t + \dots + j(C_n - C_{-n}) \sin n \omega_0 t + \dots$$

this is a trigonometric series representation of $f(t)$ derived from exponential series. Let us compare this with actual form of trigonometric series

$$a_0 = C_0$$

$$a_n = C_n + C_{-n}$$

$$b_n = j(C_n - C_{-n})$$

Above coefficients may be written as

$$C_0 = a_0$$

$$C_n = \frac{1}{2}(a_n - j b_n)$$

$$C_{-n} = \frac{1}{2}(a_n + j b_n)$$

$$\text{If } C_n = \frac{1}{2}(a_n - j b_n)$$

$$\text{then } a_n = 2 \operatorname{Re}\{C_n\}$$

$$b_n = -2 \operatorname{Im}\{C_n\}$$

This represent the coefficients of exponential series in terms of coefficients of trigonometric series.

Ex-3: Find the exponential Fourier series expansion for the rectangular wave shown in Fig. below.

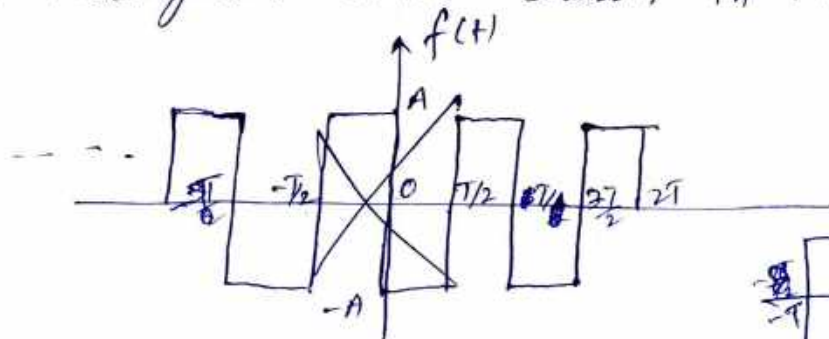
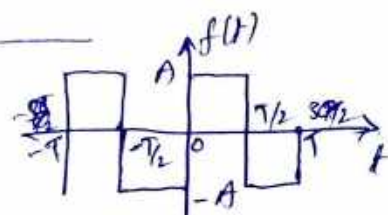


Fig. Rectangular wave



Solution:

(13)

From Fig. the expression for rectangular wave is

$$f(t) = \begin{cases} -A; & -T/2 \leq t \leq 0 \\ +A; & 0 \leq t \leq T/2 \end{cases} \quad T = \frac{2\pi}{\omega_0}$$

Time period = T sec., frequency $\omega_0 = \frac{2\pi}{T}$ rad/sec.

The exponential Fourier series is $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$,

Where $c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t} dt$

$$= \frac{1}{T} \left[\int_{-T/2}^0 -A e^{-jn\omega_0 t} dt + \int_0^{T/2} A e^{-jn\omega_0 t} dt \right]$$

$$c_n = \frac{2A}{jn\pi}$$

$$c_n = \frac{jA}{n\pi}$$

$$= \frac{A}{T} \left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0} \right]_{-T/2}^0 + \frac{A}{T} \left[-\frac{e^{-jn\omega_0 t}}{jn\omega_0} \right]_0^{T/2}$$

$$= \frac{A}{T} \left[\frac{e^{-jn\frac{2\pi}{T} t}}{jn\frac{2\pi}{T}} \right]_{-T/2}^0 + \frac{A}{T} \left[-\frac{e^{-jn\frac{2\pi}{T} t}}{jn\frac{2\pi}{T}} \right]_0^{T/2}$$

$$= \frac{A}{jn2\pi} \left[e^0 - e^{+jn\frac{2\pi}{T} \times \frac{T}{2}} \right] + \frac{A}{jn2\pi} \left[e^{-jn\frac{2\pi}{T} \times \frac{T}{2}} - e^0 \right]$$

$$= \frac{A}{jn2\pi} [1 - e^{jn\pi}] - \frac{A}{jn2\pi} [e^{-jn\pi} - 1]$$

$$= \frac{A}{jn2\pi} [2 - (e^{jn\pi} + e^{-jn\pi})] = \frac{A}{jn\pi} \left[1 - \frac{(e^{jn\pi} + e^{-jn\pi})}{2} \right]$$

$$c_n = \frac{A}{jn\pi} [1 - \cos n\pi]$$

$$c_n = \begin{cases} \frac{2A}{jn\pi}, & n \text{ is odd} \\ 0, & n \text{ even (or) } 0 \text{ (zero)} \end{cases}$$

Expansion of exponential F.S is

$$f(t) = \sum_{n(\text{odd})}^{\infty} \frac{2A}{j n \pi} e^{jn\omega t}$$

$$= \frac{2A}{j\pi} \left[e^{j\omega t} + \frac{e^{j3\omega t}}{3} + \frac{e^{j5\omega t}}{5} + \dots - \frac{e^{-j\omega t}}{1} - \frac{e^{-j3\omega t}}{3} - \dots \right]$$

Symmetry conditions of Fourier Series:

The Fourier Series of a periodic signal $f(t)$ with no symmetry contains both odd (sine) and even (dc and cosine) components. It is observed that depending on the symmetrical conditions of the function, some components of the series become zero. Some of the symmetries are

- i. Even symmetry
- ii. Odd symmetry
- iii. Hidden Symmetry
- iv. Rotation or half-wave symmetry.

i) Even symmetry: A function $f(t)$ is said to be an even function or having even symmetry if

$$f(t) = f(t) \quad \text{or}$$

$$f(t) = f(-t)$$

If $f(t)$ has even symmetry, then

$$b_n = 0, \quad a_n = \frac{4}{T} \int_0^{T/2} f(t) \cos n\omega t dt, \quad \text{and} \quad a_0 = \frac{2}{T} \int_0^{T/2} f(t) dt.$$

ii) Odd symmetry: A function $f(t)$ is said to be an odd function or having odd symmetry if

$$f(t) = -f(-t) \text{ or}$$

$$f(-t) = -f(t)$$

(15)

If $f(t)$ has odd symmetry, then

$$a_0 = 0, a_n = 0, \text{ and}$$

$$b_n = \frac{4}{T} \int_0^{T/2} f(t) \sin n\omega t dt.$$

iii) Hidden symmetry: If a periodic function which does not satisfy either of the symmetry conditions but the Fourier series consists of a dc term (A) and sine terms alone.

If a function $f(t)$ has hidden symmetry, then

$$f_0(t) = f(t) - \frac{A}{2}$$

where $f_0(t)$ is an odd function.

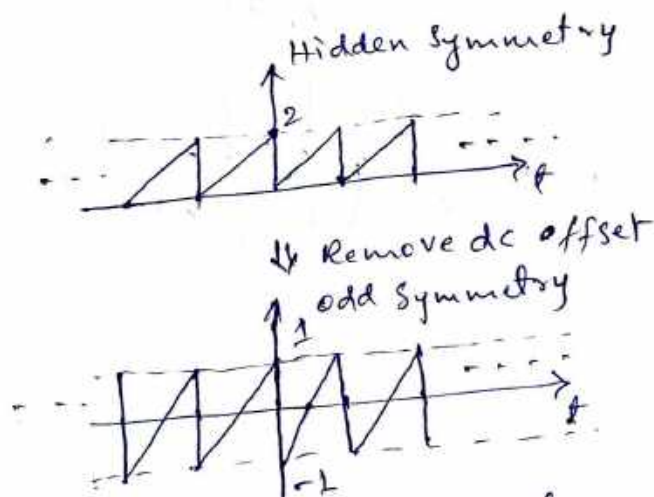
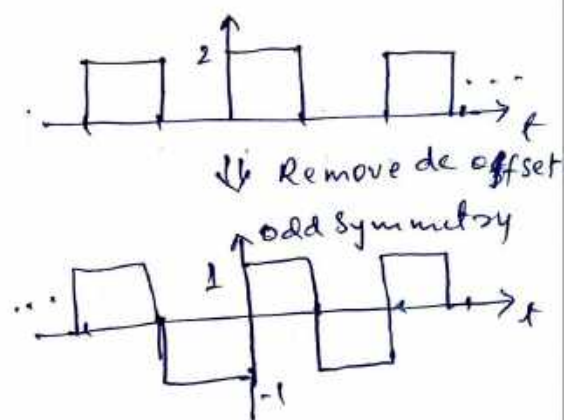


Fig: Hidden symmetry & dc offset.



iv) Rotation or Half-wave symmetry:

A function is said to be rotation or half-wave symmetry if $f(t) = -f(t \pm \frac{T}{2})$

→ In half-wave symmetry, the shape of the signal is identical, with opposite sign at every ~~instant~~ alternate half cycle.

→ The Fourier series of any function which has half-wave symmetry contains only odd harmonics of both sine and cosine terms.

If $f(t)$ has half-wave symmetry, then

$$a_n = \begin{cases} \frac{4}{T} \int_0^{T/2} f(t) \cos n\omega_0 t dt & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}$$

$$b_n = \begin{cases} \frac{4}{T} \int_0^{T/2} f(t) \sin n\omega_0 t dt & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}$$

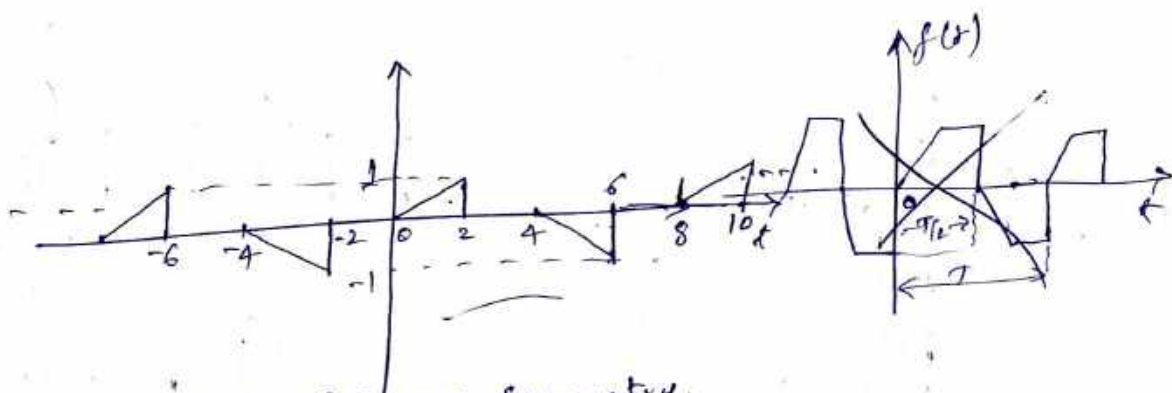


Fig. Half-wave symmetry.

The complex Fourier spectrum:

Spectrum of frequency: the range of frequencies that a signal contains is called spectrum of frequencies or frequency spectrum.

The representation of frequency components at harmonics

frequencies $\omega_0, 2\omega_0, 3\omega_0, \dots, n\omega_0, \dots$ etc is called its frequency domain representation. In the discrete function representation, also called line spectrum, only the amplitude of the components are specified at equally spaced frequency values. (17)

Exponential Fourier series is more preferable to represent frequency spectrum than trigonometric Fourier series because of its simplicity. The complex Fourier spectrum or the frequency spectrum of $f(t)$ is equal to the representation of Fourier coefficients c_n at $n = \pm 1, \pm 2, \pm 3, \pm 4, \dots$ that is the coefficients c_n , in general, are complex numbers, expressed as

$$c_n = |c_n| e^{i\phi_n}$$

where $|c_n|$ is the amplitude (or magnitude) spectrum
 ϕ_n is the phase spectrum.

The spectrum for positive frequencies is

$$c_n = |c_n| e^{i\phi_n} = \frac{2}{T} \int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t} dt$$

The spectrum for negative frequencies is

$$c_{-n} = |c_n| e^{-i\phi_n} = \frac{2}{T} \int_{-T/2}^{T/2} f(t) e^{jn\omega_0 t} dt$$

$$\text{Also } c_n^* = c_{-n} \text{ and } |c_n| = |c_{-n}|.$$

→ Therefore, the amplitude spectrum is symmetrical about the origin (even symmetry).

→ While the phase spectrum is antisymmetric (odd symmetry).

→ Since, the spectrum is extended from $-\infty$ to ∞ , Solution
is a double sided spectrum.

For trigonometric Fourier Series the amplitude spectrum is represented with either a_n or b_n or both, depending upon the symmetry.

Since the amplitude spectrum for both coefficients extended from 0 to ∞ , it is a one sided spectrum and has no negative frequencies. Also no phase spectrum exists.

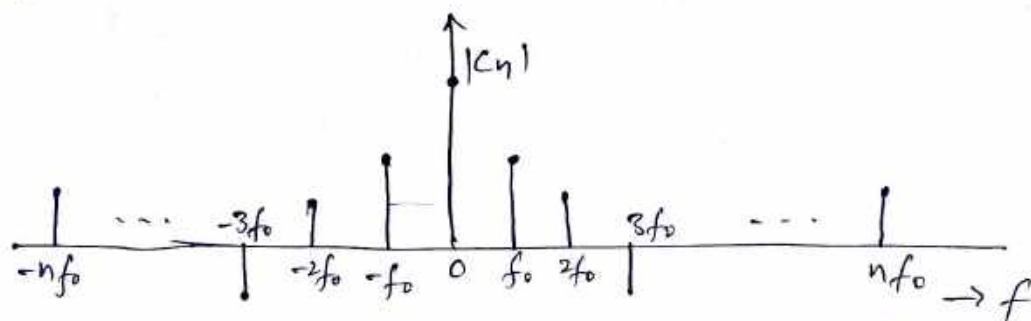


Fig. Two-sided spectrum of Exponential Fourier Series.

Ex-4: (a) Evaluate the trigonometric Fourier series expansion of a full-wave rectified cosine function shown in fig. (b) derive the corresponding exponential Fourier series, and (c) draw the complex Fourier spectrum.

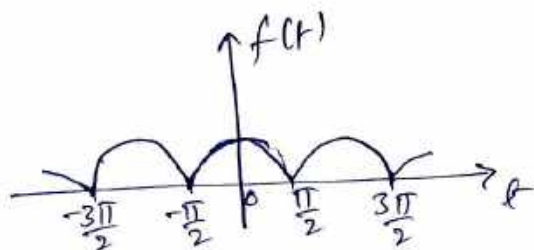


Fig. Full-wave rectified cosine signal.

Solution:

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From the fig.

$$f(t) = \cos t; \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2},$$

Time period $T = \pi$,

fundamental frequency $\omega_0 = \frac{2\pi}{T} = 2 \text{ rad/sec}$

a) The trigonometric Fourier series of function $f(t)$

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

As the waveform follows even symmetry, all the sine terms are zero and $b_n = 0$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$$

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) dt = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t dt = \frac{1}{\pi} \left[\sin t \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{2}{\pi}$$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos n\omega_0 t dt = \frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t \cdot \cos n\omega_0 t dt$$

$$= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2 \cos n\omega_0 t \cdot \cos t dt = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [\cos(2n\omega_0 t - t) + \cos(n\omega_0 t + t)] dt$$

$$= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos(2n-1)t + \cos(2n+1)t) dt = \frac{1}{\pi} \left[\frac{\sin(2n-1)t}{2n-1} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{\sin(2n+1)t}{2n+1} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{\pi} \left[\frac{\sin(2n-1)\frac{\pi}{2}}{2n-1} + \frac{\sin(2n+1)\frac{\pi}{2}}{2n-1} + \frac{\sin(2n+1)\frac{\pi}{2}}{2n+1} + \frac{\sin(2n+1)\frac{\pi}{2}}{2n+1} \right]$$

$$a_n = \frac{2}{\pi} \left[\frac{\sin(2n-1)\frac{\pi}{2}}{2n-1} + \frac{\sin(2n+1)\frac{\pi}{2}}{2n+1} \right]$$

$$\therefore \sin((2n-1)\frac{\pi}{2}) = (-1)^{n+1} \text{ and } \sin((2n+1)\frac{\pi}{2}) = (-1)^n$$

$$a_n = \frac{2}{\pi} \left[\frac{(-1)^{n+1}}{2n-1} + \frac{(-1)^n}{2n+1} \right]$$

Therefore the trigonometric Fourier Series for $f(t)$ is

$$f(t) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{2}{\pi} \left(\frac{(-1)^{n+1}}{2n-1} + \frac{(-1)^n}{2n+1} \right) \cos(2nt)$$

$$\boxed{f(t) = \frac{2}{\pi} \left[1 + \frac{2}{3} \cos 2t - \frac{2}{15} \cos 4t + \frac{2}{35} \cos 6t - \dots \right]}$$

b) Exponential Fourier Series:

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} C_n e^{j2nt}$$

From trigonometric Fourier Series, we know that

$$C_n = \frac{1}{2}(a_n - jb_n) = \frac{1}{2}(a_n - j \times 0) = \frac{a_n}{2}$$

$$C_n = \frac{a_n}{2} = \frac{1}{\pi} \left[\frac{(-1)^{n+1}}{2n-1} + \frac{(-1)^n}{2n+1} \right]$$

$$C_0 = a_0 = \frac{2}{\pi}$$

Therefore the exponential Fourier Series for $f(t)$ is

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} \frac{1}{\pi} \left(\frac{(-1)^{n+1}}{2n-1} + \frac{(-1)^n}{2n+1} \right) e^{j2nt}$$

$$f(t) = \frac{2}{\pi} + \frac{2}{3\pi} e^{j2t} - \frac{2}{15\pi} e^{j4t} + \frac{2}{35\pi} e^{j6t} - \dots - \frac{2}{3\pi} e^{-j2t} - \frac{2}{15\pi} e^{-j4t} + \dots$$

c) The frequency spectrum is

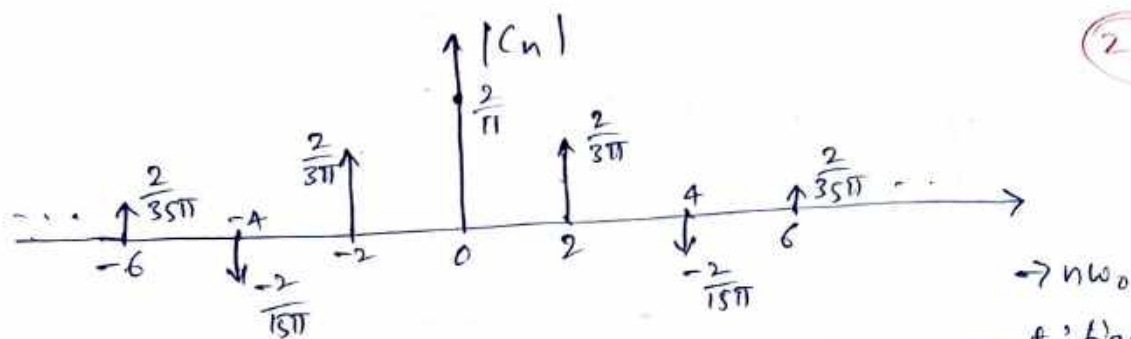


Fig. Amplitude Spectrum of full-wave rectified cosine signal.

H.W.F.

Ex-5 For the periodic signal in Fig. find the trigonometric Fourier series.

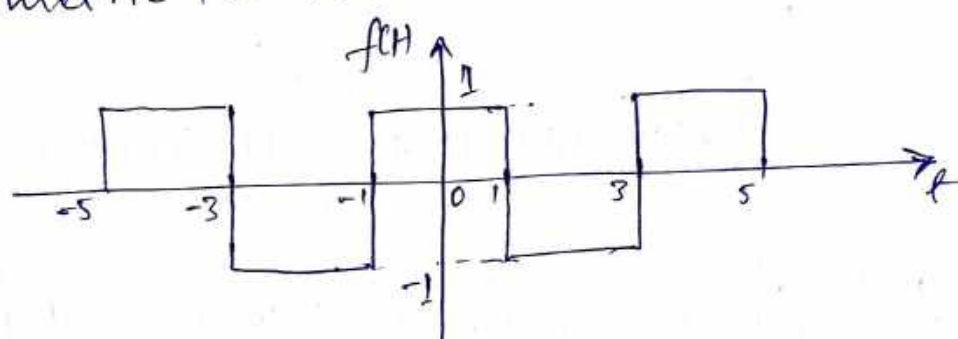


Fig. Square wave signals

Solution:

From fig, given, the expression for the square wave is

$$f(t) = \begin{cases} +1; & -1 \leq t \leq +1 \\ -1; & +1 \leq t \leq +3 \end{cases}$$

Time period $(T) = 4$;

Fundamental frequency $(\omega) = \frac{2\pi}{T} = \frac{\pi}{2}$.

The trigonometric Fourier series is given by

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \quad \text{--- (1)}$$

As the wave form follows even symmetry, all the

Sine terms are zero; $b_n = 0$, and since the t axis is positioned at the middle of the signal amplitude, the dc component, $a_0 = 0$. Then eqn ① becomes

$$① \Rightarrow f(t) = \sum_{n=1}^{\infty} a_n \cos n \times \frac{\pi}{2} t \quad \text{and}$$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos n \omega_0 t \, dt$$

$$= \frac{2}{4} \int_{-1}^3 f(t) \cos n \frac{\pi}{2} t \, dt = \frac{1}{2} \int_{-1}^1 f(t) \cos(n \frac{\pi}{2}) t \, dt + \frac{1}{2} \int_1^3 f(t) \cos(n \frac{\pi}{2}) t \, dt$$

$$= \frac{1}{2} \left[\int_{-1}^1 f(t) \cos(n \frac{\pi}{2}) t \, dt + \int_1^3 f(t) \cos(n \frac{\pi}{2}) t \, dt \right]$$

$$= \frac{1}{2} \left[\int_{-1}^1 (+1) \cos(n \frac{\pi}{2}) t \, dt + \int_1^3 (-1) \cos(n \frac{\pi}{2}) t \, dt \right]$$

$$= \frac{1}{2} \left[\frac{\sin(n \frac{\pi}{2}) t}{\frac{n \pi}{2}} \Big|_{-1}^1 + \frac{\sin(n \frac{\pi}{2}) t}{\frac{n \pi}{2}} \Big|_1^3 \right]$$

$$= \frac{1}{n \pi} \left[\sin(n \frac{\pi}{2}) + \sin(n \frac{\pi}{2}) - \sin(n \frac{\pi}{2}) + \sin(n \frac{\pi}{2}) \right]$$

$$= \frac{1}{n \pi} \left[2 \sin(n \frac{\pi}{2}) - \sin(n \pi + n \frac{\pi}{2}) \right] = \frac{1}{n \pi} \left[2 \sin(n \frac{\pi}{2}) + \sin(n \frac{\pi}{2}) \right]$$

$$a_n = \frac{4}{n \pi} \sin(n \frac{\pi}{2})$$

$$a_n = \begin{cases} \frac{4}{n \pi} & \text{for } n = 1, 5, 9, \dots \\ -\frac{4}{n \pi} & \text{for } n = 3, 7, 11, \dots \\ 0 & \text{for } n \text{ is even} \end{cases}$$

$\therefore$ The trigonometric F-S for $f(t)$ is

(23)

$$f(t) = \frac{4}{\pi} \left[\cos\left(\frac{\pi t}{2}\right) - \frac{1}{3} \cos\left(\frac{3\pi t}{2}\right) + \frac{1}{5} \cos\left(\frac{5\pi t}{2}\right) - \frac{1}{7} \cos\left(\frac{7\pi t}{2}\right) + \dots \right]$$

Ex-6: For the periodic Gate function shown in

Fig. Find

- Find the trigonometric Fourier series
- Derive the corresponding exponential Fourier series.

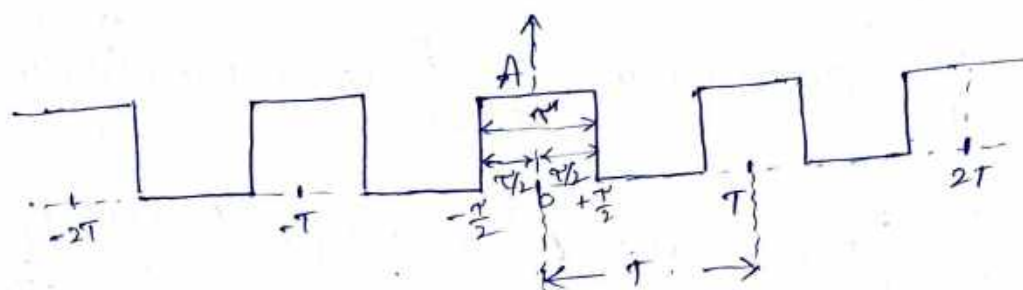


Fig. A periodic Gate function.

Solution: ~~So~~, The periodic Gate function appears like a periodic pulse with pulse width π and repetition time T .

From given Fig. $f(t)$ can be expressed as

$$f(t) = \begin{cases} A & ; -\frac{\pi}{2} < t < \frac{\pi}{2} \\ 0 & ; \frac{\pi}{2} < t < T - \frac{\pi}{2} \end{cases}$$

Time period $T = T \text{ sec}$; Fundamental $\omega_0 = \frac{2\pi}{T} \text{ rad/sec}$.

(a) Trigonometric Fourier Series is given by

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \quad \text{--- ①}$$

As the wave form follows even symmetry, sine terms are zero; $b_n = 0$;

Then equation (1) becomes

(14)

The

$$\text{eqn (1)} \Rightarrow f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$$

$$\begin{aligned} a_0 &= \frac{1}{T} \int_{-\frac{T}{2}}^{T-\frac{T}{2}} f(t) dt = \frac{1}{T} \left[\int_{-\frac{T}{2}}^{\tau_h} f(t) dt + \int_{\tau_h}^{T-\frac{T}{2}} f(t) dt \right] \\ &= \frac{1}{T} \left[\int_{-\frac{T}{2}}^{\tau_h} A dt + \int_{\tau_h}^{T-\frac{T}{2}} 0 dt \right] = \frac{A}{T} t \Big|_{-\frac{T}{2}}^{\tau_h} = \frac{A\tau}{T} \end{aligned}$$

$$\boxed{a_0 = \frac{A\tau}{T}}$$

$$\begin{aligned} a_n &= \frac{2}{T} \int_{-\frac{T}{2}}^{T-\frac{T}{2}} f(t) \cos n\omega_0 t dt = \frac{2}{T} \left[\int_{-\frac{T}{2}}^{\tau_h} f(t) \cos n\omega_0 t dt + \int_{\tau_h}^{T-\frac{T}{2}} f(t) \cos n\omega_0 t dt \right] \\ &= \frac{2}{T} \left[\int_{-\frac{T}{2}}^{\tau_h} A \cos n\omega_0 t dt + \int_{\tau_h}^{T-\frac{T}{2}} 0 \cos n\omega_0 t dt \right] = \frac{2A}{T} \frac{\sin n\omega_0 t}{n\omega_0} \Big|_{-\frac{T}{2}}^{\tau_h} \end{aligned}$$

$$= \frac{2A}{nT\omega_0} \left[\sin n\omega_0 \tau_h - \sin n\omega_0 \left(-\frac{T}{2}\right) \right] = \frac{2A}{n\omega_0 T} \left[\sin n\omega_0 \frac{\tau}{2} + \sin n\omega_0 \frac{T}{2} \right]$$

$$a_n = \frac{4A}{n\omega_0 T} \sin n\omega_0 \frac{\tau}{2} = \frac{2A\tau}{T} \frac{\sin n\omega_0 \frac{\tau}{2}}{n\omega_0 \frac{\tau}{2}} \quad \left[\because \text{multiply \& divide by } \frac{\tau}{2} \right]$$

$$a_n = \frac{2A\tau}{T} \text{Sa}(n\omega_0 \frac{\tau}{2})$$

$$\left[\because \text{Sa}(x) = \frac{\sin x}{x} \right]$$

$\therefore$ The series is given by

$$f(t) = \frac{A\tau}{T} + \frac{2A\tau}{T} \sum_{n=1}^{\infty} \text{Sa}(n\omega_0 \frac{\tau}{2}) \cos n\omega_0 t$$

(b) Derivation of exponential series from the trigonometric series. we know that series relation

$$c_0 = a_0 = \frac{A\tau}{T}$$

$$c_n = \frac{1}{2}(a_n - j b_n) = \frac{1}{2}(a_n - j 0) = \frac{a_n}{2}$$

$$c_n = \frac{A\tau}{T} \text{Sa}(n\omega_0 \frac{\tau}{2})$$

The exponential Fourier Series is given by

(25)

$$f(t) = \frac{A_T}{T} + \frac{A_T}{T} \text{Sa}\left(\frac{\omega_0 T}{2}\right) e^{j\omega_0 t} + \frac{A_T}{T} \text{Sa}\left(\frac{2\omega_0 T}{2}\right) e^{j2\omega_0 t} + \dots$$

$$+ \dots + \frac{A_T}{T} \text{Sa}\left(\frac{n\omega_0 T}{2}\right) e^{jn\omega_0 t} + \dots$$

$$\dots + \frac{A_T}{T} \text{Sa}\left(\frac{\omega_0 T}{2}\right) e^{-j\omega_0 t} + \frac{A_T}{T} \text{Sa}\left(\frac{2\omega_0 T}{2}\right) e^{-j2\omega_0 t} + \dots$$

$$\dots + \frac{A_T}{T} \text{Sa}\left(\frac{n\omega_0 T}{2}\right) e^{-jn\omega_0 t} + \dots$$

Properties of Fourier Series:

Fourier series of continuous time signals have many properties that are useful for solving problems.

The important properties are explained below.

1) Linearity:

Let $x_1(t)$ and $x_2(t)$ be two periodic signals with period T and have exponential Fourier coefficients c_n and d_n respectively.

$$\text{If } x_1(t) \xleftrightarrow{\text{FS}} c_n, \quad x_2(t) \xleftrightarrow{\text{FS}} d_n,$$

then the linear combination of the two signals with same time period T satisfy the relation

$$Ax_1(t) + Bx_2(t) \xleftrightarrow{\text{FS}} Ac_n + Bd_n,$$

where A and B are arbitrary constants.

proof:

$$\text{FS}[Ax_1(t) + Bx_2(t)] = \frac{1}{T} \int_{t_0}^{t_0+T} [Ax_1(t) + Bx_2(t)] e^{-jn\omega_0 t} dt$$

$$= A \frac{1}{T} \int_{t_0}^{t_0+T} x_1(t) e^{-jn\omega_0 t} dt + B \frac{1}{T} \int_{t_0}^{t_0+T} x_2(t) e^{-jn\omega_0 t} dt$$

$$= Ac_n + Bd_n$$

$Ax_1(t) + Bx_2(t) \xleftrightarrow{\text{FS}} Ac_n + Bd_n$

Hence proved.

2) Time Shifting:

(26)

The time shifting property states that if

$$x(t) \xleftrightarrow{FS} c_n, \text{ then}$$

$$x(t-t_0) \longleftrightarrow [e^{-jn\omega_0 t_0}] c_n.$$

proof: Let $x(t)$ be a periodic signal with time period T has exponential Fourier coefficient c_n , then

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}, \quad t_0 < t < t_0 + T$$

put $t = t - t_0$

$$x(t-t_0) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 (t-t_0)}$$

$$x(t-t_0) = \sum_{n=-\infty}^{\infty} [c_n e^{-jn\omega_0 t_0}] e^{jn\omega_0 t}$$

$$\boxed{x(t-t_0) \xleftrightarrow{FS} [e^{-jn\omega_0 t_0} c_n]}, \text{ Hence proved.}$$

(or)

$$c_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt$$

$$c_n = \frac{1}{T} \int_0^T x(t-t_0) e^{-jn\omega_0 t} dt$$

Let $\tau = t - t_0$

$$\tau + t_0 = t$$

$$d\tau = dt$$

$$c_n = \frac{1}{T} \int_{-t_0}^{T-t_0} x(\tau) e^{-jn\omega_0 (\tau+t_0)} d\tau$$

3) Time Reversal:

The time reversal states that if $x(t) \xleftrightarrow{FS} c_n$, then

$$x(-t) \xleftrightarrow{FS} c_{-n}.$$

proof: Let $x(t)$ be a periodic signal with period T has exponential Fourier coefficient c_n , then,

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}, \quad t_0 < t < t_0 + T$$

(or)

$$0 < t < T$$

$$c_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt$$

$$c_n = \frac{1}{T} \int_0^T x(-t) e^{-jn\omega_0 t} dt$$

$$\text{Let } -t = \tau$$

$$-dt = d\tau$$

$$t=0 \Rightarrow \tau=0$$

$$t=T \Rightarrow \tau=-T$$

(27)

$$c_n = \frac{1}{T} \int_0^{-T} x(\tau) e^{-jn\omega_0(-\tau)} (-d\tau)$$

$$= -\frac{1}{T} \int_0^{-T} x(\tau) e^{-j(-n)\omega_0\tau} d\tau$$

by interchanging the limits

$$c_n = +\frac{1}{T} \int_{-T}^0 x(\tau) e^{-j(-n)\omega_0\tau} d\tau$$

$\underbrace{\hspace{10em}}_{c_{-n}}$

$$\boxed{x(-t) \xleftrightarrow{FS} c_{-n}} \text{ , Hence proved.}$$

Time scaling:

The time scaling states that if $x(t) \xleftrightarrow{FS} c_n$, then $x(at) \xleftrightarrow{FS} c_n$.

proof: Let $x(t)$ be a periodic signal with period T has exponential coefficient c_n .

$$c_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt$$

$$c_n = \frac{1}{T} \int_0^T x(at) e^{-jn\omega_0 t} dt$$

If time scaling compresses time period, then

corresponding frequency spectrum will be expanded.

Time period $T' = \frac{T}{a}$

$$\omega_0' = \frac{2\pi}{T'} = \frac{2\pi}{T/a} = a \frac{2\pi}{T} = a\omega_0$$

$$\left. \begin{array}{l} \text{let } at = \tau \\ \tau = \frac{T}{a} \\ at = \frac{d\tau}{a} \end{array} \right| \begin{array}{l} t=0 \\ \tau=0 \\ t=T' \\ \tau = at' = a \times \frac{T}{a} = T \\ \tau = T \end{array}$$

Now c_n will become as

$$c_n' = \frac{1}{T'} \int_0^T x(\tau) e^{-jn(a\omega_0)\frac{\tau}{a}} \frac{d\tau}{a}$$

$$= \frac{a}{T} \int_0^T x(\tau) e^{-jn\omega_0\tau} \frac{d\tau}{a}$$

$$c_n' = \frac{1}{T} \int_0^T x(\tau) e^{-jn\omega_0\tau} d\tau$$

$\underbrace{\hspace{10em}}_{c_n}$

$$\therefore \boxed{x(at) \xleftrightarrow{F.S} c_n}$$

Note: Time Scaling of a signal will change time period and corresponding frequency, but it does not change Exponential Fourier coefficient.

5. conjugation:

conjugate property states that if $x(t) \xleftrightarrow{F.S} c_n$ then

$x^*(t) \xleftrightarrow{F.S} c_n^*$, where $x^*(t)$ is complex conjugate

of $x(t)$.

(29)

Proof: Let $x(t)$ be a periodic signal with time period T has exponential coefficient c_n given by

$$c_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt$$

and exponential coefficient of $x^*(t)$ is given by
[or take conjugate on both sides]

$$c_n^* = \frac{1}{T} \int_0^T x^*(t) e^{+jn\omega_0 t} dt$$

put $n = -n$

$$c_{-n}^* = \frac{1}{T} \int_0^T x^*(t) e^{+j(-n)\omega_0 t} dt$$

$$c_{-n}^* = \frac{1}{T} \int_0^T x^*(t) e^{-jn\omega_0 t} dt$$

$$\boxed{x^*(t) \xleftrightarrow{FS} c_{-n}^*}$$

Hence proved

Frequency shifting:

If $x(t) \xleftrightarrow{FS} c_n$, then

$$x(t) e^{jm\omega_0 t} \xleftrightarrow{FS} c_{n-m}$$

proof: Let $x(t)$ be any periodic signal with time period T . and Exponential Fourier coefficient of $x(t)$ is

$$c_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt$$

put $n = n-m$

$$c_{n-m} = \frac{1}{T} \int_0^T x(t) e^{-j(n-m)\omega_0 t} dt$$

$$= \frac{1}{T} \int_0^T \underbrace{x(t) e^{+jm\omega_0 t}} e^{-jn\omega_0 t} dt$$

$$\boxed{x(t) e^{jn\omega_0 t} \longleftrightarrow C_{n-m}} \quad \text{Hence proved.}$$

⑦ Time Differentiation

If $x(t) \xleftrightarrow{FS} C_n$, then

$$\frac{dx(t)}{dt} \xleftrightarrow{FS} jn\omega_0 C_n$$

proof:

Let $x(t)$ be any periodic function has time T , then exponential Fourier series is given by

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

differentiate on both sides w.r.t t , we get

$$\frac{d}{dt} x(t) = \frac{d}{dt} \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} C_n \frac{d}{dt} e^{jn\omega_0 t}$$

$$\frac{d}{dt} x(t) = \sum_{n=-\infty}^{\infty} (jn\omega_0 C_n) e^{jn\omega_0 t}$$

$$\boxed{\frac{d}{dt} x(t) \longleftrightarrow jn\omega_0 C_n} \quad \text{Hence proved}$$

In general

$$\boxed{\frac{d^k}{dt^k} x(t) \longleftrightarrow (jn\omega_0)^k C_n}$$

⑧ Time Integration property.

If $x(t) \xleftrightarrow{FS} C_n$, then

$$\int_{-\infty}^{\infty} x(\tau) d\tau \xleftrightarrow{FS} \frac{C_n}{jn\omega_0}$$

proof: let $x(t)$ be any periodic signal with time period, T , fundamental frequency ω_0 and FS is given by $x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$

replace t by τ (dummy variable)

$$x(\tau) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 \tau}$$

Take integration on both sides, we get

$$\int_{-\infty}^t x(\tau) d\tau = \int_{-\infty}^t \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 \tau} d\tau$$

$$= \sum_{n=-\infty}^{\infty} c_n \int_{-\infty}^t e^{jn\omega_0 \tau} d\tau$$

$$\int_{-\infty}^t x(\tau) d\tau = \sum_{n=-\infty}^{\infty} \frac{c_n}{jn\omega_0} e^{jn\omega_0 t}$$

$$\therefore \boxed{\int_{-\infty}^t x(\tau) d\tau \xleftrightarrow{FS} \frac{c_n}{jn\omega_0}} \quad \text{Hence proved}$$

convolution:

If $x_1(t)$ & $x_2(t)$ are any periodic functions with time periods

T_1 and T_2 , and $x_1(t) \xleftrightarrow{FS} c_n$ & $x_2(t) \xleftrightarrow{FS} d_n$ then

$$x_1(t) * x_2(t) \xleftrightarrow{FS} T c_n d_n$$

proof: Exponential Fourier coefficient of $x(t)$ is

$$\text{given by } FS[x(t)] = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt$$

$$FS[x_1(t) * x_2(t)] = \frac{1}{T} \int_0^T [x_1(t) * x_2(t)] e^{-jn\omega_0 t} dt$$

$$= \frac{1}{T} \int_0^T \left[\int_0^T x_1(\tau) x_2(t-\tau) d\tau \right] e^{-jn\omega_0 t} dt$$

$$\left[\because x_1(t) * x_2(t) = \int_0^T x_1(\tau) x_2(t-\tau) d\tau \right]$$

$$= \frac{1}{T} \int_0^T \left[\int_0^T x_1(\tau) x_2(t-\tau) d\tau \right] e^{-jn\omega_0 t} \cdot \frac{e^{jn\omega_0 \tau}}{e^{jn\omega_0 \tau}} dt$$

$$= \left[\frac{1}{T} \int_0^T x_1(\tau) e^{-jn\omega_0 \tau} d\tau \right] \left[\frac{1}{T} \int_0^T x_2(t-\tau) e^{jn\omega_0(t-\tau)} dt \right]$$

*

$$\begin{array}{l|l} \text{put } t-\tau = \alpha & t=0 \Rightarrow \alpha = -\tau \\ dt = d\alpha & t=T \Rightarrow \alpha = T-\tau \end{array}$$

$$= \underbrace{\left[\frac{1}{T} \int_0^T x_1(\tau) e^{-jn\omega_0 \tau} d\tau \right]}_{C_n} \cdot T \underbrace{\left[\frac{1}{T} \int_{-T}^{T-T} x_2(\alpha) e^{jn\omega_0 \alpha} d\alpha \right]}_{C_d}$$

$$\therefore \boxed{x_1(t) * x_2(t) \longleftrightarrow T C_n C_d} \quad \text{Hence proved.}$$

Note: The convolution in time domain gives multiplication in frequency domain.

⑩ Multiplication in Time:

The multiplication property states that

if $x_1(t) \longleftrightarrow C_n$ & $x_2(t) \longleftrightarrow C_d$ then

$$x_1(t) \cdot x_2(t) \longleftrightarrow C_n * C_d$$

proof: Let $x_1(t)$ & $x_2(t)$ be any periodic signals with time period T . have Fourier coefficients C_n & C_d respectively. And EFS coefficient of $x(t)$ is given by

$$C_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt$$

$$FS[x_1(t) \cdot x_2(t)] = \frac{1}{T} \int_0^T [x_1(t) x_2(t)] e^{-jn\omega_0 t} dt$$

$$\begin{aligned} &= \frac{1}{T} \int_0^T \left[\sum_{l=-\infty}^{\infty} c_l e^{j l \omega_0 t} \right] e^{j n \omega_0 t} dt \\ &= \sum_{l=-\infty}^{\infty} c_l \left[\frac{1}{T} \int_0^T e^{j (n-l) \omega_0 t} dt \right] \\ &= \sum_{l=-\infty}^{\infty} c_l \cdot d_{n-l} = c_n \ast d_n \end{aligned}$$

convolution
of Discrete
signals c_n & d_n

$\therefore x_1(t) \cdot x_2(t) \longleftrightarrow c_n \ast d_n$

Note: multiplication in time domain gives convolution in frequency domain.

Parseval's power Theorem:

If $x(t) \longleftrightarrow c_n$ then $P_{x(t)} = \sum_{n=-\infty}^{\infty} |c_n|^2$

where $x(t)$ is periodic power signal, with time period T .

Proof: Let $x(t)$ be periodic signal with period T , has exponential F-coefficient and series is given by

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j n \omega_0 t}$$

And the average power of a periodic signal across a one-ohm resistance is given by

$$\begin{aligned} P &= \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{n=-\infty}^{\infty} c_n e^{j n \omega_0 t} \right|^2 dt \\ &= \frac{1}{T} \int_{-T/2}^{T/2} \sum_{n=-\infty}^{\infty} |c_n|^2 |e^{j n \omega_0 t}|^2 dt \end{aligned}$$

$$= \sum_{n=-\infty}^{\infty} |c_n|^2 \frac{1}{T} \int_{-T/2}^{T/2} 1^2 dt$$

$$E: |e^{jn\omega_0 t}| = 1 \quad (34)$$

$$= \sum_{n=-\infty}^{\infty} |c_n|^2 \cdot \frac{1}{T} \cdot T = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Hence $\boxed{P_{x(t)} = \sum_{n=-\infty}^{\infty} |c_n|^2}$

Note: Parseval's theorem says that the total average power of a periodic signal $x(t)$ is equal to the sum of the average powers in all of its harmonic components.

The Fourier Transform:

So far, Fourier series has been used in the analysis of

- i) an arbitrary waveform over a finite interval
- ii) a periodic waveform over an entire interval $(-\infty, \infty)$

However, it is desirable to analyse any general waveform, periodic or not, over an entire interval $(-\infty, \infty)$ because a vast majority of interesting signals extend for all time $(-\infty, \infty)$ and are non-periodic in nature.

→ Any periodic or non periodic signal that can be represented by a continuous frequency in the frequency domain is called Fourier transform.

Fourier transform from Fourier Series:

(33)

Consider a periodic function $f_T(t)$ over an interval $-\infty < t < \infty$. The exponential form of Fourier Series is

$$f_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \quad \leftarrow (1)$$

where the Fourier coefficient

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f_T(t) e^{-jn\omega_0 t} dt \quad \text{and } \omega_0 = \frac{2\pi}{T} \quad \text{--- (2)}$$

The Fourier coefficient c_n represents the amplitude of the frequency components

Now the non-periodic function can be expressed in terms of a periodic function with infinite time period as

$$f(t) = \lim_{T \rightarrow \infty} f_T(t)$$

As $T \rightarrow \infty$, the fundamental frequency ω_0 becomes vanishingly small. Therefore fundamental frequency ω_0 can be written as $\omega_0 \rightarrow d\omega$ and

$$\frac{1}{T} = \frac{\omega_0}{2\pi} = \frac{d\omega}{2\pi}$$

Now, let the function c_n of eqn (2) with T be denoted as $Tc_n = F(\omega)$, a function of ω . Then the eqn becomes

$$c_n T = \int_{-T/2}^{T/2} f_T(t) e^{-jn\omega_0 t} dt$$

as $T \rightarrow \infty$

$$F(\omega) = C_n T = \lim_{T \rightarrow \infty} \int_{-T/2}^{T/2} f_T(t) e^{j\omega t} dt$$

or

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{j\omega t} dt \quad \text{--- (3)}$$

above equation (3) is a function of ω and represents the Fourier transform of $f(t)$.

From equation (1)

$$f_T(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega t}$$

divide & multiplying by T on right hand side

$$f_T(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} T C_n e^{jn\omega t}$$

and by replacing $C_n T$ by the function $F(\omega)$, the summation becomes an integral. therefore,

$$f_T(t) = \int_{-\infty}^{\infty} F(\omega) \frac{1}{T} e^{j\omega t} d\omega$$

as $T \rightarrow \infty$ the function $f_T(t) \rightarrow f(t)$. thus

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

The Fourier ~~pair~~ transform ^{pair} can be written as

$$F(\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$f(t) = \mathcal{F}^{-1}[F(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \quad \text{Inverse F.T.}$$

The Fourier transform $F(\omega)$ is in general, complex-valued and can be expressed as

$$F(\omega) = |F(\omega)| e^{i\phi(\omega)}$$

If $f(t)$ is real

$$F(-\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt = F^*(\omega)$$

$$\text{Thus } |F(-\omega)| = |F^*(\omega)|$$

$$\text{and } \phi(-\omega) = -\phi(\omega).$$

Therefore, for real signals the amplitude spectrum $|F(\omega)|$ is an even function and the phase spectrum $\phi(\omega)$ is an odd function of ω .

The Fourier transform pair can also be represented with the continuous frequency f Hz instead of ω rad/sec by substituting $\omega = 2\pi f$. That is,

$$F(f) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi f t} dt$$

$$f(t) = \int_{-\infty}^{\infty} F(f) e^{j2\pi f t} df$$

Existence of a Fourier transform:

Dirichlet's conditions:

- ① The function $f(t)$ is a single-valued function of the variable t within the interval (t_1, t_2)
- 2) The function $f(t)$ has a finite number of discontinuities in the interval (t_1, t_2)

- 3) The function $f(t)$ has a finite number of maxima and minima in the interval (t_1, t_2) .
- 4) The function $f(t)$ is absolutely integrable i.e.,

$$\int_{t_1}^{t_2} |f(t)| dt < \infty$$

Fourier Transform of Standard Signals:

Ex - 1: Find the Fourier transform of a single-sided exponential function $e^{-bt} u(t)$ shown in fig. 1. And draw the spectrum where $u(t)$ is unit step fun.

Solution: Given function $f(t) = e^{-bt} u(t)$, $b > 0$,

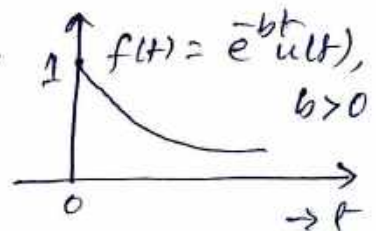


Fig. 1.

The Fourier transform of a function $f(t)$ is given by

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$F(\omega) = \int_{-\infty}^0 f(t) e^{-j\omega t} dt + \int_0^{\infty} f(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^0 e^{-bt} u(t) e^{-j\omega t} dt + \int_0^{\infty} e^{-bt} u(t) e^{-j\omega t} dt$$

$$= 0 + \int_0^{\infty} e^{-bt} e^{-j\omega t} dt = \int_0^{\infty} e^{-(b+j\omega)t} dt$$

$$= \frac{e^{-(b+j\omega)t}}{-(b+j\omega)} \Big|_0^{\infty} = \frac{1}{b+j\omega}$$

[$\because u(t)$ defined for $t > 0$]

$$F(\omega) = \frac{b-j\omega}{b^2+\omega^2} = \frac{b}{b^2+\omega^2} - \frac{j\omega}{b^2+\omega^2}$$

The complex form (magnitude & phase) of F.T

(39)

$$F(\omega) = |F(\omega)| e^{j\phi(\omega)}$$

$$|F(\omega)| = \sqrt{\left(\frac{b}{b^2 + \omega^2}\right)^2 + \left(\frac{-\omega}{b^2 + \omega^2}\right)^2} = \sqrt{\frac{(b^2 + \omega^2)}{(b^2 + \omega^2)^2}} = \frac{1}{\sqrt{b^2 + \omega^2}}$$

$$\phi(\omega) = \tan^{-1}\left(\frac{-\omega}{\frac{b}{b^2 + \omega^2}}\right) = -\tan^{-1}\left(\frac{\omega}{b}\right)$$

$$F(\omega) = \frac{1}{\sqrt{b^2 + \omega^2}} e^{-j \tan^{-1}\left(\frac{\omega}{b}\right)}$$

Spectrum: The magnitude and phase spectrum are shown in Fig. 2. Note that F.T exists only for the positive value of b , and it is not absolutely integrable for the negative value of b .

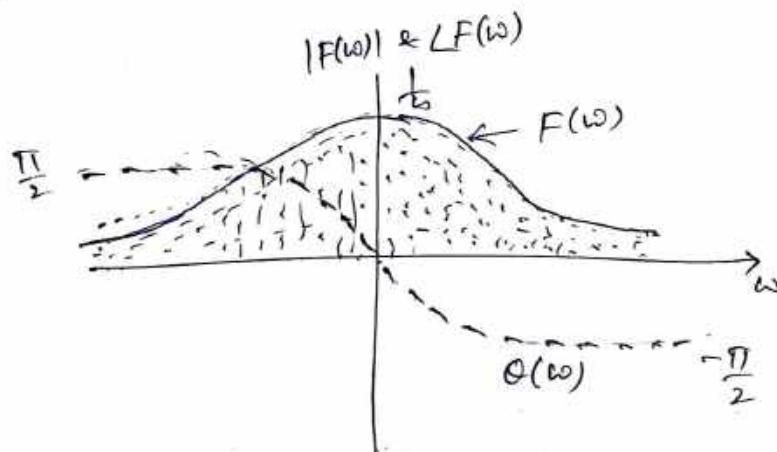


Fig. 2. F.T spectrum of single sided exp. signal.

The magnitude spectrum has a value $\frac{1}{b}$ at $\omega=0$, it decreases as ω increases. Similarly, the phase spectrum varies between $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ depending on the

values $\tan^{-1}(\frac{\omega}{b})$ at different frequencies. (40)

Since F. transform is complex, both plots are necessary.

Ex-2: Find the Fourier transform of a double-sided exponential signal $e^{-b|t|}$ shown in Fig. 1 and draw the spectrum.

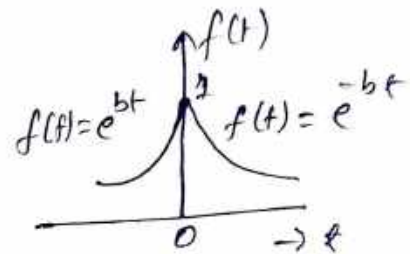


Fig. 1.

Sol: Given function $f(t) = e^{-b|t|}$

Fourier transform of given function

$f(t)$ is

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$F(\omega) = \int_{-\infty}^{\infty} e^{-b|t|} e^{-j\omega t} dt$$

For convenience, the integration can be split into two parts. Thus,

$$F(\omega) = \int_{-\infty}^0 e^{-b(-t)} e^{-j\omega t} dt + \int_0^{\infty} e^{-b(t)} e^{-j\omega t} dt$$

$$F(\omega) = \int_{-\infty}^0 e^{(b-j\omega)t} dt + \int_0^{\infty} e^{-(b+j\omega)t} dt$$

$$= \frac{e^{(b-j\omega)t}}{b-j\omega} \Big|_{-\infty}^0 + \frac{e^{-(b+j\omega)t}}{-(b+j\omega)} \Big|_0^{\infty}$$

$$= \frac{1}{b-j\omega} + \frac{1}{b+j\omega} = \frac{2b}{b^2 + \omega^2}$$

$$\boxed{F(\omega) = \frac{2b}{b^2 + \omega^2}}$$

spectrum:

$$|F(\omega)| = \frac{2b}{(\omega^2 + b^2)}$$

(41)
[∵ Imaginary part is zero]

$$\phi(\omega) = 0$$

$$\phi(\omega) = 0$$

As the $F(\omega)$ is real, only magnitude plot is desired

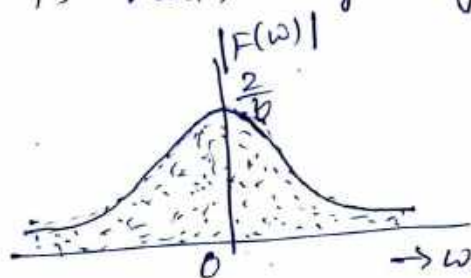


Fig. 2 Frequency domain of given function

Ex-3 Find the Fourier transform of a Gate function of amplitude A and width τ as shown in Fig. 1.

Sol: A Gate function $G_\tau(t)$ is a rectangular pulse defined by

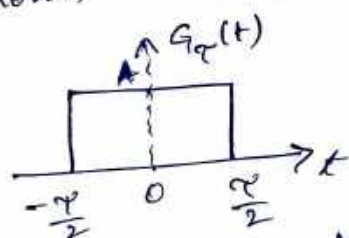


Fig. 1. Gate pulse.

$$f(t) = \begin{cases} A & |t| < \frac{\tau}{2} \\ 0 & \text{elsewhere} \end{cases}$$

$|t|$ stands for positive as well as negative values of t .

The Fourier transform of this function is obtained as follows.

$$F(\omega) = \int_{-\tau/2}^{\tau/2} A e^{-j\omega t} dt = \frac{A}{-j\omega} \left[e^{-j\omega t} \right]_{-\tau/2}^{\tau/2}$$

$$= \frac{A}{-j\omega} \left[e^{-j\omega \frac{\tau}{2}} - e^{j\omega \frac{\tau}{2}} \right] = \frac{2A}{\omega} \left[\frac{e^{j\omega \frac{\tau}{2}} - e^{-j\omega \frac{\tau}{2}}}{2j} \right]$$

$$= \frac{A}{\omega/2} \sin(\omega \frac{\tau}{2})$$

Multiplying & dividing by τ we get

$$F(\omega) = A\tau \frac{\sin(\frac{\omega\tau}{2})}{\frac{\omega\tau}{2}} = A\tau \text{Sa}(\frac{\omega\tau}{2})$$

(42)

$$F(\omega) = A\tau \text{Sa}(\frac{\omega\tau}{2}) \quad \left[\because \text{Sa}(x) = \frac{\sin x}{x} \right]$$

$$A G_{\tau}(t) \xrightarrow{\text{FT}} A\tau \text{Sa}(\frac{\omega\tau}{2})$$

Spectrum: Here $F(\omega)$ is real function, and hence, only amplitude plot is sufficient.

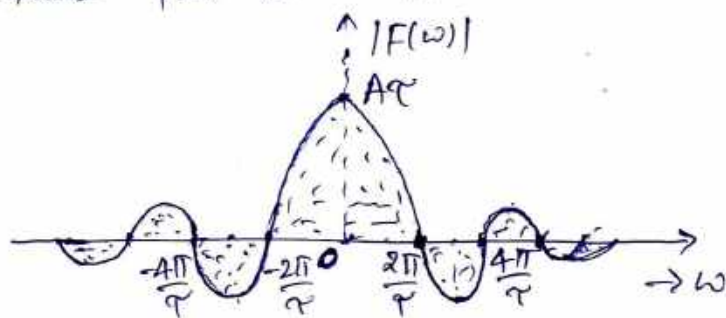


Fig-2. Frequency spectrum of given function

Ex-4: Find the Fourier transform of an impulse function $f(t) = \delta(t)$

Sol: The Fourier transform of $f(t)$ from basic definition is

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$F(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt$$

using sifting property of impulse function

$$F(\omega) = e^{-j\omega(0)} = 1. \quad \left[\begin{array}{l} \text{sifting property} \\ \int_{t_1}^{t_2} f(t) \delta(t-t_0) dt \end{array} \right]$$

$$= \begin{cases} f(t_0); & t_1 < t_0 < t_2 \\ 0; & \text{otherwise} \end{cases}$$

$$\boxed{F\{\delta(t)\} \longleftrightarrow 1}$$

or

$$\boxed{\delta(t) \xrightarrow{\text{FT}} 1}$$

The F.T of an impulse function is unity. (43)

The below Fig. clearly shows that an impulse function contains the entire frequency component $(-\infty, \infty)$ of identical magnitude and the same phase, i.e., the bandwidth of the delta function is infinite.

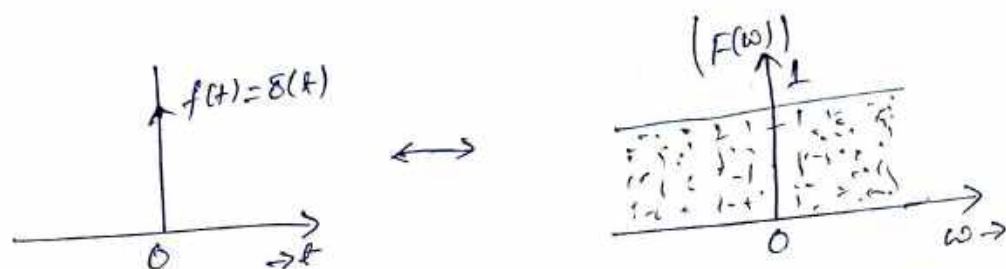
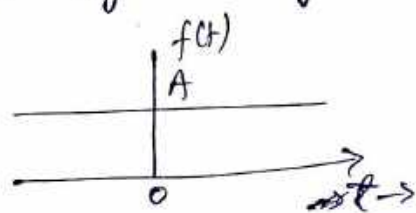


Fig. An impulse function & its spectrum.

Ex-5: Find the Fourier transform of a constant function $f(t) = A$.

Sol: A constant signal may be drawn as below



The Fourier transform of $f(t)$ is given by

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$F(\omega) = \int_{-\infty}^{\infty} A e^{-j\omega t} dt$$

This function is not absolutely integrable, but its F.T can be determined using the concept of delta function

$$\left. \begin{array}{l} \text{put } t = x \\ t = -x \\ dt = -dx \end{array} \right\} \begin{array}{l} t = -\infty \Rightarrow x = +\infty \\ t = +\infty \Rightarrow x = -\infty \end{array}$$

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$$F(\omega) = \int_{-\infty}^{\infty} A e^{j\omega x} (-dx)$$

$$= -A \int_{\infty}^{-\infty} e^{j\omega x} dx$$

Interchanging limits.

$$F(\omega) = A \int_{-\infty}^{\infty} e^{j\omega x} dx$$

$$= 2\pi A \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega x} dx = 2\pi A \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j2\omega} dx$$

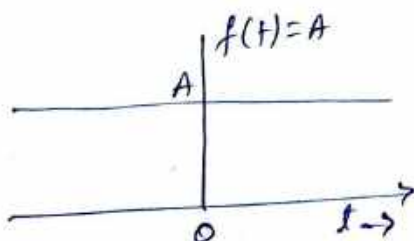
compare above with inverse F.T i.e.,

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \quad [\because \delta(t) \longleftrightarrow 1]$$

$$F(\omega) = 2\pi A \delta(\omega)$$

[Assume x plays ω role
 ω plays t role]

$$\boxed{A \longleftrightarrow 2\pi A \delta(\omega)}$$



$\longleftrightarrow$

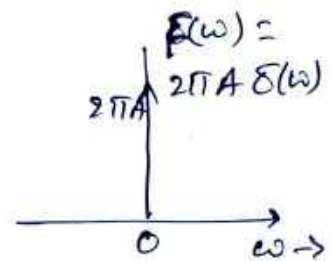


Fig. constant signal (dc) & its F.T spectrum.

Ex-6 Find the F.T of Signum functions denoted by $\text{sgn}(t)$.

Sol: The ^{Signum} function shown in fig is expressed as

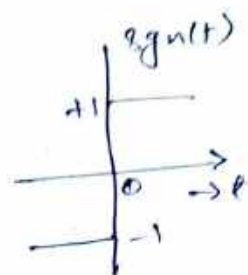


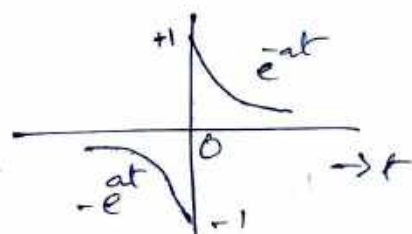
Fig. Signum fun.

$$\text{sgn}(t) = \begin{cases} +1, & t > 0 \\ -1, & t < 0 \end{cases}$$

The function is not absolutely integrable, we can obtain F.T using ^{unit} step function.

The function can also be defined in terms of unit step function as

$$\text{sgn}(t) = u(t) - u(-t)$$



and it can be modified as

$$\text{sgn}(t) = \lim_{a \rightarrow 0} [e^{-at} u(t) - e^{at} u(-t)]$$

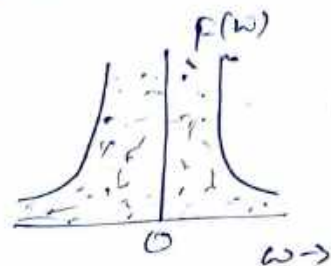
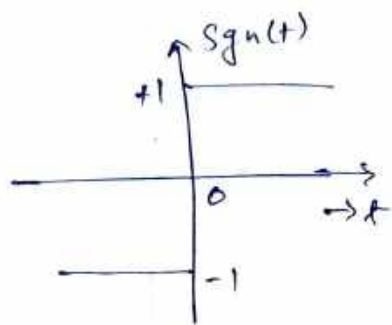
$$F[\text{sgn}(t)] = \lim_{a \rightarrow 0} \left[\int_0^{\infty} e^{-at} e^{-j\omega t} dt - \int_{-\infty}^0 e^{at} e^{-j\omega t} dt \right]$$

$$= \lim_{a \rightarrow 0} \left[\int_0^{\infty} e^{-(a+j\omega)t} dt - \int_{-\infty}^0 e^{(a-j\omega)t} dt \right]$$

$$= \lim_{a \rightarrow 0} \left[\frac{-2j\omega}{a^2 + \omega^2} \right]$$

$$= \frac{2}{j\omega}$$

$$\boxed{\text{sgn}(t) \longleftrightarrow \frac{2}{j\omega}}$$



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Ex-7. Find a Fourier transform of a unit step function shown in Fig.

Sol: The step function shown in fig. is expressed

as

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

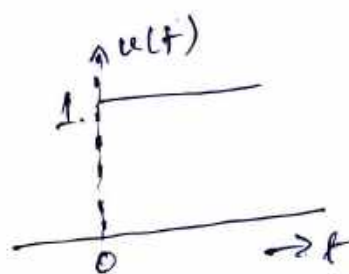


Fig. A unit step function.

The Fourier transform of $f(t)$ is given by

$$F\{f(t)\} = F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

And the F.T of a unit step function is

$$F\{u(t)\} = \int_{-\infty}^{\infty} u(t) e^{-j\omega t} dt.$$

Here unit step function is not absolutely integrable, But we can derive F-T using signum function and constant.

We know that $\text{sgn}(t) = 2u(t) - 1$

$$u(t) = \frac{1}{2} + \frac{1}{2} \text{sgn}(t).$$

Its Fourier transform is

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$$F\{u(t)\} = \frac{1}{2}F\{1\} + \frac{1}{2}F\{\text{sgn}(t)\}$$

$$\text{But } F\{1\} = 2\pi\delta(\omega) \text{ and } F\{\text{sgn}(t)\} = \frac{2}{j\omega}$$

hence

$$F\{u(t)\} = \frac{1}{2} \times 2\pi\delta(\omega) + \frac{1}{2} \times \frac{2}{j\omega}$$

$$F\{u(t)\} = \pi\delta(\omega) + \frac{1}{j\omega}$$

$$u(t) \longleftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$

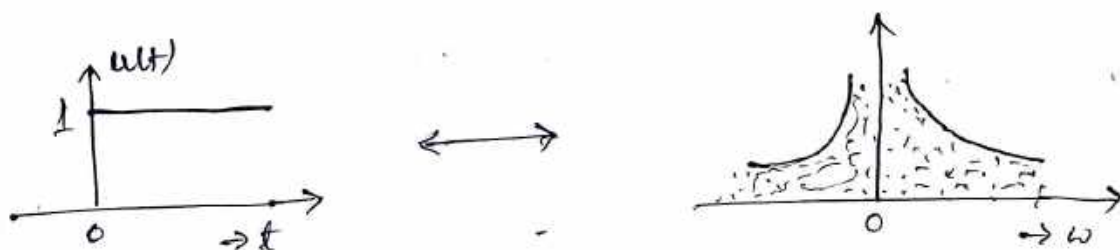


Fig. A unit step function & its spectrum.

Ex-8) Find the Fourier transform of $f(t) = te^{-at}u(t)$.

Sol:

ans: $\frac{1}{(a+j\omega)^2}$

properties of Fourier Transform

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The Fourier transform properties help in finding the effect of various time domain operations on frequency domain.

1. Linearity property:

Definition: If $f_1(t) \longleftrightarrow F_1(\omega)$ &
 $f_2(t) \longleftrightarrow F_2(\omega)$

Then $a_1 f_1(t) + a_2 f_2(t) \longleftrightarrow a_1 F_1(\omega) + a_2 F_2(\omega)$

where a_1 & a_2 are arbitrary constants.

proof: consider two signals $f_1(t)$ and $f_2(t)$ with Fourier transforms $F_1(\omega)$ and $F_2(\omega)$ respectively.

$$\begin{aligned} \text{Then } F\{a_1 f_1(t) + a_2 f_2(t)\} &= \int_{-\infty}^{\infty} [a_1 f_1(t) + a_2 f_2(t)] e^{j\omega t} dt \\ &= \int_{-\infty}^{\infty} a_1 f_1(t) e^{j\omega t} dt + \int_{-\infty}^{\infty} a_2 f_2(t) e^{j\omega t} dt \\ &= a_1 \int_{-\infty}^{\infty} f_1(t) e^{j\omega t} dt + a_2 \int_{-\infty}^{\infty} f_2(t) e^{j\omega t} dt \\ &= a_1 F_1(\omega) + a_2 F_2(\omega) \end{aligned}$$

$$\boxed{\therefore a_1 f_1(t) + a_2 f_2(t) \longleftrightarrow a_1 F_1(\omega) + a_2 F_2(\omega)}$$

In general

$$a_1 f_1(t) + a_2 f_2(t) + a_3 f_3(t) + \dots + a_n f_n(t) \longleftrightarrow a_1 F_1(\omega) + a_2 F_2(\omega) + \dots + a_n F_n(\omega)$$

It is also called as Superposition property.

2. Time scaling property:

49

Definition: If $f(t) \leftrightarrow F(\omega)$

$$\text{Then } f(at) \leftrightarrow \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

where a is a real constant.

proof: The Fourier transform of $f(at)$ is

$$F\{f(at)\} = \int_{-\infty}^{\infty} f(at) e^{-j\omega t} dt$$

$$\text{Let } at = x \Rightarrow a \frac{dt}{dx} = 1 \quad \left/ \quad \begin{array}{l} t = \infty \Rightarrow x = \infty \\ t = -\infty \Rightarrow x = -\infty \end{array} \right.$$
$$dt = \frac{dx}{a}$$

case i: $a > 0$,

$$\begin{aligned} F\{f(at)\} &= \int_{-\infty}^{\infty} f(x) e^{-j\omega \left(\frac{x}{a}\right)} \frac{dx}{a} \\ &= \frac{1}{a} \int_{-\infty}^{\infty} f(x) e^{-j\left(\frac{\omega}{a}\right)x} dx \\ &= \frac{1}{a} F\left(\frac{\omega}{a}\right) \end{aligned}$$

case ii: $a < 0$,

$$F\{f(at)\} = \int_{-\infty}^{\infty} f(at) e^{-j\omega t} dt$$

$$\text{Let } -at = x \Rightarrow -a \frac{dt}{dx} = 1 \quad \left/ \quad \begin{array}{l} t = +\infty, x = -\infty \\ t = -\infty, x = +\infty \end{array} \right.$$
$$dt = -\frac{dx}{a}$$

$$\begin{aligned} F\{f(at)\} &= \int_{\infty}^{-\infty} f(x) e^{-j\omega \left(-\frac{x}{a}\right)} \left(-\frac{dx}{a}\right) \\ &= -\frac{1}{a} \int_{\infty}^{-\infty} f(x) e^{-j\left(-\frac{\omega}{a}\right)x} dx \end{aligned}$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} f(x) e^{-j(-\frac{\omega}{a})x} dx \quad \left[\because \int_a^b x dx = - \int_b^a x dx \right]$$

$$= \frac{1}{a} F\left(-\frac{\omega}{a}\right)$$

From above two cases

$$f(at) \longleftrightarrow \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

significance: compression of a function in the time domain by a factor 'a' is equivalent to expansion of its frequency domain by the same factor 'a' or vice-versa.

3. Duality ~~and symmetry~~ property:

statement: If $f(t) \longleftrightarrow F(\omega)$

then $F(t) \longleftrightarrow 2\pi f(-\omega)$

or
 $F(t) \longleftrightarrow f(-f)$

proof: The inverse Fourier transform of $F(\omega)$ is

given by

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

Therefore

$$2\pi f(t) = \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

put $t = -t$

$$2\pi f(-t) = \int_{-\infty}^{\infty} F(\omega) e^{-j\omega t} d\omega$$

(51)

By interchanging the variables t and ω i.e.,

$$2\pi f(-\omega) = \int_{-\infty}^{\infty} F(t) e^{-j\omega t} dt$$

$$= F[F(t)] \quad \left[\because F[f(t)] = \int_{-\infty}^{\infty} f(t) e^{j\omega t} dt \right]$$

Thus the Fourier transform of the time function

$F(t)$ is $2\pi f(-\omega)$

$$\boxed{F(t) \longleftrightarrow 2\pi f(-\omega)}$$

For an even function

$$f(-\omega) = f(\omega)$$

and hence

$$\boxed{F(t) \longleftrightarrow 2\pi f(\omega)}$$

Ex-9 Find the inverse Fourier transform of

a) $\text{sgn}(\omega)$ b) $u(\omega)$

Sol: (a)

$$\text{sgn}(t) \longleftrightarrow \frac{2}{j\omega}$$

or

$$\frac{2}{j\omega} \longleftrightarrow \text{sgn}(t)$$

By duality property

replace t by $-t$

(52)

$$\frac{2}{j\omega} \leftrightarrow \text{sgn}(-t)$$

replace by interchanging
 t and ω

$$\frac{2}{j\omega} \leftrightarrow \text{sgn}(-\omega)$$

$$\frac{2}{j\omega} \leftrightarrow 2\pi \text{sgn}(-\omega)$$

[$\because$ By duality property
 $F(t) \leftrightarrow 2\pi f(\omega)$]

$$\frac{2}{j\omega} \leftrightarrow -\pi \text{sgn}(\omega)$$

[$\because$ signum function
is odd function]

$$\frac{j}{\pi t} \leftrightarrow \text{sgn}(\omega)$$

$$\boxed{\text{sgn}(\omega) \leftrightarrow \frac{j}{\pi t}}$$

(b) $u(\omega)$ can be written as

$$u(\omega) = \frac{1}{2} + \frac{1}{2} \text{sgn}(\omega) \quad [\because \text{sgn}(t) = 2u(t) - 1]$$

by taking inverse F.T

$$\begin{aligned} F^{-1}\{u(\omega)\} &= F^{-1}\left\{\frac{1}{2}\right\} + F^{-1}\left\{\frac{1}{2} \text{sgn}(\omega)\right\} \\ &= \frac{1}{2} F^{-1}\{1\} + \frac{1}{2} F^{-1}\{\text{sgn}(\omega)\} \\ &= \frac{1}{2} \delta(t) + \frac{1}{2} \left[\frac{j}{\pi t}\right] \\ &= \frac{1}{2} \delta(t) + \frac{j}{2\pi t} \end{aligned}$$

$$u(\omega) \leftrightarrow \frac{1}{2} \delta(t) + \frac{j}{2\pi t}$$

④ Time shifting property:

statement: If $f(t) \longleftrightarrow F(\omega)$

$$\text{Then } f(t-b) \longleftrightarrow F(\omega) e^{-j\omega b}$$

proof:

the Fourier transform of $f(t-b)$ is given by

$$F\{f(t-b)\} = \int_{-\infty}^{\infty} f(t-b) e^{-j\omega t} dt$$

put $t-b = y$,

$$\Rightarrow t = y+b \quad \left/ \quad \begin{array}{l} t = \infty \Rightarrow y = \infty \\ t = -\infty \Rightarrow y = -\infty \end{array} \right.$$

$$dt = dy$$

$$\begin{aligned} F\{f(y)\} &= \int_{-\infty}^{\infty} f(y) e^{-j\omega(y+b)} dy \\ &= \int_{-\infty}^{\infty} f(y) \cdot e^{-j\omega y} \cdot e^{-j\omega b} dy \end{aligned}$$

$$= e^{-j\omega b} \int_{-\infty}^{\infty} f(y) e^{-j\omega y} dy \quad \left[\because \text{integration w.r.t } y \right]$$

* Since y is a dummy variable

$$\boxed{f(t-b) \longleftrightarrow F(\omega) e^{-j\omega b}}$$

Significance: A shift in the time domain by an amount b is equivalent to multiplication

by $e^{-j\omega b}$ in the frequency domain. i.e., magnitude spectrum ~~is~~ ^{remains} unchanged, but phase spectrum is

(34)

changed by $- \omega b$.

Note:

Ex-(10) Find the Fourier transform of pulse function

$f(t)$ shown in Fig. 1.

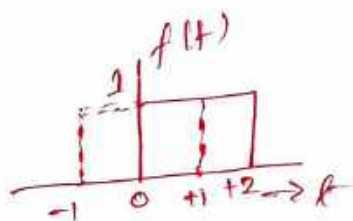


Fig. 1.

Sol: The function $f(t)$ shown in Fig. 1 is a Gate function $G_{\tau}(t)$, shifted by $+1$ sec. (where $G_{\tau}(t)$ is identified by dotted lines)

The Fourier transform of $AG_{\tau}(t)$ is given by

$$AG_{\tau}(t) \longleftrightarrow A \tau \text{Sa}\left(\frac{\omega \tau}{2}\right)$$

Here $A=1$, $\tau=2$ sec

$$\text{Hence } G_2(t) \longleftrightarrow 1 \times 2 \times \text{Sa}\left(\frac{\omega \times 2}{2}\right)$$

$$G_2(t) \longleftrightarrow 2 \text{Sa}(\omega)$$

Since ~~But~~ $f(t) = G_2(t-1)$

~~using time shifting property~~

$$F\{f(t)\} = F\{G_2(t-1)\}$$

using time shifting property

$$F\{g_2(t-1)\} = F\{g(t)\} e^{-j\omega \times 1}$$

(55)

$$= 2 \text{sinc}(\omega) \cdot e^{-j\omega}$$

$$\therefore f(t-b) \leftrightarrow F(\omega) e^{-j\omega b}$$

$$\boxed{f(t) \leftrightarrow 2 \text{sinc}(\omega) e^{-j\omega}}$$

5) Frequency shifting (or) Frequency translation property.

Statement: If $f(t) \xleftrightarrow{F.T} F(\omega)$
 then $e^{j\omega_c t} f(t) \xleftrightarrow{F.T} F(\omega - \omega_c)$

proof: The Fourier transform of $f(t)$ is given by
 definition

$$F\{f(t)\} = F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

therefore

$$\begin{aligned} F\{e^{j\omega_c t} f(t)\} &= \int_{-\infty}^{\infty} e^{j\omega_c t} f(t) e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} f(t) e^{-j(\omega - \omega_c)t} dt \end{aligned}$$

$$\therefore \boxed{e^{j\omega_c t} f(t) \leftrightarrow F(\omega - \omega_c)}$$

significance: According to this property, multiplication of a function $f(t)$ by $e^{j\omega_c t}$ is equivalent to shifting its F.T $F(\omega)$ in the positive direction by an amount of ω_c . i.e., the spectrum is translated

by an amount ω_c . Therefore this theorem (property) is also known as frequency translation theorem. (56)

Note: $F\{f(t) \cos \omega_c t\} = \frac{1}{2} [F(\omega + \omega_c) + F(\omega - \omega_c)]$
 $F\{f(t) \sin \omega_c t\} = \frac{j}{2} [F(\omega + \omega_c) - F(\omega - \omega_c)]$

Ex-11 Find the Fourier transform of Eternal sinusoids

a) $\cos \omega_c t$ b) $\sin \omega_c t$ shown in Fig 1.(a) & (b)

respectively.

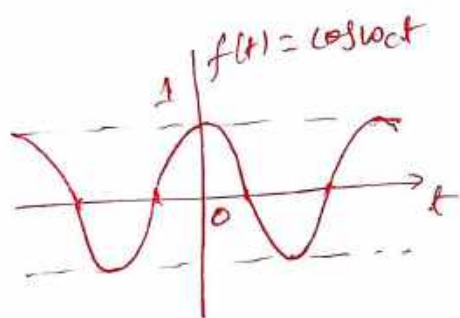


Fig 1.(a)

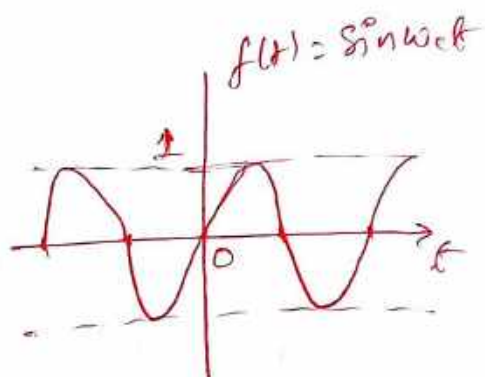


Fig 1.(b)

Sol:

(a) $f(t) = \cos \omega_c t$.

using Euler's identity to express the cosine function $f(t) = \cos \omega_c t = \frac{e^{j\omega_c t} + e^{-j\omega_c t}}{2}$

Note that eternal sinusoid exists forever $(-\infty, \infty)$.

By applying F.T

$$F\{f(t)\} = F\{\cos \omega_c t\} = F\left\{\frac{e^{j\omega_c t} + e^{-j\omega_c t}}{2}\right\}$$

By linearity property of the F.T

we have

$$F[\cos \omega_c t] = F\left[\frac{1}{2} e^{j\omega_c t}\right] + F\left[\frac{1}{2} e^{-j\omega_c t}\right]$$

(57)

above expression is in the form of $F[f(t) e^{j\omega_c t}]$

by frequency shifting property

$$e^{j\omega_c t} f(t) \longleftrightarrow F(\omega - \omega_c)$$

and Fourier transform of a constant

$$F[A] = 2\pi A \delta(\omega)$$

and

$$F[\cos \omega_c t] = \frac{1}{2} 2\pi \delta(\omega - \omega_c) + \frac{1}{2} 2\pi \delta(\omega + \omega_c)$$

$$\boxed{\cos \omega_c t \longleftrightarrow \pi \delta(\omega + \omega_c) + \pi \delta(\omega - \omega_c)}$$

Spectrum:

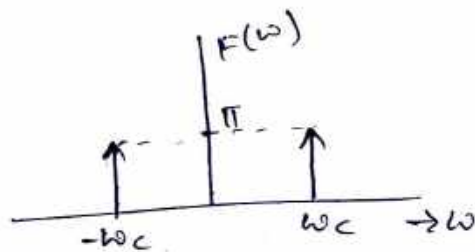


Fig 2

b)

$$\sin \omega_c t = \frac{1}{2j} (e^{j\omega_c t} - e^{-j\omega_c t})$$

using linearity property of F-T

$$\begin{aligned} F[\sin \omega_c t] &= F\left[\frac{1}{2j} e^{j\omega_c t}\right] - F\left[\frac{1}{2j} e^{-j\omega_c t}\right] \\ &= \frac{1}{j} [F[\frac{1}{2} e^{j\omega_c t}] - F[\frac{1}{2} e^{-j\omega_c t}]] \end{aligned}$$

using frequency shifting property

$$F[\sin \omega_c t] = \frac{1}{j} [\pi \delta(\omega - \omega_c) - \pi \delta(\omega + \omega_c)]$$

$$= j\pi [\delta(\omega + \omega_c) - \delta(\omega - \omega_c)]$$

$$\boxed{\sin \omega_c t \leftrightarrow j\pi \delta(\omega + \omega_c) - j\pi \delta(\omega - \omega_c)}$$

Spectrum:

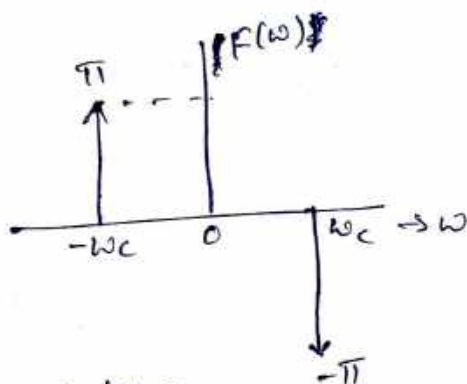
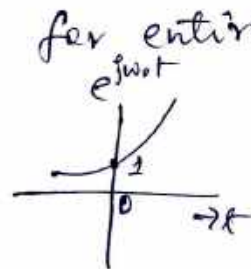


Fig. 3.

Ex-12:

Find the Fourier transform of an ~~external~~ Exponential signal $e^{j\omega_0 t}$.

not absolutely integrable
Sol: The ~~external~~ exponential exists for entire period $(-\infty, \infty)$.



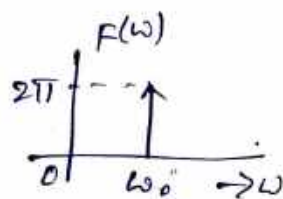
$$F[e^{j\omega_0 t}] = F[1 \cdot e^{j\omega_0 t}]$$

The Fourier transform of a constant 1 is given by $2\pi \delta(\omega)$ and ~~the~~ ^{using} frequency shifting property

we have

$$F[e^{j\omega_0 t}] = 2\pi \delta(\omega - \omega_0)$$

$$e^{j\omega_0 t} \leftrightarrow 2\pi \delta(\omega - \omega_0)$$



6) (a) Area under the curve (wave form) or function.

(59)

Statement: If $f(t) \leftrightarrow F(\omega)$

$$\text{then } \int_{-\infty}^{\infty} f(t) dt = F(0)$$

meaning: Area under a function $f(t)$ is equal to the value of its Fourier transform $F(\omega)$ at $\omega=0$.

proof: The Fourier transform of function $f(t)$

is given by

$$F[f(t)] = F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

put $\omega=0$
then $\boxed{F(0) = \int_{-\infty}^{\infty} f(t) dt}$

or

$$\boxed{\int_{-\infty}^{\infty} f(t) dt = F(0)}$$

(6)

(b) Area under the Fourier transform.

If $f(t) \leftrightarrow F(\omega)$

$$\text{then } \int_{-\infty}^{\infty} F(\omega) d\omega = 2\pi f(0).$$

meaning: Area under the Fourier transform $F(\omega)$ of a function $f(t)$ is equal to the value of product of 2π and the function $f(t)$ at $t=0$.

proof: Inverse Fourier transform of $F(\omega)$ is given by

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

(60)

put $t = 0$

$$f(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) d\omega$$

$$\boxed{\int_{-\infty}^{\infty} F(\omega) d\omega = 2\pi \cdot f(0)}$$

7) Time convolution property.

$$\text{If } f(t) \longleftrightarrow F(\omega) \text{ \& } g(t) \longleftrightarrow G(\omega)$$

$$g(t) \longleftrightarrow G(\omega)$$

$$\text{Then } f(t) * g(t) \longleftrightarrow F(\omega) \cdot G(\omega)$$

Meaning: A convolution operation is transformed to modulation in frequency domain (multiplication).

Proof: The Fourier transform of $f(t) * g(t)$ is

$$F[f(t) * g(t)] = \int_{-\infty}^{\infty} [f(t) * g(t)] e^{j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau \right] e^{j\omega t} dt$$

$$= \int_{-\infty}^{\infty} f(\tau) \left[\int_{-\infty}^{\infty} g(t-\tau) e^{j\omega t} dt \right] d\tau$$

$$\because f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau$$

$$= \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau$$

put $t - \tau = \alpha \Rightarrow t = \alpha + \tau$
 $dt = d\alpha$

$$= \int_{-\infty}^{\infty} f(\tau) \left[\int_{-\infty}^{\infty} g(\alpha) e^{j\omega(\tau+\alpha)} d\alpha \right] d\tau$$

$$= \int_{-\infty}^{\infty} f(\tau) e^{j\omega\tau} d\tau \int_{-\infty}^{\infty} g(x) e^{-j\omega x} dx$$

(6)

$$F[f(t) * g(t)] = F(\omega) \cdot G(\omega) \quad \left[\because F[f(t)] = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \right]$$

$$\boxed{f(t) * g(t) \longleftrightarrow F(\omega) \cdot G(\omega)}$$

8) Frequency convolution (or) multiplication property

$$If \ f(t) \longleftrightarrow F(\omega)$$

$$g(t) \longleftrightarrow G(\omega)$$

$$Then \ f(t) \cdot g(t) \longleftrightarrow \frac{1}{2\pi} [F(\omega) * G(\omega)]$$

Meaning: multiplication in time domain corresponds to convolution of spectrums in frequency domain.

proof: The Fourier transform of $f(t) \cdot g(t)$ is given by

$$F[f(t) \cdot g(t)] = \int_{-\infty}^{\infty} f(t) g(t) e^{-j\omega t} dt \quad \text{--- (1)}$$

Inverse Fourier transform states that

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \quad \text{--- (2)}$$

Substituting eqn (2) in eqn (1)

$$and \ F[f(t) \cdot g(t)] = \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \ g(t) e^{-j\omega t} dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \left[\int_{-\infty}^{\infty} e^{j\omega t} g(t) e^{-j\omega t} dt \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) G(\omega - \omega) d\omega \quad \left[\because e^{j\omega t} f(t) \longleftrightarrow F(\omega - \omega_c) \right]$$

$$F[f(t) \cdot g(t)] = \frac{1}{2\pi} [F(\omega) * G(\omega)]$$

(62)

$$[\because f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau]$$

$$\therefore \boxed{f(t) g(t) \longleftrightarrow \frac{1}{2\pi} [F(\omega) * G(\omega)]}$$

⑨ Integration property:

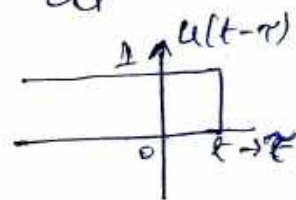
$$\text{If } f(t) \longleftrightarrow F(\omega)$$

$$\text{then } \int_{-\infty}^{\infty} f(\tau) d\tau \longleftrightarrow \pi F(0) \delta(\omega) + \frac{F(\omega)}{j\omega}$$

proof: The Fourier transform of $\int_{-\infty}^t f(\tau) d\tau$ is given by

$$F\left[\int_{-\infty}^t f(\tau) d\tau\right] = \int_{-\infty}^{\infty} \left[\int_{-\infty}^t f(\tau) d\tau\right] e^{j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau) u(t-\tau) d\tau\right] e^{j\omega t} dt$$



$$u(-\tau+t) = 1, -\infty < \tau < t$$

$$\boxed{= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau) u(t-\tau) d\tau\right] e^{j\omega t} \times e^{j\omega \tau} \times e^{-j\omega \tau} d\tau \times e^{j\omega t} dt}$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau) u(t-\tau) e^{-j\omega(t-\tau)} dt\right] e^{j\omega \tau} d\tau$$

$$= \int_{-\infty}^{\infty} f(\tau) \left[\int_{-\infty}^{\infty} u(t-\tau) e^{-j\omega(t-\tau)} dt\right] e^{j\omega \tau} d\tau$$

$$= \int_{-\infty}^{\infty} f(\tau) \left[\int_{-\infty}^{\infty} u(t-\tau) e^{-j\omega(t-\tau)} dt\right] e^{j\omega \tau} d\tau$$

$$= \int_{-\infty}^{\infty} f(\tau) \left[\pi \delta(\omega) + \frac{1}{j\omega}\right] e^{j\omega \tau} d\tau$$

$$= \left[\pi \delta(\omega) + \frac{1}{j\omega}\right] \int_{-\infty}^{\infty} f(\tau) e^{j\omega \tau} d\tau$$

$$\left[\pi \delta(\omega) + \frac{1}{j\omega} \right] \times F(\omega)$$

(63)

$$\pi F(\omega) \delta(\omega) + \frac{F(\omega)}{j\omega}$$

$$\pi F(0) \delta(\omega) + \frac{F(\omega)}{j\omega}$$

$\therefore$ product property of $\delta(t)$

$$f(t) \delta(t-t_0) = f(t_0) \times \delta(t-t_0)$$

$$\boxed{\int_{-\infty}^t f(\tau) d\tau \longleftrightarrow \pi F(0) \delta(\omega) + \frac{F(\omega)}{j\omega}}$$

(or)

$$F\left[\int_{-\infty}^t f(\tau) d\tau\right] = F\left[\int_{-\infty}^{\infty} f(\tau) u(t-\tau) d\tau\right]$$

$$= F[f(t) * u(t)]$$

$$= F(\omega) \cdot U(\omega) \quad \left[\because f(t) * u(t) = \int_{-\infty}^{\infty} f(\tau) u(t-\tau) d\tau \right]$$

$$\left[\because f(t) * g(t) \longleftrightarrow F(\omega) \cdot \tilde{G}(\omega) \right]$$

$$F\left[\int_{-\infty}^t f(\tau) d\tau\right] = F(\omega) \left[\pi \delta(\omega) + \frac{1}{j\omega} \right]$$

$$\left[\because F[u(t)] = U(\omega) = \pi \delta(\omega) + \frac{1}{j\omega} \right]$$

$$= \pi F(\omega) \delta(\omega) + \frac{F(\omega)}{j\omega}$$

$$= \pi F(0) \delta(\omega) + \frac{F(\omega)}{j\omega}$$

$\therefore$ product property of $\delta(t)$.

$$f(t) \times \delta(t-t_0) = f(t_0) \times \delta(t-t_0)$$

$$\boxed{\int_{-\infty}^t f(\tau) d\tau \longleftrightarrow \pi F(0) \delta(\omega) + \frac{F(\omega)}{j\omega}}$$

⑩ Frequency Differentiation property

(64)

$$\text{If } f(t) \longleftrightarrow F(\omega)$$

$$\text{Then } -j\omega f(t) \longleftrightarrow \frac{d}{d\omega} F(\omega)$$

proof: The Fourier transform of $f(t)$ is given by

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$\begin{aligned} \therefore \frac{d}{d\omega} F(\omega) &= \int_{-\infty}^{\infty} f(t) \frac{d}{d\omega} e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} [-j\omega f(t)] e^{-j\omega t} dt \end{aligned}$$

$$\frac{dF(\omega)}{d\omega} = F[-j\omega f(t)]$$

$$\therefore \boxed{-j\omega f(t) \longleftrightarrow \frac{dF(\omega)}{d\omega}}$$

Ingeneral $\boxed{(-j\omega)^n f(t) \longleftrightarrow \frac{d^n F(\omega)}{d\omega^n}}$

⑪ Time Differentiation property:

$$\text{If } f(t) \longleftrightarrow F(\omega)$$

$$\text{then } \frac{df(t)}{dt} \longleftrightarrow j\omega F(\omega)$$

proof:

Inverse Fourier transform is defined as

$$\mathcal{F}^{-1}[F(\omega)] = f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{+j\omega t} d\omega \quad \text{--- (1)}$$

differentiate above w.r.t to t

$$\frac{df(t)}{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \frac{d}{dt} e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} [F(\omega) \cdot j\omega] e^{j\omega t} d\omega$$

$$= j\omega f(\omega) \quad [\because \text{eqn (1)}]$$

$$\therefore \boxed{\frac{d}{dt} f(t) \longleftrightarrow j\omega f(\omega)}$$

In general:

$$\frac{d^n}{dt^n} f(t) \longleftrightarrow (j\omega)^n F(\omega)$$

Parseval's Theorem for Energy signals
(or)

Rayleigh's Energy theorem.

$$\text{If } f(t) \xleftrightarrow{F.T} F(\omega)$$

$$\text{Then } E = \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

proof: Let E be the energy of the function $f(t)$, & is given by

$$E = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} f(t) \cdot f(t) dt$$

But $f(t)$ inverse Fourier transform of $F(\omega)$ is given by

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

$$\text{Therefore } E = \int_{-\infty}^{\infty} f(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \right] dt$$

By interchanging the order of integration

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \left[\int_{-\infty}^{\infty} f(t) e^{j\omega t} dt \right] d\omega$$

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \cdot F(-\omega) d\omega$$

$$[\because F(\omega) = \int_{-\infty}^{\infty} f(t) e^{j\omega t} dt]$$

For a real function $f(t)$, the Fourier transforms $F(\omega)$ and $F(-\omega)$ are complex conjugates i.e., $F(\omega) = F^*(-\omega)$

and $F(\omega) \cdot F(-\omega) = |F(\omega)|^2$

as magnitude of $F(\omega)$ & $F(-\omega)$ is same. Hence

$$E = \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\text{Area under } |F(\omega)|^2]$$

put $\omega = 2\pi f \Rightarrow d\omega = 2\pi df$

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(2\pi f)|^2 \cdot 2\pi df$$

$$E = \int_{-\infty}^{\infty} |F(f)|^2 df$$

Ex-13:

Show that a normalised Gaussian pulse is its own Fourier transform.
(or)

Find the Fourier transform of function $f(t) = e^{-\pi t^2}$

Sol: Let us consider a Gaussian pulse which is similar to a Gaussian probability density function.

The pulse is shown in Fig 1. may be expressed as $f(t) = e^{-\pi t^2}$ ——— (1)

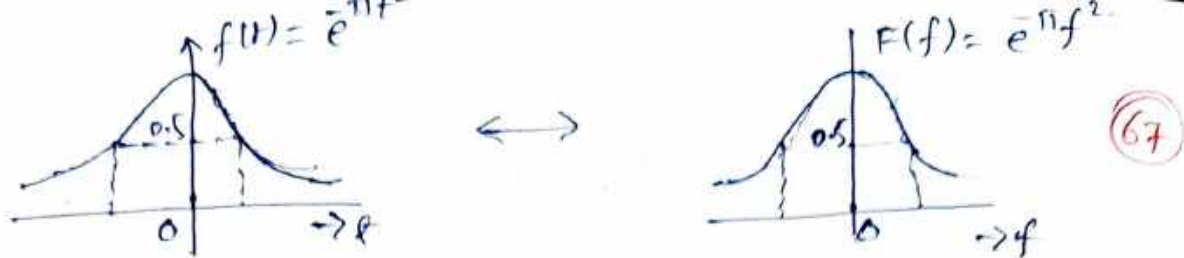


Fig-1. Gaussian function & its Spectrum.

Differentiate eqn (1) w.r.t t , we get,

$$\frac{df(t)}{dt} = -2\pi t e^{-\pi t^2}$$

and $\frac{df(t)}{dt} = -2\pi t f(t) \quad [\because \text{eqn (1)}]$

$$\frac{df(t)}{dt} = 2\pi j [j t f(t)]$$

Multiplying by -1 on both the sides.

$$-\frac{df(t)}{dt} = 2\pi j [-j t f(t)]$$

By taking time differentiation property on the left and frequency differentiation property on the right-side of the above eqn, we get

$$-j\omega F(\omega) = 2\pi j \frac{dF(\omega)}{d\omega} \quad \left[\because \begin{aligned} \frac{d}{dt} f(t) &\leftrightarrow j\omega F(\omega) \\ -j t f(t) &\leftrightarrow \frac{d}{d\omega} F(\omega) \end{aligned} \right]$$

$$-\frac{\omega}{2\pi} F(\omega) = \frac{dF(\omega)}{d\omega}$$

$$\frac{dF(\omega)}{d\omega} + \frac{\omega}{2\pi} F(\omega) = 0$$

The above differential equation yields the solution

$$F(\omega) = 0$$

$$F(\omega) e^{\int \frac{\omega}{2\pi} d\omega} = \int 0 \times e^{\int \frac{\omega}{2\pi} d\omega} + C$$

$$F(\omega) e^{\frac{\omega^2}{4\pi}} = 0 + C$$

put $C = 1$ [normalized]

$$F(\omega) e^{\frac{\omega^2}{4\pi}} = 1$$

$$F(\omega) = e^{-\frac{\omega^2}{4\pi}} = e^{-\frac{(2\pi f)^2}{4\pi}} = e^{-\frac{4\pi^2 f^2}{4\pi}}$$

$$F(f) = e^{-\pi f^2}$$

or

$$F(f) = e^{-\pi f^2} \quad \text{--- (2)}$$

From eqn (1) & eqn (2) it is proved

$$e^{-\pi t^2} \longleftrightarrow e^{-\pi f^2}$$

The Fourier transform of periodic functions:

Let us now find Fourier transform of a periodic function $f(t)$ with period T . The function $f(t)$ can be expressed as in terms of complex Fourier series.

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T}$$

The Fourier transform of both sides yields

$$F[f(t)] = F\left[\sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}\right]$$

(69)

$$= \sum_{n=-\infty}^{\infty} c_n F[1 \cdot e^{jn\omega_0 t}]$$

using frequency shifting property and the Fourier transform of a constant '1'

$$[\because f(t) e^{jn\omega_0 t} \leftrightarrow F(\omega - n\omega_0)]$$

$$1 \leftrightarrow 2\pi \delta(\omega)$$

$$F[f(t)] = \sum_{n=-\infty}^{\infty} c_n 2\pi \delta(\omega - n\omega_0)$$

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t} dt$$

therefore

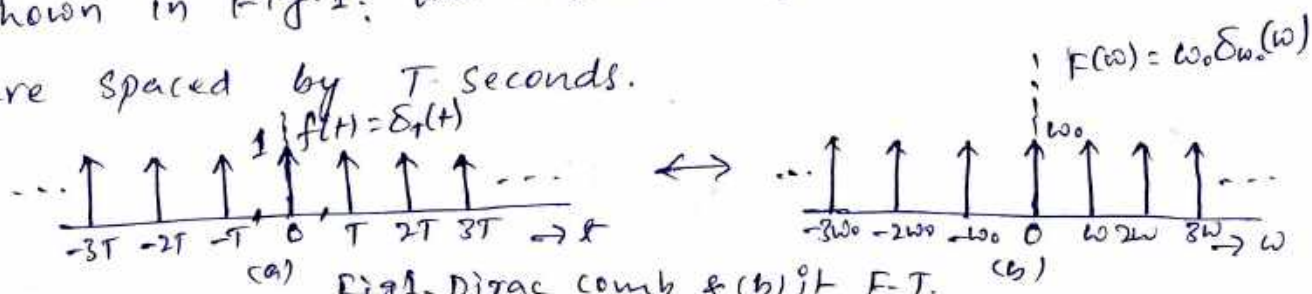
$$F[f(t)] = 2\pi \sum_{n=-\infty}^{\infty} c_n \delta(\omega - n\omega_0)$$

Thus, the Fourier transform of a periodic function consists of a train of equally spaced impulses. These are located at the harmonic frequencies of a signal, and the strength of each pulse is given by as $2\pi c_n$. It is discrete spectrum.

Ex - (14)

prove that a Dirac comb is its own Fourier transform

Sol: A Dirac comb is a comb-like wave form consisting of a sequence of equidistant impulses shown in Fig. 1. The sequence of unit impulses are spaced by T seconds.



The function may be expressed as

(70)

$$\delta_T(t) = \delta(t) + \delta(t-T) + \delta(t-2T) + \delta(t-3T) + \dots + \delta(t-nT) + \dots \\ \delta(t+T) + \delta(t+2T) + \delta(t+3T) + \dots + \delta(t+nT) + \dots$$

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

The Fourier transform of may be obtained as follows
The function can be expressed in terms of complex Fourier series

$$f(t) = \delta_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

applying F-T on both the sides

$$F[f(t)] = F[\delta_T(t)] = F\left[\sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}\right]$$

$$= \sum_{n=-\infty}^{\infty} c_n F[1 \cdot e^{jn\omega_0 t}]$$

$$= \sum_{n=-\infty}^{\infty} c_n \times 2\pi \delta(\omega - n\omega_0)$$

$$\left[\because f(t) e^{jn\omega_0 t} \longleftrightarrow F(\omega - n\omega_0) \right. \\ \left. \& 1 \longleftrightarrow 2\pi \delta(\omega) \right]$$

$$\boxed{F[\delta_T(t)] = 2\pi \sum_{n=-\infty}^{\infty} c_n \delta(\omega - n\omega_0)} \quad \text{--- (1)}$$

The value of c_n is evaluated as follows

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} \delta_T(t) e^{-jn\omega_0 t} dt$$

Function $\delta_T(t)$ has only one impulse in the interval $(-T/2, T/2)$, and hence is equivalent to $\delta(t)$. The c_n can be written as

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jn\omega_0 t} dt$$

using sifting property of unit impulse function

therefore $c_n = \frac{1}{T}$

$$\left[\because \int_{t_1}^{t_2} f(t) \delta(t-t_0) dt = \begin{cases} f(t_0), & t_1 < t_0 < t_2 \\ 0 & \text{otherwise} \end{cases} \right]$$

Substituting c_n in eqn ①

$$\text{eqn ①} \Rightarrow F[\delta_T(t)] = 2\pi \sum_{n=-\infty}^{\infty} \frac{1}{T} \delta(\omega - n\omega_0)$$

or

$$\omega_0 = \frac{2\pi}{T}$$

$$\delta_T(t) \longleftrightarrow \omega_0 \delta_{\omega_0}(\omega)$$

The spectrum shown in Fig 1(b), δ_T consists of a sequence of impulse located at harmonics of ω_0 and each impulse has constant strength of $\frac{2\pi}{T} = \omega_0$. Thus, the Fourier transform of the periodic unit impulse function is also a periodic unit impulse. In other words, the periodic unit impulse function (Dirac-comb) has its own transform.

Ex-15 ^{Imp}

Find the Inverse Fourier transform of $\delta(\omega - \omega_0)$

Sol: Given function $F(\omega) = \delta(\omega - \omega_0)$, the inverse Fourier transform of a signal $F(\omega)$ can be expressed as

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega \quad (72)$$

$$= \frac{1}{2\pi} e^{j\omega_0 t}$$

$$\because \int_{t_1}^{t_2} f(t) \delta(t - t_0) dt = \begin{cases} f(t_0), & t_1 < t_0 < t_2 \\ 0, & \text{otherwise} \end{cases}$$

$$\boxed{\therefore F^{-1}[\delta(\omega - \omega_0)] = \frac{1}{2\pi} e^{j\omega_0 t}}$$

Signal Transmission through Linear Systems:

Linear system: A system is said to be linear if it satisfies the principle of superposition. This implies that the response of a linear system to a weighted sum of input signals is equal to the same weighted sum of output signals, each output signal being associated with particular input signal acting on the system independently of all other input signals.

If the system is linear, then

$$y(t) = \sum_{i=1}^n a_i y_i(t) = T \left[\sum_{i=1}^n a_i f_i(t) \right]$$

where $T[\]$ is a transformation operator

and $y_i(t) = T[f_i(t)]$ is the output of the system in response to the input $f_i(t)$ acting alone.

Linear Time-Invariant Systems:

A system is said to be linear time invariant (LTI) system if it satisfies linearity and time invariance properties.

$$\text{If } y(t) = T[f(t)], \text{ then } y(t-t_0) = T[f(t-t_0)]$$

It is a time-invariant system.

For an LTI systems, if the input is delayed by t_0 seconds, the system satisfies superposition and homogeneity principles. Also, the output is delayed by the same time t_0 seconds.

Linear Time Variant Systems:

(2)

A system is said to be linear time variant (LTV) if it satisfies the linear property but not the time invariant property.

ex: $y(t) = f(t-2) + f(2-t)$

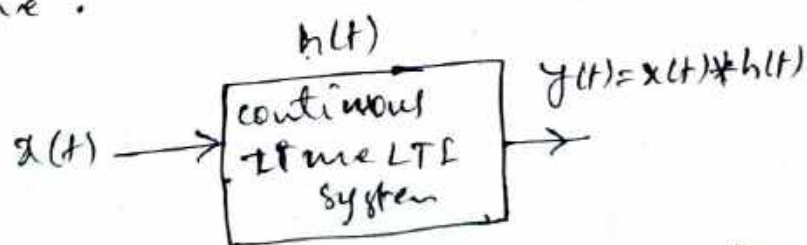
Response of a Linear System

Impulse Response:

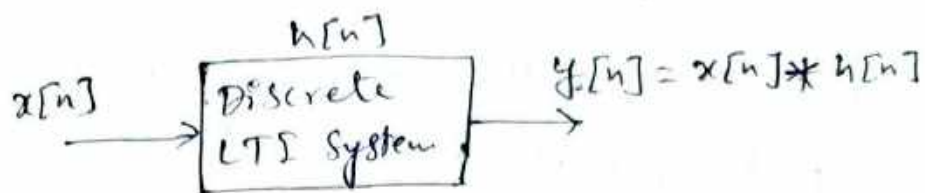
When the input signal to a system is the impulse signal, that is $x(t) = \delta(t)$, then the response of the system is known as impulse response, denoted by $h(t)$. The impulse response of an LTI system due to an impulse input applied at time $t=0$ is

$$h(t) = T[\delta(t)].$$

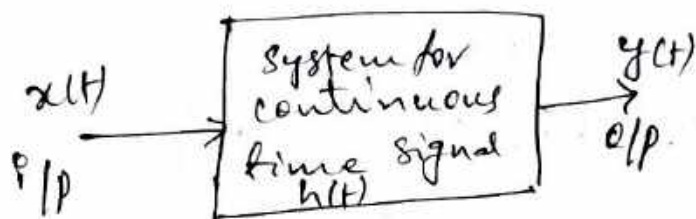
- * The impulse response denoted with $h(t)$ for continuous time.
- * The impulse response denoted with $h[n]$ for discrete time.



For continuous time signal.



Derivation of $y(t) = x(t) * h(t)$.



output $y(t) = T[x(t)]$

where $x(t)$ is input signal.

$y(t)$ is output/response.

$T[\]$ is transformation

A continuous-time signal can be expressed as the weighted superposition of the time-shifted impulses:

$$x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t-\tau) d\tau$$

the response $y(t)$ can be expressed as

$$\begin{aligned} y(t) = T[x(t)] &= T\left[\int_{-\infty}^{\infty} x(\tau) \delta(t-\tau) d\tau\right] \\ &= \int_{-\infty}^{\infty} x(\tau) T[\delta(t-\tau)] d\tau \end{aligned}$$

for time invariant system ~~h(t)~~ $h(t-\tau) = T[\delta(t-\tau)]$

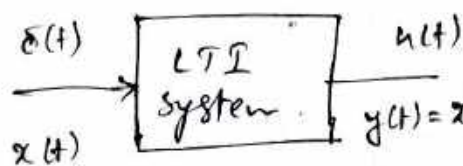
$$\text{then } y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \rightarrow \textcircled{1}$$

above equation $\textcircled{1}$ indicates that a continuous time

LTI system is completely characterized by its impulse response. This equation is called a convolution integral and is denoted by $x(t) * h(t)$. (2)

Thus the system response is

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau.$$



$y(t) = x(t) * h(t)$ → response of a system is given by input is convolved with impulse response.

Transfer function of the system:

Response of the LTI system is given by

$$y(t) = x(t) * h(t)$$

applying F.T on both the sides.

$$F[y(t)] = F[x(t) * h(t)]$$

[∵ Time convolution property of F.T]

$$Y(\omega) = X(\omega) H(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

Transfer function of the system = $\frac{\text{output}}{\text{input}}$.

$H(\omega)$ represents the frequency response of the system. It is complex valued function and can be expressed as

$$H(\omega) = |H(\omega)| \angle H(\omega)$$

or

(or)

$$H(\omega) = |H(\omega)| e^{j\phi(\omega)}$$

5

where $|H(\omega)|$ is amplitude response (symmetrical)

$\phi(\omega)$ is phase response (anti symmetrical)

$$|H(\omega)| = |H(-\omega)| \rightarrow \text{symmetrical}$$

$$\phi(-\omega) = -\phi(\omega) \rightarrow \text{anti symmetrical}$$

properties of LTI system:

① i) commutative property:

$$x(t) * h(t) = h(t) * x(t)$$

$$\int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau = \int_{-\infty}^{\infty} x(t-\tau) h(\tau) d\tau$$

ii) Associative property:

$$\{x(t) * h_1(t)\} * h_2(t) = x(t) * \{h_1(t) * h_2(t)\}$$

iii) Distributive property:

$$x(t) * \{h_1(t) + h_2(t)\} = x(t) * h_1(t) + x(t) * h_2(t).$$

② Causality: $\rightarrow$ see page number (29), further properties.

A continuous time LTI system is said to be causal if and only if it satisfies impulse response

$$h(t) = 0, \text{ for } t < 0, \quad y(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau = \int_0^{\infty} x(t-\tau) h(\tau) d\tau.$$

Note: All physical systems are causal since it is impossible to predict their future.

Q-1: A system produces an output $y(t) = e^{-t}u(t)$ for an input $x(t) = e^{-2t}u(t)$. Determine the impulse response and frequency response of the system. (6)

Sol: Given output ~~and~~ $y(t) = e^{-t}u(t)$ and
input $x(t) = e^{-2t}u(t)$

and their Fourier transforms are

$$Y(\omega) = F[y(t)] = F[e^{-t}u(t)] = \frac{1}{1+j\omega}$$

$$X(\omega) = F[x(t)] = F[e^{-2t}u(t)] = \frac{1}{2+j\omega}$$

The transfer function of the system is

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\frac{1}{1+j\omega}}{\frac{1}{2+j\omega}} = \frac{2+j\omega}{1+j\omega}$$

$$H(\omega) = 1 + \frac{1}{1+j\omega}$$

the impulse response of the system is

$$\begin{aligned} h(t) &= F^{-1}[H(\omega)] = F^{-1}\left[1 + \frac{1}{1+j\omega}\right] \\ &= F^{-1}[1] + F^{-1}\left[\frac{1}{1+j\omega}\right] \\ &= \delta(t) + e^{-t}u(t) \end{aligned}$$

$$\boxed{h(t) = \delta(t) + e^{-t}u(t)}$$

The frequency response of the system is

$$H(\omega) = \frac{2+j\omega}{1+j\omega}$$

$$\text{magnitude response } |H(\omega)| = \frac{\sqrt{4+\omega^2}}{\sqrt{1+\omega^2}}$$

$$\text{phase response } \angle H(\omega) = \phi(\omega) = \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}(\omega)$$

Ex-② consider a causal LTI system with frequency response $H(\omega) = \frac{1}{j\omega + 4}$. For a particular input $x(t)$, the system is observed to produce an output $y(t) = e^{-2t}u(t) - e^{-4t}u(t)$. Find the input signal $x(t)$.

Solution:

Given the frequency response $H(\omega) = \frac{1}{4 + j\omega}$ and output $y(t) = e^{-2t}u(t) - e^{-4t}u(t)$, The Fourier transform is

$$Y(\omega) = F[e^{-2t}u(t)] - F[e^{-4t}u(t)]$$

$$Y(\omega) = \frac{1}{2 + j\omega} - \frac{1}{4 + j\omega} = \frac{2}{(2 + j\omega)(4 + j\omega)}$$

the transfer function of the system is given by

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} \Rightarrow X(\omega) = \frac{Y(\omega)}{H(\omega)} = \frac{\frac{2}{(2 + j\omega)(4 + j\omega)}}{\frac{1}{(4 + j\omega)}} = \frac{2}{2 + j\omega}$$

$$X(\omega) = \frac{2}{2 + j\omega}$$

Taking the Inverse Fourier transform

$$\text{Input } x(t) = F^{-1}[X(\omega)] = F^{-1}\left[\frac{2}{2 + j\omega}\right]$$

$$\boxed{x(t) = 2e^{-2t}u(t)}$$

Ex-③ consider the system described by $y''(t) + 5y'(t) + 6y(t) = -x'(t)$. Find the impulse response $h(t)$ of the system.

Sol:

Given the differential equation

$y''(t) + 5y'(t) + 6y(t) = -x'(t)$ • Taking F.T on both sides

$$(j\omega)^2 Y(\omega) + 5(j\omega)Y(\omega) + 6Y(\omega) = -j\omega X(\omega)$$

$$\Rightarrow Y(\omega) [(j\omega)^2 + 5(j\omega) + 6] = -j\omega X(\omega) \quad \left[\because \begin{array}{l} \text{Time diff. property} \\ \text{Frequency} \\ \text{and } \frac{d^n f(t)}{dt^n} = (j\omega)^n F(\omega) \end{array} \right]$$

$$\frac{Y(\omega)}{X(\omega)} = \frac{-j\omega}{(j\omega)^2 + 5(j\omega) + 6}$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{-j\omega}{(2+j\omega)(3+j\omega)} = \frac{2}{2+j\omega} - \frac{3}{3+j\omega}$$

Taking inverse F.T, we obtain the impulse response

$h(t)$ of the system

$$h(t) = 2e^{-2t}u(t) - 3e^{-3t}u(t).$$

Distortionless Transmission through a system.

Transmission of a signal through a system is said to be distortionless if the output signal is an exact replica of the input signal.

→ However, a constant change in magnitude and a constant time delay in output replica are not considered as distortion.

→ The magnitude of the output replica may increase or decrease during transmission depending on whether the system offers a gain or attenuation, respectively.

Therefore output signal $y(t)$ can be written ^⑨ in terms of input $x(t)$ as

$$y(t) = K x(t - t_0) \quad \text{--- ①}$$

where K is constant, represents change in magnitude (Gain or Attenuation)

and t_0 is time delay in transmission of signal through a system.

and its F.T is

$$Y(\omega) = K e^{-j\omega t_0} X(\omega)$$

$$\frac{Y(\omega)}{X(\omega)} = H(\omega) = K e^{-j\omega t_0}$$

The magnitude and phase angle of this transfer function $H(\omega)$ is given by

$$\left. \begin{aligned} |H(\omega)| &= K \\ \angle H(\omega) = \theta(\omega) &= -\omega t_0 \end{aligned} \right\} \text{--- ②}$$

or more generally

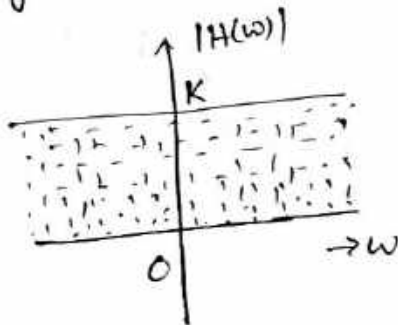
$$\theta(\omega) = n\pi - \omega t_0, \quad (n \text{ is +ve or -ve integer})$$

Equation ② reveals that

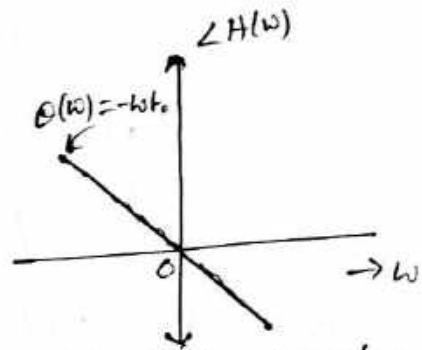
① magnitude $|H(\omega)|$ should be constant for a distortionless transmission. It means that all the frequency present in $x(t)$ are equally attenuated or amplified and system bandwidth is infinite.

② The ~~pas~~ phase spectrum $\angle H(\omega)$ or $\phi(\omega)$ should be directly proportional to the frequency as shown in Fig 1(b)

Note: No system has infinite bandwidth in practice and, hence distortionless conditions are never met exactly.



(a) Magnitude plot.



(b) phase spectrum

Fig. 1. A distortionless Transmission system.

Band width of a system:

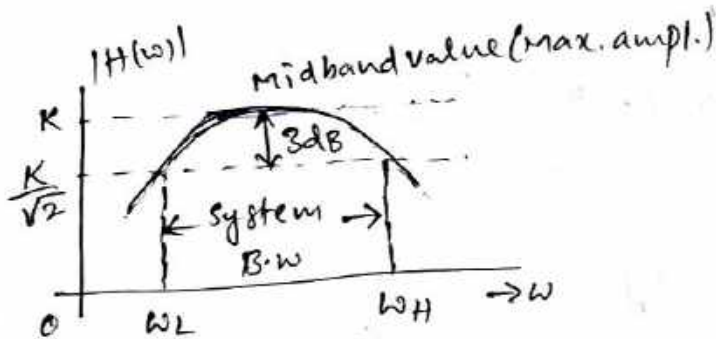


Fig. 2 Bandwidth of a system.

→ The bandwidth ^{of system} is defined as the interval of frequencies over which $|H(\omega)|$ remains within $\frac{1}{\sqrt{2}}$ times of its midband value.

A typical $|H(\omega)|$ plot having bandwidth of $(\omega_H - \omega_L)$ is shown in Fig. 2.

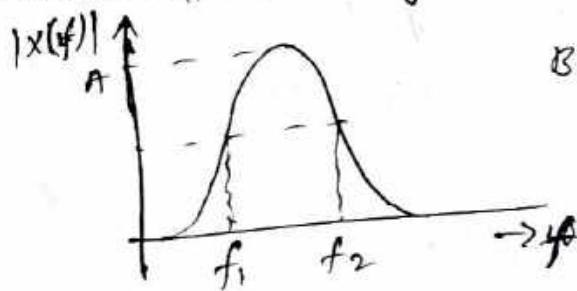
For distortionless transmission, the band width ~~is~~ should be infinite, but physical systems are limited to finite band width. The system with finite (11) band width can provide distortionless transmission for a band-limited signal.

The band limited signals can be transmitted without distortion if the system bandwidth is atleast equal to the signal band width.

Signal Bandwidth:

It is ^{the} range of significant frequency components present in a signal. A signal may have frequency components in the entire frequency range from $-\infty$ to ∞ . However, since all practical signals have finite energy, only some frequency components of the signals have significant amplitude within a certain frequency band. Outside this band the signals have negligible amplitude.

→ It is the range of frequencies that contains most of the signal energy is known as bandwidth of the signal.



$$B.W = f_2 - f_1$$

Filter: A filter is a frequency selective network that permits a band of frequencies to pass through it

with little attenuation, the attenuation increases severely for the remaining frequencies.

the band allowed is called passband and the band attenuated severely is called stopband.

Ideal filter? Ideal filter has sharp cut-off frequency.

Types of filters: [LTI systems may be classified into five types of filters]

- 1) ~~LPF~~ Low pass Filter (LPF)
- 2) High pass Filter (HPF)
3. Band pass Filter (BPF)
4. Band Reject Filter (BRF)
or
Band Elimination Filter (BEF)
5. All pass Filter.

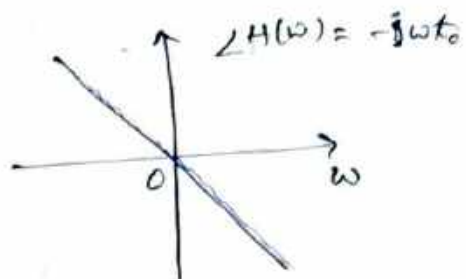
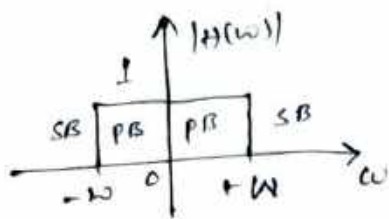
1) Ideal Lowpass Filter:

The ideal lowpass filter transmits all signals frequency components below a certain frequency ω rad/s called cutoff frequency, without distortion. The ~~above~~ signal above this frequency is attenuated completely.

The transfer function of an ideal LPF is given by

$$H(\omega) = \begin{cases} e^{-j\omega t_0} & |\omega| < \omega_c \\ 0 & , \quad |\omega| > \omega_c \end{cases}$$

The response is shown the magnitude & phase chara. of the filter is $|H(\omega)| = 1$ & $\angle H(\omega) = -\omega t_0$ for $|\omega| < \omega_c$
The response is shown in Fig 1.



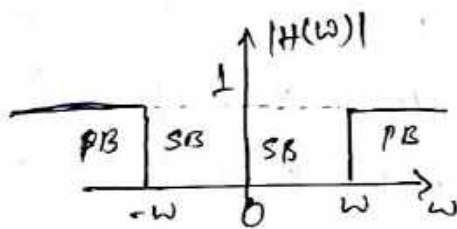
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Fig 1. Ideal LPF characteristics: (a) magnitude response
(b) phase response.

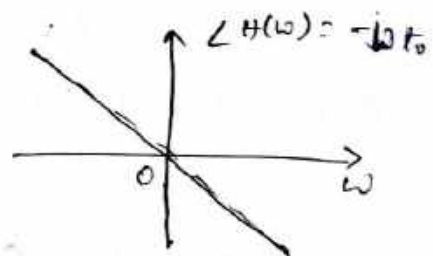
2) Ideal HPF: It passes the frequency components above a particular cutoff frequency and rejects or attenuates below the cut off frequency.

$$H(\omega) = \begin{cases} 0 & : |\omega| < \omega_c \\ 1 & : |\omega| > \omega_c \end{cases}$$

PB → pass Band
SB → stop Band



(a)



(b)

Fig 2. Ideal HPF characteristics: (a) magnitude response and
(b) phase response.

3) Ideal BPF: An ideal bandpass filter transmits all signal frequency components within a certain frequency band ω_1 to ω_2 rad/sec without distortion. The signal with frequency outside this band is stopped completely. The transfer function of an ideal BPF is given by (Magnitude & phase)

$$|H(\omega)| = \begin{cases} 1 & \text{for } \omega_1 < |\omega| < \omega_2 \\ 0 & \text{otherwise} \end{cases}$$

(14)

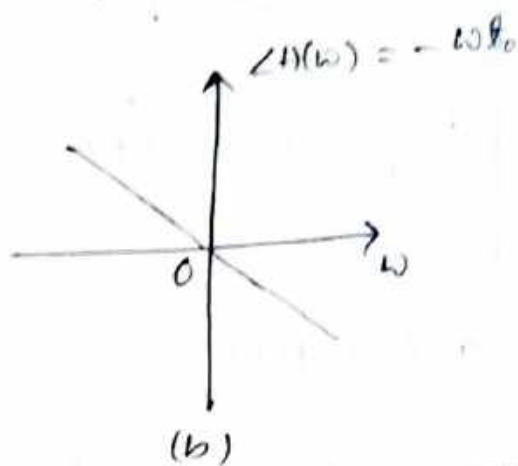
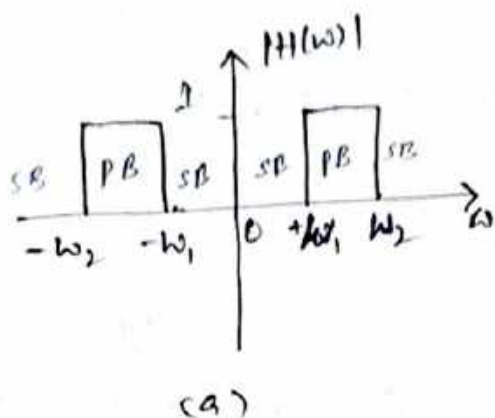


Fig 3. Ideal BPF characteristics: a) magnitude response b) phase response.

4) Ideal BRF:

An ideal band-rejection filter rejects or stops all signals frequency components within certain frequency function band of $\omega_2 - \omega_1$ rad/sec. The signal outside this band is transmitted without distortion.

$$|H(\omega)| = \begin{cases} 0 & \text{for } \omega_1 < |\omega| < \omega_2 \\ 1 & \text{otherwise} \end{cases}$$

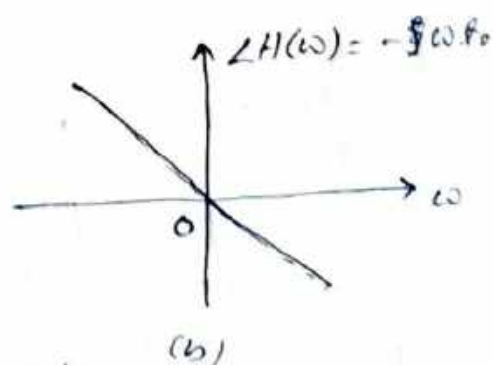
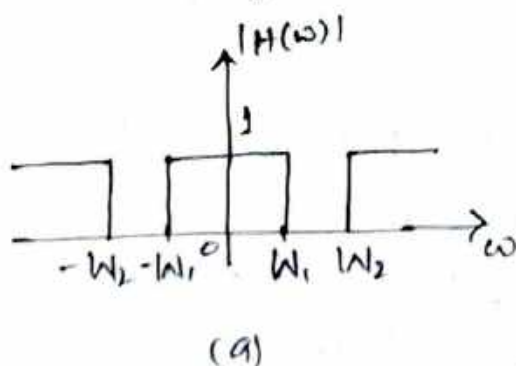


Fig. 4. Ideal ^{band} rejection filter characteristics: a) magnitude response b) phase response

Note: 1) All ideal filters are non-causal and distortionless transmission systems.

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2) Filters are used to eliminate the unwanted frequency components of a signal or the noise components from a signal.

3) Filters are also used to recover the input signals from a noisy signal.

Causality and physical Realisability:

1) The Paley-Wiener criterion:

For real time applications, all systems should be physically realisable. The impulse response $h(t)$ of a physically realisable system must be causal. That is $h(t) = 0$ for $t < 0$. This condition, called the causal condition, ensures that the system produces an output response only ^{after} an input signal is applied.

In the frequency domain approach, the necessary and sufficient condition for the magnitude response function $|H(\omega)|$ to be physically realizable is:

$$\int_{-\infty}^{\infty} \frac{|\ln |H(\omega)||}{1 + \omega^2} d\omega < \infty$$

This condition is known as the Paley-Wiener criterion. To satisfy this condition the function $|H(\omega)|$ must be ~~satisfied~~ square integrable, i.e.,

$$\int_{-\infty}^{\infty} |H(\omega)|^2 d\omega < \infty$$

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Note: All causal systems that satisfy Paley-Wiener criterion are physically realizable.

Ideal filters are not physically realizable. But it is possible to construct physically realizable filters close to the ideal filter characteristics. As an example consider LPF having transfer function

$$|H(\omega)| = \begin{cases} 1, & |\omega| < W \\ \epsilon, & |\omega| > W \end{cases}$$

where ϵ is an arbitrarily small value, produces very nearly ideal filter characteristics. Shown in Fig.

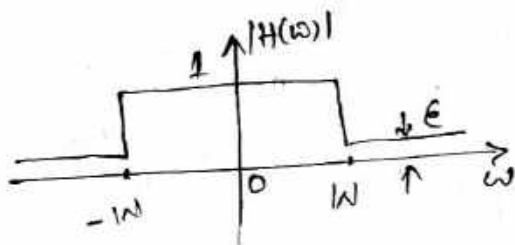


Fig. 1 magnitude response of a physically realizable LPF

Bandwidth and Rise time:

→ The rise time (t_r) of the output is defined as the time the response takes to reach from 10% to 90% of the maximum value of the signal, or in general, it is the time of the response to reach from zero to the final value of the signal.

$$\left. \frac{dy(t)}{dt} \right|_{t=t_0} = \frac{1}{t_r}$$

Relationship between Bandwidth and Rise time.

Show that impulse response of ideal low pass filter is non-causal. (17)

Consider an ideal LPF. Its transfer function is given by

$$H(\omega) = \begin{cases} e^{j\omega t_0} & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases} \quad H(\omega) = |H(\omega)| e^{j\omega t_0}$$

$\downarrow$
K

By where ω_c is the cut off frequency or 3-dB B.W of the filter. By applying inverse Fourier transform, the impulse response is

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)| e^{j\omega t_0} e^{j\omega t} d\omega \quad [\because H(\omega) = |H(\omega)| e^{j\omega t_0}]$$

$|H(\omega)| = 1 \text{ for } |\omega| < \omega_c$

$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega(t-t_0)} d\omega$$

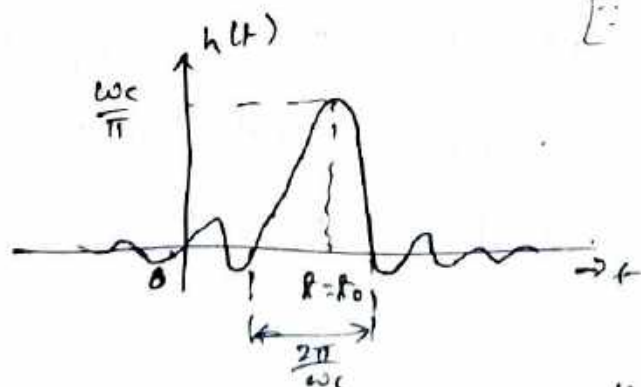
$$= \frac{1}{2\pi} \left[\frac{e^{j\omega(t-t_0)}}{j\omega(t-t_0)} \right]_{-\omega_c}^{\omega_c} = \frac{1}{2\pi} \left[\frac{e^{j\omega_c(t-t_0)}}{j\omega_c(t-t_0)} - \frac{e^{-j\omega_c(t-t_0)}}{j\omega_c(t-t_0)} \right]$$

$$= \frac{1}{\pi} \left[\frac{e^{j\omega_c(t-t_0)} - e^{-j\omega_c(t-t_0)}}{2j} \right] \times \frac{1}{(t-t_0)}$$

$$= \frac{1}{\pi(t-t_0)} \times \sin(\omega_c(t-t_0)) = \frac{\omega_c}{\pi} \left[\frac{\sin \omega_c(t-t_0)}{\omega_c(t-t_0)} \right]$$

$$h(t) = \frac{\omega_c}{\pi} \text{sinc } \omega_c(t-t_0) \quad \text{or} \quad h(t) = \frac{\omega_c}{\pi} \left[\frac{\sin \omega_c(t-t_0)}{\omega_c(t-t_0)} \right]$$

$$\text{sinc}(x) = \frac{\sin x}{x}$$



$$\because \text{sinc } x = \frac{\sin \pi x}{\pi x}$$

Fig. Impulse response of the ideal LPF.

From above Fig., we can conclude that impulse response of ideal low pass filter is non-causal.

Now consider the step response of the ideal LPF.

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$$y(t) = h(t) * u(t)$$

$$= \int_{-\infty}^{\infty} h(\tau) u(t-\tau) d\tau$$

$$= \int_{-\infty}^t h(\tau) 1 d\tau \quad [\because u(t-\tau) = 1, -\infty < \tau < t]$$

$$h(t) = \frac{\omega_c}{\pi} \frac{\sin \omega_c(t-t_0)}{\omega_c(t-t_0)}$$

$$y(t) = \int_{-\infty}^t \frac{\omega_c}{\pi} \frac{\sin \omega_c(\tau-t_0)}{\omega_c(\tau-t_0)} d\tau$$

$$\text{put } x = \omega_c(\tau-t_0)$$

$$dx = \omega_c d\tau \Rightarrow d\tau = \frac{dx}{\omega_c}$$

$$\tau = t \Rightarrow x = \omega_c(t-t_0)$$

$$\tau = -\infty \Rightarrow x = -\infty$$

$$y(t) = \int_{-\infty}^{\omega_c(t-t_0)} \frac{\omega_c}{\pi} \frac{\sin x}{x} \frac{dx}{\omega_c}$$

$$y(t) = \frac{1}{\pi} \int_{-\infty}^{\omega_c(t-t_0)} \frac{\sin x}{x} dx = \frac{1}{\pi} \int_{-\infty}^{\omega_c(t-t_0)} \text{Si}(x) dx$$

$$y(t) = \frac{1}{\pi} [\text{Si}(x)]_{-\infty}^{\omega_c(t-t_0)}$$

where $\text{Si}(x)$ is the Sine Integral function.

The properties of Sine Integral function are:

i) $\text{Si}(x)$ is an odd function, that is ~~Si~~ $\text{Si}(-x) = -\text{Si}(x)$

ii) $\text{Si}(0) = 0$.

iii) $\text{Si}(\infty) = \frac{\pi}{2}$; and $\text{Si}(-\infty) = -\frac{\pi}{2}$

The step response is expressed as

$$y(t) = \frac{1}{\pi} [\text{Si}(\omega_c(t-t_0)) - \text{Si}(-\infty)]$$

$$= \frac{1}{\pi} [\text{Si}(\omega_c(t-t_0))] + \frac{\pi}{2}$$

$$y(t) = \frac{1}{2} + \frac{1}{\pi} \text{Si}[\omega_c(t-t_0)]$$

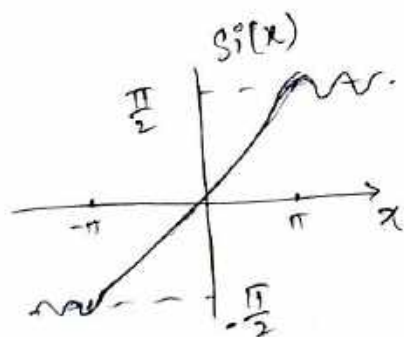
If $\omega_c \rightarrow \infty$, then the response is

$$\begin{aligned} y(t) &= \frac{1}{2} + \frac{1}{\pi} \text{Si}[\infty] \\ &= \frac{1}{2} + \frac{1}{\pi} \times \frac{\pi}{2} = 1 \end{aligned}$$

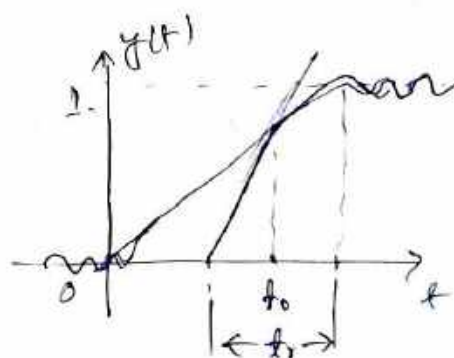
If $\omega_c \rightarrow -\infty$, then the response is

$$\begin{aligned} y(t) &= \frac{1}{2} + \frac{1}{\pi} \text{Si}[-\infty] \\ &= \frac{1}{2} - \frac{1}{\pi} \times \frac{\pi}{2} = 0 \end{aligned}$$

The $\text{Si}(x)$ function & step response can be drawn as below.



(a) Sine function.
Fig.



(b) Step response of an ideal LPF.

From Fig. above, we can observe that $y(t)$ approaches a delayed unit step $u(t-t_0)$

To find the rise time, draw a tangent at $t=t_0$.

With line $y(t)=0$ & $y(t)=1$. from fig (b) we have

$$\frac{1}{t_r} = \left. \frac{dy(t)}{dt} \right|_{t=t_0} = \frac{\omega_c}{\pi} \left. \frac{\sin \omega_c(t-t_0)}{\omega_c(t-t_0)} \right|_{t=t_0}$$

$$\frac{1}{t_r} = \frac{\omega_c}{\pi}$$

$$t_r = \frac{\pi}{\omega_c}$$

or

$$t_r \times \omega_c = \pi$$

for LPF, cutoff frequency = Bandwidth

$\therefore$ Bandwidth $\times$ rise time = constant.

So, the rise time is inversely proportional to the bandwidth.

convolution: Convolution is a mathematical way of combining two signals to form a third signal.

Convolution is important ~~for~~ mathematical tool because it relates the input signal and impulse response to the output of the system.

Let $x_1(t) * x_2(t)$ be two continuous-time signals. then convolution of $x_1(t)$ and $x_2(t)$ can be expressed as

$$y(t) = x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau$$

where τ is independent time variable & denotes convolution process.

The convolution integral can be applied to any LTI system. However, the convolution integral can be simplified if we assume the following: (21)

First, if $x(t)$ is a positive signal i.e., $x(t) = 0$ for $t < 0$, then

$$y(t) = \int_0^{\infty} x(\tau) h(t-\tau) d\tau.$$

Second if the system is causal, i.e., $h(t) = 0$ for $t < 0$, then $h(t-\tau) = 0$ for $t-\tau < 0$ or $\tau > t$ and the convolution is

$$y(t) = \int_0^t x(\tau) h(t-\tau) d\tau$$

Third if $x(t)$ is a positive signal and the system $h(t)$ is causal, then

$$y(t) = \int_0^t x(\tau) h(t-\tau) d\tau$$

properties of the convolution Integral:

Let $x_1(t)$ & $x_2(t)$ be two continuous-time signals

i) commutative property

$$x_1(t) * x_2(t) = x_2(t) * x_1(t)$$

proof: By definition, we know that the convolution is

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau$$

$$\begin{array}{l} \text{put } t-\tau = x \\ \tau = t-x \\ d\tau = -dx \end{array} \quad \left/ \quad \begin{array}{l} \tau = \infty \Rightarrow x = -\infty \\ \tau = -\infty \Rightarrow x = +\infty \end{array} \right.$$

$$x_1(t) * x_2(t) = \int_{\infty}^{-\infty} x_1(t-x) x_2(x) (-dx) = \int_{-\infty}^{\infty} x_2(x) x_1(t-x) dx$$

$$= x_2(t) * x_1(t)$$

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ii) Distributive property:

$$x_1(t) * [x_2(t) + x_3(t)] = x_1(t) * x_2(t) + x_1(t) * x_3(t)$$

iii) Associative property:

$$[x_1(t) * x_2(t)] * x_3(t) = x_1(t) * [x_2(t) * x_3(t)] = \underline{x_1(t) * x_2(t) * x_3(t)}$$

$$= x_1(t) * x_2(t) * x_3(t).$$

iv) Convolution of a function with impulse

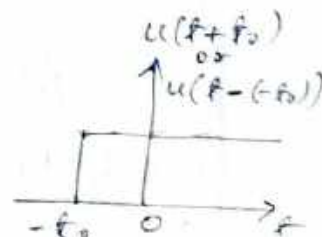
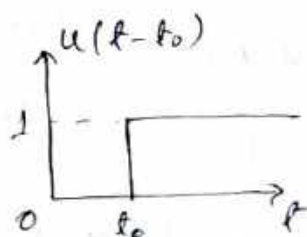
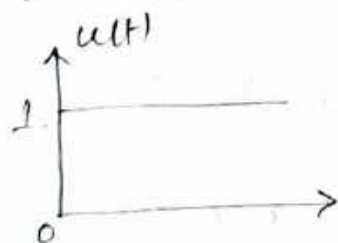
$$a) f(t) * \delta(t - t_0) = f(t - t_0)$$

$$b) f(t) * \delta(t) = f(t)$$

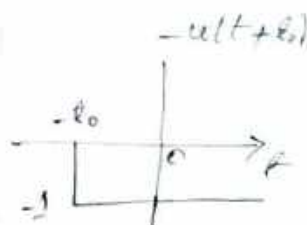
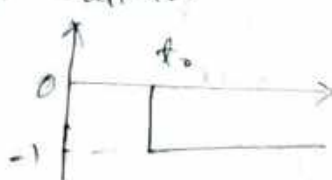
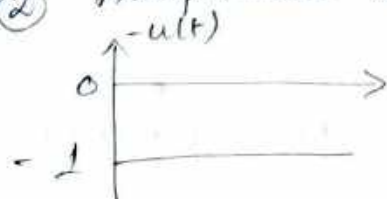
$$c) f(t) * \delta'(t - t_0) = f'(t - t_0)$$

Transformation of independent variables:

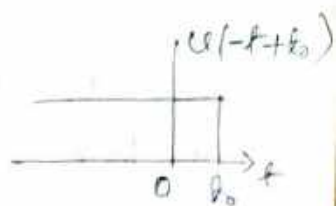
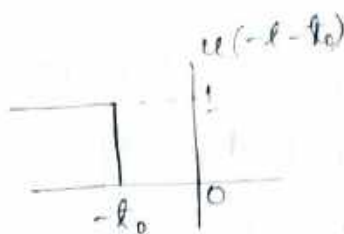
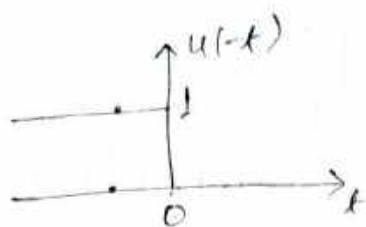
1. Shifting: Let us take unit step function.



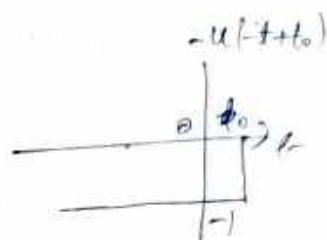
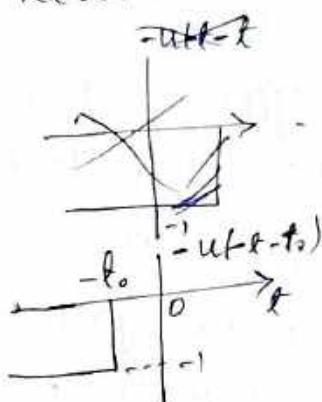
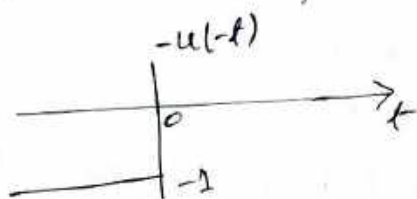
② Amplitude Reversal: $-u(t - t_0)$



③ Time Reversal

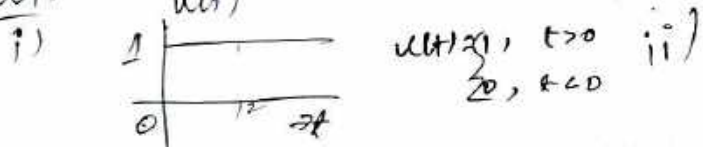


④ Time and Amplitude Reversal

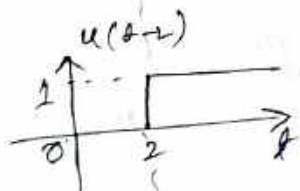
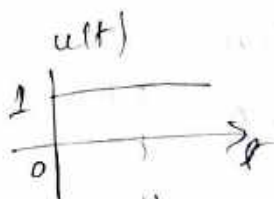


Ex: ① Draw the following signals using graphical representation - i) $x(t) = u(t) - u(t-2)$ ii) $x(t) = u(t) + u(t-1)$

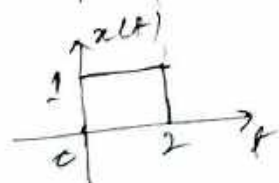
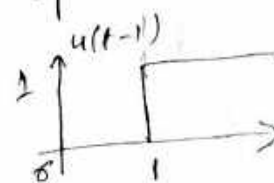
Sol:



$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$



$$u(t-2) = \begin{cases} 1; & t > 2 \\ 0; & t < 2 \end{cases}$$



$$x(t) = \begin{cases} 1; & 0 < t < 2 \\ 0; & \text{otherwise} \end{cases}$$



properties of LTI system:

③ Stability of LTI system:

(24)

A continuous time LTI system is bounded input, bounded output (BIBO) stable if and only if the impulse response is absolutely integrable.

consider an LTI system with impulse response $h(t)$.

The output response is

$$y(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau.$$

$$\text{Therefore } |y(t)| \leq \int_{-\infty}^{\infty} |h(\tau)| |x(t-\tau)| d\tau$$

If ~~an~~ input $x(t)$ is bounded and $|x(t-\tau)| < \infty$ and bounded output, i.e., $|y(t)| < \infty$, the impulse response should be absolutely integrable. Hence

$$\boxed{\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty} \quad \text{or} \quad \boxed{\int_{-\infty}^{\infty} |h(t-\tau)| d\tau < \infty}$$

Therefore, for LTI systems above equation gives the necessary and sufficient condition for BIBO stability.

④ Static and Dynamic Systems:

A system S is static or memoryless if its output at any time depends only on the value of ~~the~~ its input at that instant of time. For an LTI system, this property can only hold if its impulse response is itself an impulse.

But from the convolution property, we know that the output depends on the previous samples (values) of the input.

Therefore, an LTI system has memory and hence it is a dynamic system.

Ex-2 Test the stability of the LTI systems, whose impulse responses are given below.

(a) $h(t) = e^{-5|t|}$

Sol: Given that, $h(t) = e^{-5|t|}$

For stability, $\int_{-\infty}^{\infty} |h(t)| dt < \infty$

or $\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$

$\therefore \tau$ is a dummy variable, which can be replaced by t

$$\therefore \int_{-\infty}^{\infty} |h(t)| dt = \int_{-\infty}^{\infty} |e^{-5|t|}| dt = \int_{-\infty}^{\infty} e^{-5|t|} dt$$

$$= \int_{-\infty}^0 e^{5t} dt + \int_0^{\infty} e^{-5t} dt = \left. \frac{e^{5t}}{5} \right|_{-\infty}^0 + \left. \frac{e^{-5t}}{-5} \right|_0^{\infty}$$

$$= \frac{e^0}{5} - \frac{e^{-\infty}}{5} + \frac{e^{-\infty}}{-5} - \frac{e^0}{-5} = \frac{1}{5} - 0 + 0 + \frac{1}{5} = \frac{2}{5}$$

Here $\int_{-\infty}^{\infty} |h(t)| dt = \frac{2}{5} = \text{constant (finite value)}$, Hence the

system is stable.

(b) Given that, $h(t) = e^{4t} u(t)$

Sol: For stability, $\int_{-\infty}^{\infty} |h(t)| dt < \infty$

$$\therefore \int_{-\infty}^{\infty} |h(t)| dt = \int_{-\infty}^{\infty} |e^{4t} u(t)| dt = \int_{-\infty}^{\infty} e^{4t} u(t) dt$$

$$= \int_0^{\infty} e^{4t} dt = \left. \frac{e^{4t}}{4} \right|_0^{\infty} = \frac{e^{\infty}}{4} - \frac{e^0}{4} = \infty - \frac{1}{4} = \infty$$

Here, $\int_{-\infty}^{\infty} |h(t)| dt = \infty$ (infinity), Hence the system is unstable.

Homework (c) $h(t) = e^{-4t} u(t)$ (stable) (d) $h(t) = t e^{-3t} u(t)$ (stable)

26

Ex-3

perform convolution of the following causal signals.

(a) Given that, $x_1(t) = 2u(t)$, $x_2(t) = u(t)$

Sol:

$$x_1(t) = 2u(t) = 2; t \geq 0$$

$$x_2(t) = 1u(t) = 1; t \geq 0$$

Let $x_3(t) = x_1(t) * x_2(t)$

By definition of convolution,

$$x_3(t) = x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau$$

$$= \int_{-\infty}^{\infty} 2u(\tau) \cdot u(t-\tau) d\tau$$

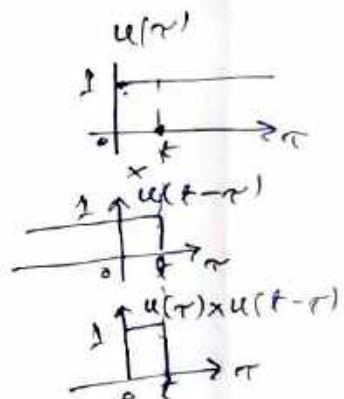
$$= \int_0^t 2 \cdot 1 d\tau$$

$$= 2 \times \tau \Big|_0^t$$

$$= 2t$$

$$= 2t u(t)$$

$$\therefore x_3(t) = 2t u(t).$$



$$[\because u(\tau) = 1 \text{ for } \tau \geq 0]$$

$$u(t-\tau) = 0 \text{ for } \tau > t$$

$$u(t-\tau) = 1 \text{ for } \tau < t$$

or

$$u(t-\tau) = 1 \text{ for } -\infty < \tau < t$$

Since $x_1(t)$ & $x_2(t)$ are causal limits will become 0 for $t < 0$

(b) $x_1(t) = e^{-2t} u(t)$, $x_2(t) = e^{-5t} u(t)$

Sol:

Given that, $x_1(t) = e^{-2t} u(t)$

$$x_2(t) = e^{-5t} u(t)$$

By definition of convolution,

(27)

$$x_3(t) = x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2\tau} u(\tau) \cdot e^{-5(t-\tau)} u(t-\tau) d\tau$$

$$= \int_0^t e^{-2\tau} e^{-5t+5\tau} d\tau$$

[$\because$ Given signals are causal signals

So, limits will become 0 to t]

$$= e^{-5t} \int_0^t e^{3\tau} d\tau$$

$$= e^{-5t} \left[\frac{e^{3\tau}}{3} \right]_0^t = \frac{e^{-5t}}{3} [e^{3t} - 1]$$

$$= \frac{1}{3} (e^{-2t} - e^{-5t});$$

$$x_3(t) = \frac{1}{3} (e^{-2t} - e^{-5t}) u(t)$$

Note: convolution of two causal signals is always causal.

Unit Step Response using convolution:

$\rightarrow$ In an LTI system, if the input $x(t)$ is a unit step signal, then that response is called a unit step response.

In general the response $y(t)$ of a system is given by convolution of input $x(t)$ and impulse

response $h(t)$ of the system is

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

(28)

Let the input $x(t)$ be unit step input $u(t)$, and the corresponding response be $S(t)$. Now the unit step response $S(t)$ is given by

$$S(t) = u(t) * h(t) \quad \text{or}$$

$$= h(t) * u(t)$$

$$= \int_{-\infty}^{\infty} h(\tau) u(t-\tau) d\tau$$

In the above convolution operation,

$$u(t-\tau) = 1 \quad \text{for } -\infty < \tau < t$$

Therefore the unit step ~~function~~ response $S(t)$ is given by

$$S(t) = \int_{-\infty}^t h(\tau) d\tau$$

Ex-④

Determine the unit step response of the following systems whose impulse responses are given below.

① $h(t) = 3u(t)$

Sol: Given that, $h(t) = 3t u(t)$

unit step response, $s(t) = \int_{-\infty}^t h(\tau) d\tau$

$$s(t) = \int_{-\infty}^t 3\tau u(\tau) d\tau$$

$$= \int_0^t 3\tau d\tau \quad [\because u(\tau) = 1, \tau \geq 0]$$

$$= 3 \frac{\tau^2}{2} \Big|_0^t = 3 \frac{t^2}{2}$$

$$s(t) = \frac{3t^2}{2} u(t); \quad u(t) = 1; \quad t \geq 0$$

(b) $h(t) = e^{5t} u(t)$

Sol: unit step response $s(t) = \int_{-\infty}^t h(\tau) d\tau$

$$s(t) = \int_{-\infty}^t e^{5\tau} u(\tau) d\tau = \int_0^t e^{5\tau} d\tau$$

$$= \frac{e^{5\tau}}{5} \Big|_0^t = \frac{1}{5} (1 - e^{5t})$$

$$\therefore s(t) = \frac{1}{5} (1 - e^{5t}) u(t).$$

(c) $h(t) = u(t+2)$

Sol: unit step response, $s(t) = \int_{-\infty}^t h(\tau) d\tau$

$$s(t) = \int_{-\infty}^t u(\tau+2) d\tau = \int_{-2}^t 1 d\tau.$$

$$= \tau \Big|_{-2}^t = t + 2$$

$$s(t) = (t+2) u(t+2)$$



Unit - IV

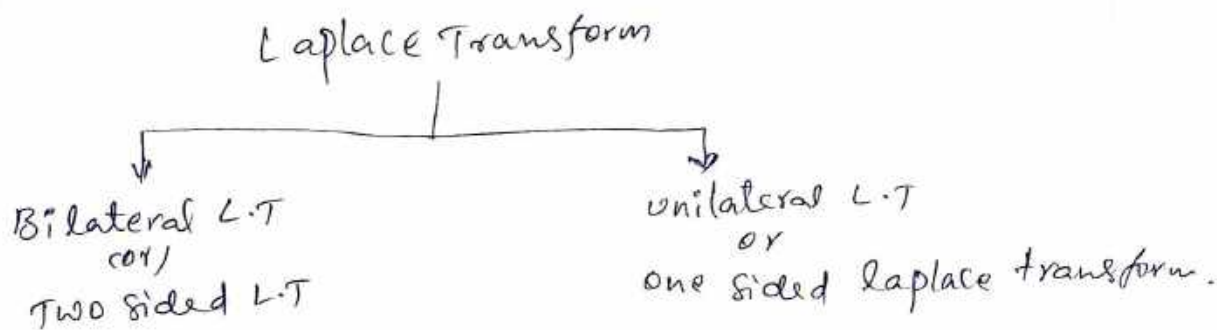
Laplace Transform & Z-Transform

(1)

The major limitations of Fourier transform are:

- i) The function $x(t)$ should be absolutely integrable.
- ii) The Fourier transform does not exist for most of the functions having complex variable $\sigma + j\omega$ encountered in LTI systems.

Laplace transform: Laplace transform is also for continuous time signals. It is also called as complex Fourier transform. The Laplace transform converts continuous time domain functions to s -domain (frequency domain).



unilateral L.T.

For any real function $x(t)$, $t \geq 0$, the L.T is defined as

$$X(s) = \int_0^{\infty} x(t) e^{-st} dt$$

where $s = \sigma + j\omega$ is a complex frequency with real numbers σ and ω

Notation for Laplace transform is $\mathcal{L}[x(t)]$, where $\mathcal{L}$ is Laplace transform operator. (2)

Inverse Laplace transform is defined as

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

$$x(t) = \mathcal{L}^{-1}[X(s)]$$

And the Laplace transform pair is $x(t) \leftrightarrow X(s)$.

Bilateral Laplace transform:

$$\mathcal{L}[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

Inverse Laplace transform is defined as

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

The relation between Laplace transform and F.T.

The Laplace transform of $x(t)$ is given by

$$\mathcal{L}[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

where $s = \sigma + j\omega$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt = \int_{-\infty}^{\infty} [x(t) e^{-\sigma t}] e^{-j\omega t} dt$$

$$\boxed{\mathcal{L}[x(t)] = F[x(t) e^{-\sigma t}]}$$

If $\sigma = 0$, Laplace transform = The Fourier transform, the Fourier transform is special class of Laplace transform.

convergence of the Laplace transform:

(8)

The necessary condition for convergence the Laplace transform is absolute integrability of $x(t) e^{\sigma t}$.

Mathematically, this can be stated as

$$\int_{-\infty}^{\infty} |x(t) e^{\sigma t}| dt < \infty$$

Laplace transform exists only for the signals which satisfy above equation, in the given region.

Region of convergence (ROC)

The range of values of the complex variable $s = \sigma + j\omega$ for which the Laplace transform converges is called the region of convergence (ROC).

→ The ROC is important to understand because it defines the region where the Laplace transform exists.

→ The ROC of $x(s)$ can be represented graphically with a pole-zero plot on the s -plane, s being given by $s = \sigma + j\omega$. Where σ is represented along the x -axis and $j\omega$ along the y -axis.

→ The ROC is a shaded area on the pole-zero plot (vertical bars)

of s -plane;

→ $x(s)$, the Laplace transform, exists for all the values of s in the shaded area.

poles and zeros of $X(s)$:

The general polynomial form of $X(s)$ can be expressed as

$$X(s) = \frac{K(s-z_1)(s-z_2)\dots(s-z_m)}{(s-p_1)(s-p_2)\dots(s-p_n)}$$

where K is a real constant.

$z_1, z_2, z_3 \dots z_m$ are zeros of $X(s)$ because $X(s) = 0$

for those values of s . [roots of numerator polynomial]

$p_1, p_2, p_3 \dots p_n$ are ^{called the} poles [the roots of the denominator polynomial] of $X(s)$, because $X(s)$ is infinite for those values of s .

$X(s)$ can be represented graphically by locating poles and zeros in the s -plane as shown below Fig.

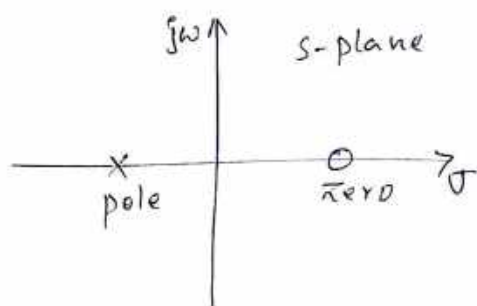


Fig. 1. pole-zero plot in the s -plane

properties of the ROC:

property 1: The ROC does not contain any poles.

property 2: If $x(t)$ is a finite duration function, that is, the signal $x(t)$ exists only between two finite time intervals (t_1, t_2) , then the ROC is the entire s -plane except at $s=0$, and $s=\infty$ as shown in Fig. 2(a).

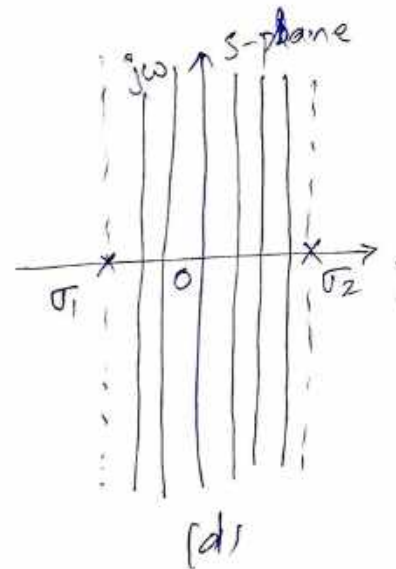
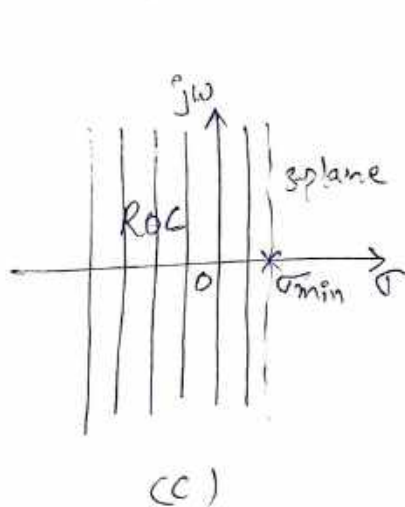
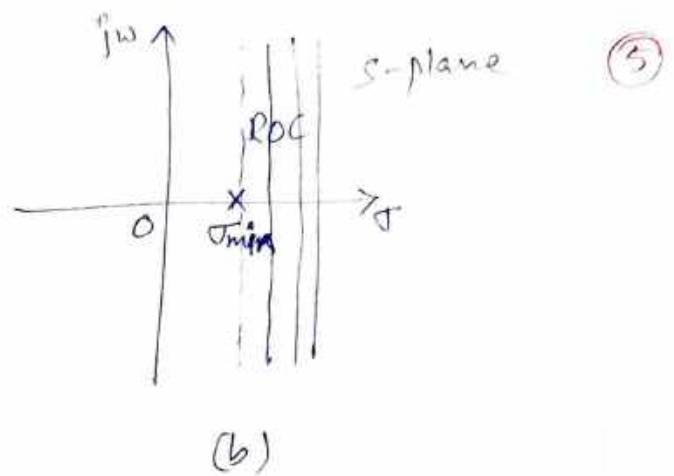
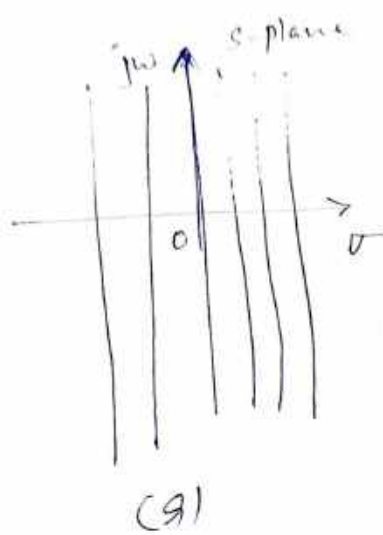


Fig 2. (a) ROC is the entire s-plane, (b) ROC for right-sided signal
 (c) ROC for left-sided signal (d) ROC for both sided signal.

property 3: If $x(t)$ is a right sided signal, that is $x(t)$ exists only in the interval $(0 < t < \infty)$, then the ROC is $\text{Re}\{s\} > \sigma_{\min}$. ROC shown in Fig 2(b)

property 4: If $x(t)$ is a left-sided signal, that is $x(t)$ exists only in the interval $(-\infty < t < 0)$, then the ROC is $\text{Re}\{s\} < \sigma_{\min}$. ROC is shown in Fig 2(c).

property 5: If $x(t)$ is a two-sided signal, that is $x(t)$ is an infinite duration signal existing in the time interval $(-\infty < t < \infty)$ then the ROC is of the form $\sigma_1 < \text{Re}\{s\} < \sigma_2$.

Ex-1: obtain Laplace transform of $x_1(t) = e^{-at}u(t)$ and (6)

$x_2(t) = -e^{-at}u(-t)$, compare their ROCs.

Sol:

Given the signal $x_1(t) = e^{-at}u(t)$. Its Laplace transform is

$$X_1(s) = \int_{-\infty}^{\infty} e^{-at}u(t) e^{-st} dt = \int_0^{\infty} e^{-(s+a)t} dt$$

$$X_1(s) = \frac{e^{-(s+a)t}}{-(s+a)} \Big|_0^{\infty} = \frac{1}{s+a}$$



$$X_1(s) = \frac{1}{a+s+j\omega}, \quad \sigma+a > 0$$

The Laplace transform converges if $\sigma+a > 0$ or $\text{Re}\{s\} > -a$
the ROC is $\text{Re}\{s\} > -a$

$$X_1(s) = \frac{1}{a+s+j\omega} = \frac{1}{s+a}, \quad \text{Re}\{s\} > -a.$$

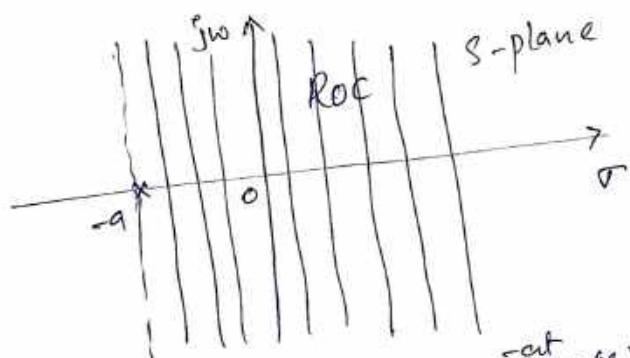


Fig.3. ROC for $x_1(t) = e^{-at}u(t)$

The function $X_1(s)$ has only one pole at $s = -a$. Since the signal is a right-sided signal, the ROC is $\text{Re}\{s\} > -a$ and is shown in Figs. shaded area right side of the line $\text{Re}\{s\} > -a$ on s-plane.

Now for the signal $x_2(t) = -e^{-at}u(-t)$

$$X_2(s) = - \int_{-\infty}^0 e^{-at-st} u(-t) dt = - \int_{-\infty}^0 e^{-(s+a)t} dt$$

$$X_1(s) = \frac{1}{s+a}, \quad \text{Re}\{s\} < -a$$

and the Laplace transform converges if $\text{Re}\{s\} < -a$

$$\mathcal{L}\{-e^{-at}u(t)\} = \frac{1}{s+a}, \quad \text{Re}\{s\} < -a$$

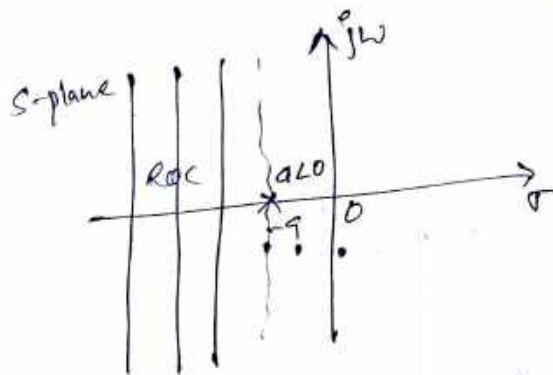


Fig. 4. ROC for $x(t) = -e^{-at}u(t)$

$$X_2(s) = \frac{1}{s+a} + 0$$

$$= \frac{1}{\sigma + j\omega + a} = \frac{1}{(\sigma + a) + j\omega}$$

If $\sigma < -a$, then only $X_2(s)$ converges.

(or)

$$\text{if } k = \sigma + a = \sigma - (-a)$$

when $\sigma > -a$, $k = \sigma - (-a) =$ positive

$$\therefore e^{k\omega} = e^{\infty} = \infty$$

when $\sigma < -a$, $k = \sigma - (-a) =$ negative

$$\therefore e^{k\omega} = e^{-\infty} = 0.$$

The $X_2(s)$ also has only one pole at $s = -a$. Since the signal is left-sided, the ROC, $\text{Re}\{s\} < -a$, is the shaded area (line bars) on the left side of the line $\text{Re}\{s\} = -a$ on s-plane.

Note: It is observed that the algebraic expression for $X_1(s)$ and $X_2(s)$ for the given two different signals are identical except for the ROCs. So, the ROC must be specified as part of the transformation.

Ex-2, Obtain the Laplace transform of $f(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$ with its ROC.

Sol: Given the signal $f(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$

From the basic properties, Laplace transform for signals are

$$e^{-2t}u(t) \leftrightarrow \frac{1}{s+2} \text{ and } e^{-t}u(t) \leftrightarrow \frac{1}{s+1} \quad (8)$$

therefore laplace transform of $f(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$ is

$$F(s) = \frac{3}{s+2} - \frac{2}{s+1} = \frac{s-1}{(s+2)(s+1)}$$

To det. Roc, the poles are at $p_1 = -1$ & $p_2 = -2$ ~~since~~
~~the~~ since the signal is right sided signal, the Roc
 is $\text{Re}\{s\} > -1$

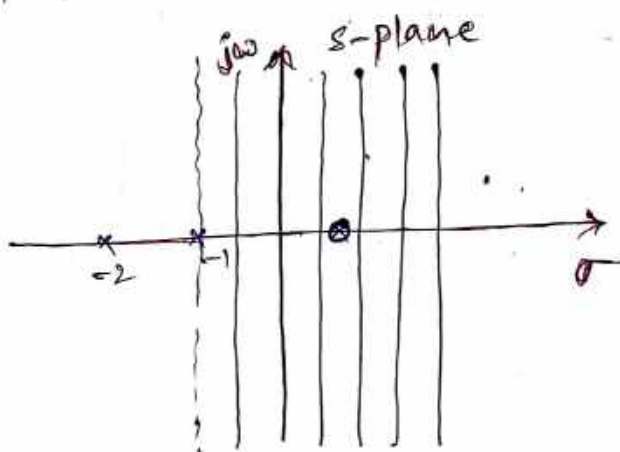


Fig 5. Roc for $f(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$ on pole-zero plot.

Unilateral Laplace Transform of simple signals

i) Laplace transform of unit impulse function $x(t) = \delta(t)$.

sol: $X(s) = \mathcal{L}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-st} dt = 1$. [bilateral & unilateral]

Thus $\delta(t) \xleftrightarrow{\text{L.T.}} 1$. The Roc is the entire s-plane

ii) unit step function $f(t) = u(t)$.

proof: Laplace transform of unit step function $x(t) = u(t)$.

$$X(s) = \int_{-\infty}^{\infty} u(t) e^{-st} dt = \int_0^{\infty} e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^{\infty} = \frac{1}{s}$$

$$X(s) = \frac{1}{s}, \quad \text{Re}\{s\} > 0$$

Thus $u(t) \leftrightarrow \frac{1}{s}$, the ROC is right half of the s-plane.

iii) $x(t) = \cos(\omega_0 t) u(t)$.

Sol: we know that $\cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$

$$X(s) = \int_{-\infty}^{\infty} \cos(\omega_0 t) u(t) e^{-st} dt = \int_0^{\infty} \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \cdot 1 \cdot e^{-st} dt$$

$$= \frac{1}{2} \int_0^{\infty} e^{-(s-j\omega_0)t} dt + \frac{1}{2} \int_0^{\infty} e^{-(s+j\omega_0)t} dt$$

$$= \frac{1}{2} \left(\frac{1}{s-j\omega_0} \right) + \frac{1}{2} \left(\frac{1}{s+j\omega_0} \right) = \frac{1}{2} \frac{2s}{s^2 + \omega_0^2}, \text{Re}\{s\} > 0$$

thus $\cos(\omega_0 t) u(t) \leftrightarrow \frac{s}{s^2 + \omega_0^2}, \text{Re}\{s\} > 0$.

iv) Laplace transform for sine function $x(t) = \sin(\omega_0 t) u(t)$

Sol: we know that $\sin(\omega_0 t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$

$$X(s) = \int_{-\infty}^{\infty} \sin(\omega_0 t) u(t) e^{-st} dt = \int_0^{\infty} \left(\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \right) \cdot 1 \cdot e^{-st} dt$$

$$= \frac{1}{2j} \int_0^{\infty} e^{-(s-j\omega_0)t} dt - \frac{1}{2j} \int_0^{\infty} e^{-(s+j\omega_0)t} dt$$

$$= \frac{1}{2j} \times \frac{1}{(s-j\omega_0)} - \frac{1}{2j} \times \frac{1}{s+j\omega_0} = \frac{1}{2j} \left(\frac{2j\omega_0}{s^2 + \omega_0^2} \right) = \frac{\omega_0}{s^2 + \omega_0^2}, \text{Re}\{s\} > 0$$

Thus $\sin(\omega_0 t) u(t) \leftrightarrow \frac{\omega_0}{s^2 + \omega_0^2}, \text{Re}\{s\} > 0$

v) Laplace transform of unit ramp function $x(t) = t u(t)$

Sol: $X(s) = \int_{-\infty}^{\infty} t u(t) e^{-st} dt = \int_0^{\infty} t e^{-st} dt = 0 \times 0 - 0 - \frac{1}{s^2} e^{-st} \Big|_0^{\infty}$
 $= \frac{1}{s^2}, \text{Re}\{s\} > 0$ [$\int u v dx = u \int v dx - \int \frac{du}{dx} \int v dx dx$]

properties of Laplace transform:

(10)

1) Linearity property.

$$\text{If } f_1(t) \longleftrightarrow F_1(s) \text{ \& } f_2(t) \longleftrightarrow F_2(s)$$

$$\text{then } a_1 f_1(t) + a_2 f_2(t) \longleftrightarrow a_1 F_1(s) + a_2 F_2(s).$$

2) In general,

$$a_1 f_1(t) + a_2 f_2(t) + \dots + a_n f_n(t) \longleftrightarrow a_1 F_1(s) + a_2 F_2(s) + \dots + a_n F_n(s).$$

② Time shifting property:

$$\text{If } f(t) \xleftrightarrow{\text{L.T.}} F(s).$$

$$\text{then } f(t-t_0) \xleftrightarrow{\text{L.T.}} e^{-st_0} F(s)$$

or

$$f(t+t_0) \xleftrightarrow{\text{L.T.}} e^{st_0} F(s).$$

③ Frequency shifting property:

$$\text{If } f(t) \xleftrightarrow{\text{L.T.}} F(s)$$

$$\text{then } e^{-at} f(t) \longleftrightarrow F(s+a)$$

or

$$e^{at} f(t) \longleftrightarrow F(s-a)$$

④ Scaling property

$$\text{If } f(t) \xleftrightarrow{\text{L.T.}} F(s)$$

$$\text{then } f(at) \xleftrightarrow{\text{L.T.}} \frac{1}{|a|} F\left(\frac{s}{a}\right); a > 0$$

⑤ Time Differentiation property of unilateral L.T.

$$\text{If } f(t) \xleftrightarrow{\text{L.T.}} F(s)$$

$$\text{Then } \frac{d}{dt} f(t) \longleftrightarrow sF(s) - f(0)$$

proof:

consider U.L.T of $f(t)$ is given by

$$L[f(t)] = F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

differentiate w.r.t s

$$\text{But } L\left[\frac{d}{dt} f(t)\right] = \int_0^{\infty} \frac{d}{dt} [f(t)] e^{-st} dt$$

$$= \int_0^{\infty} e^{-st} \frac{d}{dt} f(t) dt$$

$$[\because \int u dv = u \int dv - \int u \frac{dv}{dx} \int dx]$$

$$= e^{-st} f(t) \Big|_0^{\infty} + \int_0^{\infty} s e^{-st} f(t) dt$$

$$= 0 - f(0) + s \int_0^{\infty} f(t) e^{-st} dt$$

$$= -f(0) + sF(s)$$

$$[\because F(s) = \int_0^{\infty} f(t) e^{-st} dt \text{ by definition}]$$

$$= sF(s) - f(0)$$

$$L\left[\frac{d}{dt} f(t)\right] = sF(s) - f(0)$$

$$\text{Thus } \boxed{\frac{d}{dt} f(t) \xleftrightarrow{\text{L.T.}} sF(s) - f(0)}$$

⑥ Initial value theorem:

It states that if $f(t)$ and its first derivative are Laplace transformable, then the initial value of $f(t)$ is given by

$$f(0) = \lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} [sF(s)]$$

proof: Let us consider time differentiation property of unilateral L.T. i.e.,

$$\frac{d}{dt} f(t) \longleftrightarrow sF(s) - f(0)$$

or

$$L\left[\frac{d}{dt} f(t)\right] = sF(s) - f(0) \quad \text{--- (1)}$$

Now apply $L.T.$ on both sides $s \rightarrow \infty$

$$L.T. \lim_{s \rightarrow \infty} L\left[\frac{d}{dt} f(t)\right] = \lim_{s \rightarrow \infty} [sF(s) - f(0)] \quad \text{--- (2)}$$

no 's' terms so limit is not applicable.

consider LHS

$$L.T. \lim_{s \rightarrow \infty} L\left[\frac{d}{dt} f(t)\right] = L.T. \lim_{s \rightarrow \infty} \int_0^{\infty} \frac{d}{dt} f(t) e^{-st} dt = \int_0^{\infty} \frac{d}{dt} f(t) L.T. e^{-st} dt$$

$$L.T. \lim_{s \rightarrow \infty} L\left[\frac{d}{dt} f(t)\right] = 0 \quad \text{--- (3)}$$

from eq. (2) & (3)

$$0 = \lim_{s \rightarrow \infty} [sF(s) - f(0)]$$

$$f(0) = \lim_{s \rightarrow \infty} sF(s)$$

or

$$f(0) = \lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s) \quad \text{Hence proved.}$$

Note ① This theorem is applied only when initial value exists. (13)

②. The initial value theorem is used to calculate $f(0)$ directly from the Laplace transform $F(s)$ without the need of inverse transform.

Ex: ③ Find initial value of $F(s) = \frac{1}{s+1}$

Sol: initial value $f(0) = \lim_{s \rightarrow \infty} s F(s)$

$$= \lim_{s \rightarrow \infty} s \cdot \frac{1}{s+1} = \lim_{s \rightarrow \infty} \frac{s \times \frac{1}{s}}{s(1+\frac{1}{s})} = \frac{1}{1+\frac{1}{\infty}} = \frac{1}{1+0} = 1.$$

⑦ Final value theorem:

It states that if $f(t)$ and its first derivative are Laplace transformable, then the final value of $f(t)$ is given by

$$f(\infty) = \lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} (s F(s)).$$

proof: consider a signal $f(t)$ whose Laplace transform is $F(s)$. Then by property of differentiation.

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0) \quad \text{--- ①}$$

Now apply $\lim_{s \rightarrow 0}$ on both sides

$$\lim_{s \rightarrow 0} \mathcal{L}\left[\frac{df(t)}{dt}\right] = \lim_{s \rightarrow 0} [sF(s) - f(0)] \quad \text{--- ②}$$

consider LHS

$$\lim_{s \rightarrow 0} \left[s \frac{d}{dt} f(t) \right] = \lim_{s \rightarrow 0} \int_0^{\infty} \frac{d}{dt} f(t) e^{-st} dt$$

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$$= \int_0^{\infty} \frac{d}{dt} f(t) \lim_{s \rightarrow 0} e^{-st} dt$$

$$= \int_0^{\infty} \frac{d}{dt} f(t) dt$$

$$= f(t) \Big|_0^{\infty} = \lim_{t \rightarrow \infty} f(t) - f(0) \quad \text{--- (8)}$$

From eq (2) & (3)

$$\lim_{t \rightarrow \infty} f(t) - f(0) = \lim_{s \rightarrow 0} [s F(s) - f(0)]$$

Therefore $\boxed{f(\infty) = \lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} [s F(s)]}$

Ex-④ Find the Final value of $F(s) = \frac{s-1}{s(s^2-1)}$

Sol: consider Final value theorem

$$f(\infty) = \lim_{s \rightarrow 0} s F(s)$$

$$= \lim_{s \rightarrow 0} s \times \frac{s-1}{s(s^2-1)} = \lim_{s \rightarrow 0} \frac{\cancel{s} (s-1)}{(s+1)(\cancel{s}-1)} = \frac{1}{0+1} = 1$$

Inverse Laplace transform:

Ex-⑤ Find the Inverse Laplace transform of the function

$$F(s) = \frac{s(s+1)}{s(s+2)(s+4)}$$

Sol:

Since $F(s)$ is a proper rational function and has simple poles, using partial fraction expansion,

$$F(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+4} \quad \text{--- (1)}$$

(15)

$$\frac{5(s+1)}{s(s+2)(s+4)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+4}$$

$$5(s+1) = A(s+2) + Bs(s+4) + Cs(s+2)$$

put $s=0$

$$5(0+1) = A \times 2 + 0 + 0$$

$$A = 5/2$$

put $s=-2$

$$5(-1) = 0 + B(-2)(-2) + 0$$

$$B = 5/4$$

put $s=-4$

$$-15 = C(-4)(-2)$$

$$C = -15/8$$

Substituting A, B & C values in eqn (1)

$$\text{Thus } F(s) = \frac{5}{2s} + \frac{5}{4(s+2)} - \frac{15}{8(s+4)}$$

$$\text{Then } f(t) = L^{-1}\{F(s)\} = \frac{5}{2}u(t) + \frac{5}{4}e^{-2t}u(t) - \frac{15}{8}e^{-4t}u(t)$$

$$= \frac{5}{8} [u(t) + 2e^{-2t}u(t) - 3e^{-4t}u(t)]$$

$$f(t) = \frac{5}{8} [1 + 2e^{-2t} - 3e^{-4t}]u(t).$$

Ex - (6) Find the inverse Laplace transform of the function $F(s) = \frac{s^2+3}{(s+2)(s^2+2s+5)}$

$$F(s) = \frac{A}{s+2} + \frac{Bs+C}{s^2+2s+5} \quad \text{--- (1)}$$

$$\frac{s^2+3}{(s+2)(s^2+2s+5)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+2s+5}$$

$$s^2 + 3 = A(s^2 + 2s + 5) + (Bs + C)(s + 2)$$

(16)

put $s = -2$

$$7 = A(4 - 4 + 5)$$

$$A = 7/5 \text{ and comparing } s^2 \text{ terms}$$

$$1 = A + B$$

$$\Rightarrow 1 = \frac{7}{5} + B \Rightarrow B = 1 - \frac{7}{5} = -\frac{2}{5}$$

comparing the constant terms

$$3 = 5A + C \Rightarrow 3 = 5 \times \frac{7}{5} + C$$

$$-\frac{4}{2} = C \Rightarrow C = -2$$

$$3 = 7 + C$$

$$-\frac{4}{2} = C$$

$$C = -2$$

Substituting A, B & C values in eqn ①

$$\textcircled{1} \Rightarrow F(s) = \frac{7}{s+2} + \frac{-\frac{2}{5}s - 2}{s^2 + 2s + 5}$$

$$= \frac{7}{5(s+2)} + \frac{-2s - 10}{5(s^2 + 2s + 5)} = \frac{1}{5} \left[\frac{7}{s+2} + \frac{-2s - 10}{(s+1)^2 + 4} \right]$$

$$= \frac{1}{5} \left[\frac{7}{s+2} + \frac{-2s - 2 - 8}{(s+1)^2 + 4} \right] = \frac{1}{5} \left[\frac{7}{s+2} + \frac{-2(s+1) - 8}{(s+1)^2 + 4} \right]$$

$$= \frac{1}{5} \left[\frac{7}{s+2} - 2 \frac{(s+1)}{(s+1)^2 + 4} - \frac{8}{(s+1)^2 + 4} \right]$$

$$= \frac{1}{5} \left[\frac{7}{s+2} - 2 \frac{(s+1)}{(s+1)^2 + 2^2} - 4 \times \frac{2}{(s+1)^2 + 2^2} \right]$$

Taking inverse L.T

$$f(t) = \frac{1}{5} \left[7 e^{-2t} u(t) - 2 e^{-t} \cos 2t - 4 e^{-t} \sin 2t \times u(t) \right]$$

$$\therefore f(t) = \frac{7}{5} e^{-2t} u(t) - 2 e^{-t} \left[\frac{1}{5} \cos 2t + \frac{2}{5} e^{-t} \sin 2t u(t) \right]$$

Ex-⑦ Find the inverse Laplace transform of the function $F(s) = \frac{(s+1)(s+3)}{(s+2)(s+4)}$ (17)

Sol: Given function $F(s)$ has equal number of poles and zeros,

$$F(s) = \frac{s^2+4s+3}{s^2+6s+8} = \frac{s^2+6s+8-2s-5}{s^2+6s+8} = 1 - \frac{2s+5}{(s+2)(s+4)} \quad \text{--- (1)}$$

Using partial fraction expansion,

$$\frac{2s+5}{(s+2)(s+4)} = \frac{A}{s+2} + \frac{B}{s+4} \quad \text{--- (2)}$$

$$2s+5 = A(s+4) + B(s+2)$$

$$\text{put } s = -2$$

$$-4+5 = 2A + 0$$

$$A = \frac{1}{2}$$

$$\text{put } s = -4$$

$$-3 = 0 - 2B$$

$$B = \frac{3}{2}$$

Substituting A & B values in eqn (2)

$$\text{eqn (2)} \Rightarrow \frac{2s+5}{(s+2)(s+4)} = \frac{1/2}{s+2} + \frac{3/2}{s+4}$$

$$\text{eqn (1)} \Rightarrow F(s) = 1 - \left(\frac{1/2}{s+2} + \frac{3/2}{s+4} \right)$$

$$F(s) = 1 - \frac{1}{2(s+2)} - \frac{3}{2(s+4)}$$

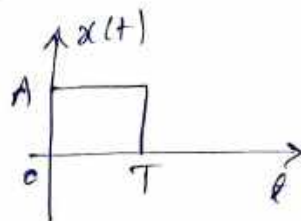
Taking ILT on both sides

$$L^{-1}\{F(s)\} = L^{-1}\left[1 - \frac{1}{2(s+2)} - \frac{3}{2(s+4)}\right]$$

$$f(t) = \delta(t) - \frac{1}{2}e^{-2t}u(t) - \frac{3}{2}e^{-4t}u(t).$$

Laplace Transform of certain signals using waveform synthesis:

Ex-⑤ pulse waveform:



pulse wave form.

the above given pulse can be represented graphically as follows

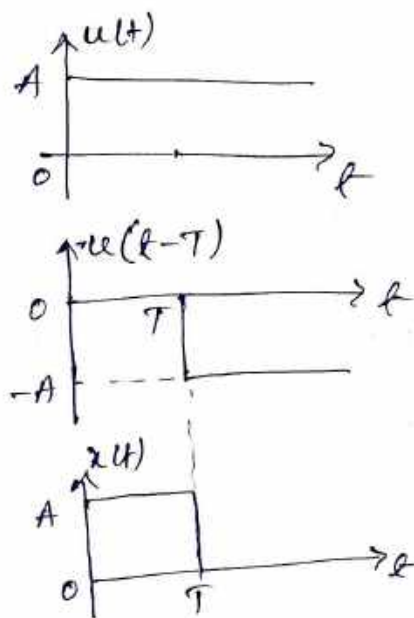


Fig. Synthesis of pulse waveform.

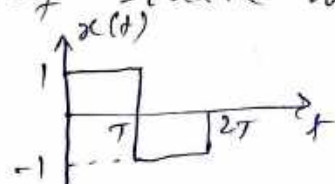
And $x(t)$ can be expressed mathematically as

$$x(t) = A u(t) - A u(t-T)$$

using Laplace transform

$$X(s) = A \frac{1}{s} - A \frac{e^{-sT}}{s} = \frac{A}{s} (1 - e^{-sT})$$

Ex-⑨ Find the Laplace transform of square wave



Sol: Given signal $x(t)$ is defined as

(19)

$$x(t) = \begin{cases} 1, & 0 \leq t \leq T \\ -1, & T \leq t \leq 2T \end{cases}$$

consider Laplace transform of $x(t)$

$$L[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

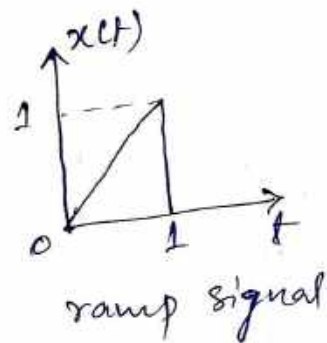
$$= \int_0^T 1 e^{-st} dt + \int_T^{2T} (-1) e^{-st} dt$$

$$= \left[\frac{e^{-st}}{-s} \right]_0^T + \left[\frac{e^{-st}}{+s} \right]_T^{2T}$$

$$= \frac{e^{-sT}}{-s} + \frac{1}{s} + \frac{e^{-2sT}}{s} - \frac{e^{-sT}}{s}$$

$$= \frac{1}{s} [1 - 2e^{-sT} + e^{-2sT}]$$

Ex-10 Find the Laplace transform of ramp signal shown in Fig.



Sol:

$$x(t) = \begin{cases} t, & 0 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Laplace transform of $x(t)$

$$L[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$= \int_0^1 t e^{-st} dt = t \int_0^1 e^{-st} dt - \int_0^1 \left[\frac{d}{dt} t \right] e^{-st} dt$$

$$= \left[\frac{t e^{-st}}{-s} \right]_0^1 + \int_0^1 \frac{e^{-st}}{s} dt = \left[\frac{t e^{-st}}{-s} \right]_0^1 - \left[\frac{e^{-st}}{s^2} \right]_0^1$$

$$= \frac{e^{-s}}{-s} - 0 + \frac{e^{-s}}{s^2} + \frac{1}{s^2} = \frac{1}{s^2} - e^{-s} \left[\frac{1}{s^2} + \frac{1}{s} \right]$$

The transfer function of LTI System:

(20)

The transfer function of a system completely describes the behaviour of the system with any type of input.

Let $x(t)$, $y(t)$ and $h(t)$ have Laplace transforms $X(s)$, $Y(s)$ and $H(s)$ respectively.

→ The transfer function of a system is defined as the ratio of the Laplace transform of the o/p signal to the Laplace transform of the input signal when all initial conditions are zero.

$$H(s) = \frac{Y(s)}{X(s)}$$

We know that if $x(t)$ is the input signal, then the output can be obtained from convolution given by

$$y(t) = x(t) * h(t)$$

The Laplace transform of the output signal is

$$L[y(t)] = L[x(t) * h(t)]$$

$$Y(s) = X(s) \cdot H(s)$$

$$\boxed{H(s) = \frac{Y(s)}{X(s)}}$$

Causality: It is observed that $h(t) = 0$ for $t < 0$, a causal system and thus is right-sided. Therefore, the ROC associated with the transfer function of a causal system is a right half plane. The converse, however, is not true.

Stability:

(2)

- 1) An LTI system is stable if and only if the ROC of its transfer function contains the $j\omega$ axis.
- 2) A causal system with a proper rational transfer function $H(s)$ is stable if and only if all of its poles are in the left half of the s -plane. That is all the poles ~~are~~ have negative real parts.

● Note: 1. To find out system transfer function, all initial conditions of the input signal should be zero. the transfer function is independent of the input signal.

$$\text{transfer function } H(s) = \frac{Y(s)}{X(s)} = \frac{Y(s)}{1} = Y(s)$$

$$[\because x(t) = \delta(t) \\ \delta(t) \xleftrightarrow{L.T} 1.]$$

● 2. The transfer function is a rational function in s .

3. Frequency response can be obtained by setting

$$H(\omega) = H(s) \Big|_{s=j\omega}$$

4. The steady state frequency response of LTI system x_s can be obtained as

$$H(\omega) = \frac{Y(s)}{X(s)} \Big|_{s=j\omega} = \frac{Y(j\omega)}{X(j\omega)}$$

$$H(\omega) = |H(\omega)| \angle \phi(\omega)$$

Advantages of Laplace Transform:

(22)

1. The higher order differential equations can be easily solved by using simple algebraic equations.
2. LT is a powerful tool for analysing system properties in the form of transform function.
3. LT can be used to analyse many classes of signals and systems which are not absolutely integrable.
4. LT provides solutions for many unstable systems such as impulse functions.
5. Fourier transform can be obtained from Laplace transform by substituting $\sigma = 0$, i.e., $s = j\omega$

Limitations:

1. Laplace transform does not converge for some type of signals whose amplitude grows faster than time.
2. The region of convergence is needed to obtain inverse Laplace transform.
3. LT is very difficult to solve complex integrals directly in the process of inverse Laplace transform.

$$\rightarrow [e^{-at} \cos(\omega_0 t)] u(t) \longleftrightarrow \frac{s+a}{(s+a)^2 + \omega_0^2}, \quad \text{Re}\{s\} > -\text{Re}\{a\}$$

$$\rightarrow [e^{-at} \sin(\omega_0 t)] u(t) \longleftrightarrow \frac{\omega_0}{(s+a)^2 + \omega_0^2}, \quad \text{Re}\{s\} < -\text{Re}\{a\}$$

Z-Transform

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The basic mathematical tool for analysing and synthesising discrete-time signals and systems is the Z-transform. The Z-transform can be derived directly by applying Laplace transform on discrete time signals; it is the discrete-time counterpart of the Laplace transform.

The Z-transform of a Discrete sequence:

The Z-transform of a discrete-time sequence $x[n]$ is defined as

$$X(z) = \mathcal{Z}[x[n]] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

where $z = re^{j\omega}$ is a complex variable defined on a complex Z-plane in the region of convergence. The Z-plane is a polar plot with radius r and ω the angular frequency.

~~$$\mathcal{Z}^{-1}[X(z)] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$~~

$$\text{Z-transform pair: } x[n] \xleftrightarrow{\mathcal{Z}} X(z)$$

Above defined Z-transform is Bilateral Z-transform or two sided Z-transform. And unilateral or one sided Z-transform of $x[n]$ is defined as

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

Ex-1

Determine the Z-transform of the sequence

a) $x[n] = [1, 2, 3, 0, 4]$, and b) $x[n] = \{3, 2, 1, 0, 1, 2, 3\}$

Sol:

a) Given $x[n] = [1, 2, 3, 0, 4]$. Taking the z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=0}^4 x[n] z^{-n}$$

$$X(z) = 1z^0 + 2z^{-1} + 3z^{-2} + 0z^{-3} + 4z^{-4}$$

$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-4}$$

b) $x[n] = \{3, 2, 1, 0, 1, 2, 3\}$

Taking z-transform,

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-3}^3 x[n] z^{-n}$$

$$X(z) = 3z^3 + 2z^2 + 1z^1 + 0z^0 + 1z^{-1} + 2z^{-2} + 3z^{-3}$$

$$= 3z^3 + 2z^2 + z + z^{-1} + 2z^{-2} + 3z^{-3}$$

Distinction between Laplace, Fourier and z-transform:

The z-transform can be derived from the Laplace transform applied to the sampled signal. Let $x(t)$ be a continuous-time signal sampled by an ideal impulse train $\delta_T(t)$. The sampled signal is

$$x_s(t) = x(t) \delta_T(t) = \sum_{n=-\infty}^{\infty} x(t) \delta(t - nT)$$

Since the $\delta(t - nT)$ is non-zero only at sampling instants $t = nT$, $x(t)$ can be replaced by $x[nT]$.

$$x_s(t) = \sum_{n=-\infty}^{\infty} x[nT] \delta(t - nT)$$

Taking Laplace transform on both sides yields

$$X_s(s) = L[x_s(t)] = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[nT] \delta(t - nT) e^{st} dt$$

Interchanging Integration & summation

(25)

$$X_s(s) = \sum_{n=-\infty}^{\infty} x[nT] \int_{-\infty}^{\infty} \delta(t-nT) e^{-st} dt$$

$$= \sum_{n=-\infty}^{\infty} x[nT] e^{-s n T}$$

$\therefore$ Sifting property of impulse.

put the complex variable $z = e^{sT}$. Then

$$\int_{-\infty}^{\infty} x(t) \delta(t-t_0) dt = \begin{cases} x(t_0), & t_0 \text{ is finite} \\ 0 & \text{otherwise} \end{cases}$$

$$L[x_s(t)] = X(s) = \sum_{n=-\infty}^{\infty} x[nT] z^{-n}$$

Thus the Laplace transform of a sampled signal $x[nT]$ is equal to the z-transform, where $z = e^{sT}$. For convenience, assume the sample time $T = 1$ sec.

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

For causal signals, $x(t) = 0$, for $t < 0$.

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

We know that Laplace operator $s = \sigma + j\omega$ then the z-transform operator will be

$$z = e^{(\sigma + j\omega)T} = e^{\sigma T} e^{j\omega T} = r e^{j\theta}$$

the magnitude is $r = |z| = e^{\sigma T}$

and the angle is $\theta = \angle z = \omega T$

for $\sigma = 0$ or $s = j\omega$, $z = e^{j\omega}$ and z-transform becomes

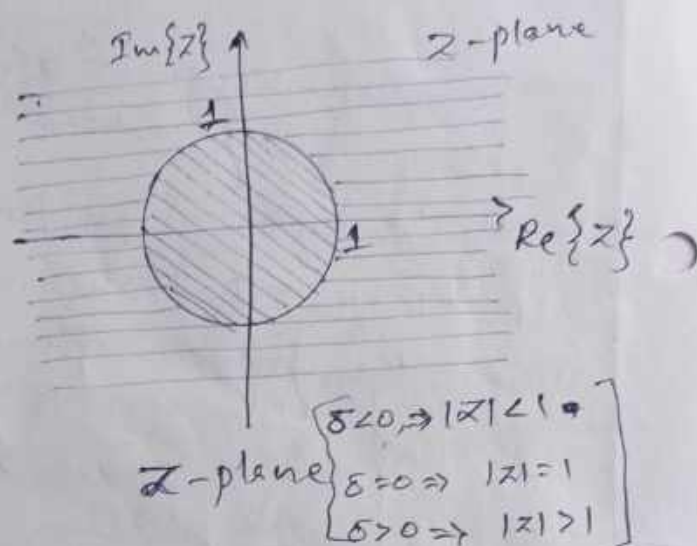
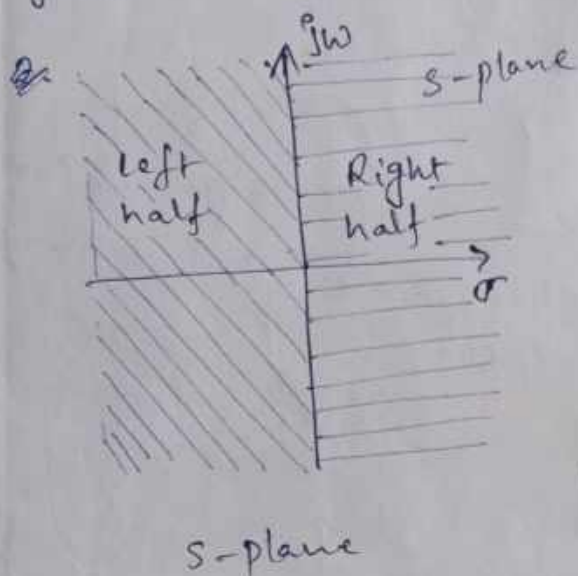
$$X(\omega) = \sum_{n=0}^{\infty} x[n] e^{-j\omega n}$$

This represents the Discrete time Fourier transform

Mapping of s-plane and z-plane:

The s-plane is a rectangular plot, whereas the z-plane is polar plot with radius $r = |z| = e^{\sigma T}$, angle $\theta = \angle z = \omega T$

Therefore the mapping of s-plane and z-plane is as follows:
1. The right half of the s-plane where $\sigma > 0$ corresponds to $|z| > 1$, while the left half of the s-plane where $\sigma < 0$ corresponds to $|z| < 1$.



Mapping of the s-plane and the z-plane

2. The line $\sigma = 0$, or $s = j\omega$ on the s-plane, corresponds to $|z| = 1$, a unit circle on the z-plane.

- $\rightarrow$ The left half of the s-plane is mapped to the interior of the unit circle and
- $\rightarrow$ the right half of the s-plane is mapped to the outside of unit circle in the z-plane.

Region of convergence (ROC) in z-Transforms

The range of values of the complex variable z for which the z-transform converges is called the region of convergence (ROC).

For a given sequence $x[n]$, the z-transform exists if the infinite sum of series $x[n]z^{-n}$ converges. the (27) region of convergence of $X(z)$ can be represented graphically with a pole-zero plot on the z-plane, where $z = re^{j\omega T}$, r is a radius and ωT is an angle in polar form.

poles and zeros of $X(z)$

The general form of $X(z)$ is given by

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_n z^{-n}}$$

The roots of the numerator polynomial $N(z)$ are called the zeros of $X(z)$. Similarly, the roots of the denominator polynomial $D(z)$ are called the poles of $X(z)$.

properties of the ROC:

property 1: The ROC does not contain any poles. Since at all pole values $X(z)$ becomes infinite and does not converge.

property 2: If $x[n]$ is a finite sequence and $X(z)$ converges for some values of z , then the ROC is the entire z-plane except possibly $z=0$ and/or $z=\infty$.

a) Right Sided Sequence, finite duration. ROC is entire z-plane except at $z=0$

b) Left sided, finite ROC is entire z-plane except at $z=\infty$

c) Two-sided, finite ROC entire z-plane except at $z=0$ & $z=\infty$.

property 3: If $x[n]$ is right sided, $N \leq n < \infty$.

(28)

ROC is in the form $|z| > r_{\max}$ or $\infty > |z| > r_{\max}$.

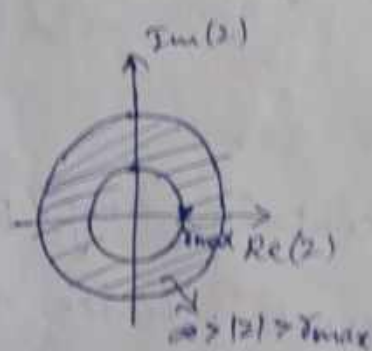
The ROC is exterior of the circle $|z| = r_{\max}$, possible exception $z = \infty$.

property 4 If $x[n]$ is left-sided, $-\infty \leq n \leq N$, where N is a finite number.

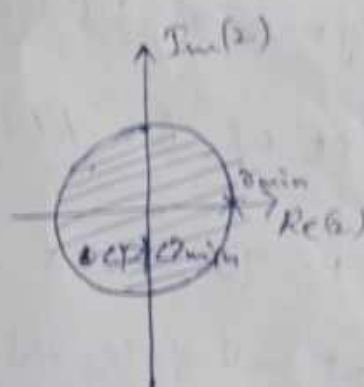
ROC is in the form $|z| < r_{\min}$, or $0 < |z| < r_{\min}$

The ROC is interior of the circle $|z| = r_{\min}$, possibly exception $z = 0$

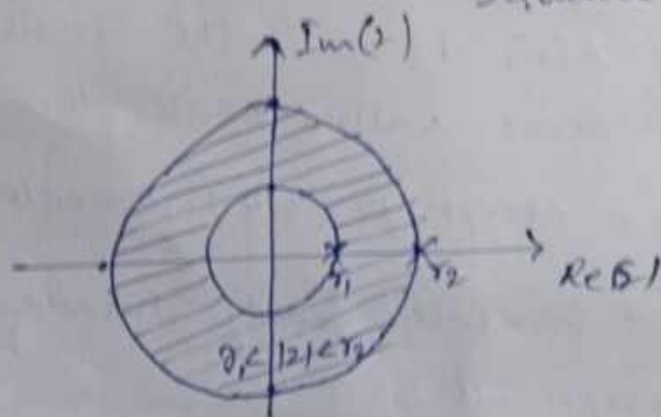
property 5: If $x[n]$ is a two-sided sequence of infinite length, ROC is of the form $r_1 < |z| < r_2$ where r_1, r_2 are the magnitudes of the two poles of $X(z)$.



(a) Right Sided Sequence



(b) Left-Sided Sequence



(c) Two Sided

Z-transform for some basic sequences:

(29)

1. Unit Impulse Sequence.

Sol: Unit Impulse Sequence is defined as

$$\delta[n] = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Z-transform of ~~unit~~ unit impulse function $x[n] = \delta[n]$ is given by

$$Z[\delta[n]] = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n}$$
$$= 1, \quad 0 < |z| < \infty$$

$\delta[n] \longleftrightarrow 1$
The ROC is entire z-plane except $z = \infty$

2. Unit Step Sequence

Sol: we know that unit step sequence

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Its Z-transform is

$$Z[u[n]] = \sum_{n=0}^{\infty} u[n] z^{-n}$$
$$= 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + \dots$$
$$= \frac{1}{1-z^{-1}} = \frac{z}{z-1}, \quad |z| > 1$$

$$\left[\because 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}, \quad |x| < 1 \right]$$

The ROC is $|z| > 1$, $u[n] \longleftrightarrow \frac{z}{z-1}$

3) Unit Ramp Sequence.

Sol: we know that the unit ramp sequence

$$r[n] = \begin{cases} n & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and also $x[n] = nu[n]$

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the z-transform is

$$\begin{aligned} Z[x[n]] &= \sum_{n=0}^{\infty} nu[n] \bar{z}^{-n} = \sum_{n=0}^{\infty} n \bar{z}^{-n} = \bar{z}^{-1} + 2\bar{z}^{-2} + 3\bar{z}^{-3} + \dots \\ &= \bar{z}^{-1}(1 + \bar{z}^{-1} + \bar{z}^{-2} + \dots) + \bar{z}^{-2}(1 + \bar{z}^{-1} + \bar{z}^{-2} + \dots) + \bar{z}^{-3}(1 + \bar{z}^{-1} + \bar{z}^{-2} + \dots) + \dots \\ &= \bar{z}^{-1}(1 + \bar{z}^{-1} + \bar{z}^{-2} + \dots)(\bar{z}^{-1} + \bar{z}^{-2} + \bar{z}^{-3} + \dots) \\ &= \bar{z}^{-1}(1 + \bar{z}^{-1} + \bar{z}^{-2} + \dots)(\bar{z}^{-1} + \bar{z}^{-2} + \bar{z}^{-3} + \dots) \\ &= \bar{z}^{-1} \left(\frac{1}{1-\bar{z}^{-1}} \right) \left(\frac{1}{1-\bar{z}^{-1}} \right) = \bar{z}^{-1} \left(\frac{1}{1-\bar{z}^{-1}} \right)^2 \\ &= \bar{z}^{-1} \left(\frac{\bar{z}^2}{(\bar{z}-1)^2} \right) = \frac{\bar{z}}{(\bar{z}-1)^2} \text{ for } |\bar{z}| > 1, \end{aligned}$$

$$x[n] \longleftrightarrow \frac{\bar{z}}{(\bar{z}-1)^2}, \text{ ROC is } |\bar{z}| > 1.$$

4) Find the z-transform of unilateral function
 $x[n] = a^n u[n]$

Sol: From basic definition of z-transform

$$Z[x[n]] = \sum_{n=-\infty}^{\infty} x[n] \bar{z}^{-n}$$

$$\text{and } x[n] = a^n u[n]$$

$$X(\bar{z}) = Z[a^n u[n]] = \sum_{n=-\infty}^{\infty} a^n u[n] \bar{z}^{-n}$$

$$= \sum_{n=0}^{\infty} (a\bar{z}^{-1})^n$$

$$X(\bar{z}) = 1 + (a\bar{z}^{-1}) + (a\bar{z}^{-1})^2 + (a\bar{z}^{-1})^3 + \dots + (a\bar{z}^{-1})^n + \dots$$

Sum of infinite series yields

$$X(\bar{z}) = \frac{1}{1-a\bar{z}^{-1}}, \text{ if } |a\bar{z}^{-1}| < 1 \text{ or } |\bar{z}| > |a|$$

$$\Rightarrow X(\bar{z}) = \frac{\bar{z}}{\bar{z}-a}$$

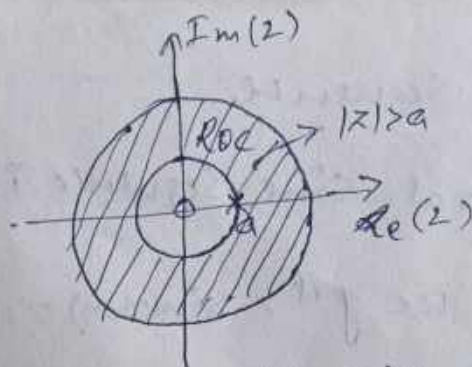


Fig. The pole-zero plot with ROC on z -plane of right sided Sequence

⑤ unilateral cosine Sequence.

sol: The unilateral cosine Sequence is $x[n] = \cos(\omega_0 n) u[n]$

Expanding we get

$$\cos(\omega_0 n) = \frac{1}{2} e^{jn\omega_0} + \frac{1}{2} e^{-jn\omega_0}$$

The z -transform is

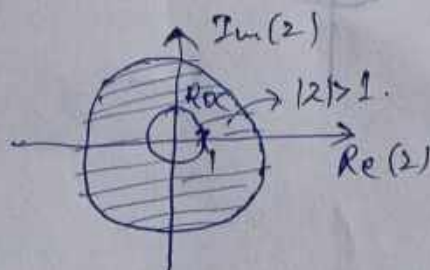
$$Z[\cos(\omega_0 n) u[n]] = \sum_{n=0}^{\infty} \left[\frac{1}{2} e^{jn\omega_0} + \frac{1}{2} e^{-jn\omega_0} \right] z^{-n}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} (e^{j\omega_0} z^{-1})^n + \frac{1}{2} \sum_{n=0}^{\infty} (e^{-j\omega_0} z^{-1})^n$$

$$= \frac{1}{2} \left(\frac{z}{z - e^{j\omega_0}} \right) + \frac{1}{2} \left(\frac{z}{z - e^{-j\omega_0}} \right) = \frac{z}{2} \left[\frac{z - e^{-j\omega_0} + z - e^{j\omega_0}}{z^2 - z e^{-j\omega_0} - z e^{j\omega_0} + 1} \right]$$

$$= \frac{z \left[2z - (e^{j\omega_0} + e^{-j\omega_0}) \right]}{z^2 - z(e^{j\omega_0} + e^{-j\omega_0}) + 1} = \frac{z(z - \cos \omega_0)}{z^2 - 2z \cos \omega_0 + 1}$$

$$\cos(\omega_0 n) u[n] \longleftrightarrow \frac{z(z - \cos \omega_0)}{z^2 - 2z \cos \omega_0 + 1}, \text{ ROC is } |z| > 1$$



⑥ unilateral sine sequence:

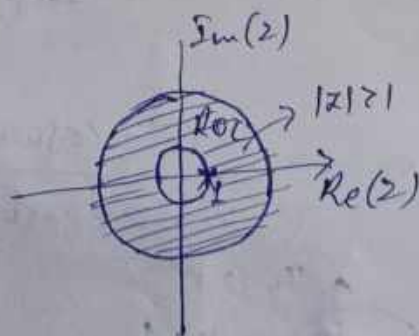
Sol: The unilateral sine sequence is $x[n] = \sin(\omega_0 n) u[n]$,

Expanding, we get, $\sin(\omega_0 n) = \frac{1}{2j} e^{jn\omega_0} - \frac{1}{2j} e^{-jn\omega_0}$

The z-transform is

$$\begin{aligned} Z[\sin(\omega_0 n) u[n]] &= \sum_{n=0}^{\infty} \left[\frac{1}{2j} e^{jn\omega_0} - \frac{1}{2j} e^{-jn\omega_0} \right] z^{-n} \\ &= \frac{1}{2j} \sum_{n=0}^{\infty} (e^{j\omega_0} z^{-1})^n - \frac{1}{2j} \sum_{n=0}^{\infty} (e^{-j\omega_0} z^{-1})^n \\ &= \frac{1}{2j} \left[\frac{1}{1 - e^{j\omega_0} z^{-1}} - \frac{1}{1 - e^{-j\omega_0} z^{-1}} \right] \\ &= \frac{1}{2j} \left(\frac{z}{z - e^{j\omega_0}} - \frac{z}{z - e^{-j\omega_0}} \right) = \frac{z}{2j} \left[\frac{z - e^{-j\omega_0} + e^{j\omega_0}}{z^2 - z e^{-j\omega_0} - z e^{j\omega_0} + 1} \right] \\ &= \frac{z \left[\frac{e^{j\omega_0} - e^{-j\omega_0}}{2j} \right]}{z^2 - 2z \left(\frac{e^{j\omega_0} + e^{-j\omega_0}}{2} \right) + 1} = \frac{z \sin(\omega_0)}{z^2 - 2z \cos \omega_0 + 1} \end{aligned}$$

$$\sin(\omega_0 n) u[n] \longleftrightarrow \frac{z \sin \omega_0}{z^2 - 2z \cos \omega_0 + 1}, \text{ Roc is } |z| > 1.$$



properties of Z-Transform:

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1. Linearity property:

If $x_1[n] \xleftrightarrow{Z.T} X_1(z)$ then

$$a x_1[n] + b x_2[n] \xleftrightarrow{Z.T} a X_1(z) + b X_2(z)$$

where a & b are arbitrary constants

proof: The Z-transform of the sequence $x[n]$ is given by

$$Z[x[n]] = X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$Z[a x_1[n] + b x_2[n]] = \sum_{n=-\infty}^{\infty} [a x_1[n] + b x_2[n]] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a x_1[n] z^{-n} + \sum_{n=-\infty}^{\infty} b x_2[n] z^{-n}$$

$$= a \sum_{n=-\infty}^{\infty} x_1[n] z^{-n} + b \sum_{n=-\infty}^{\infty} x_2[n] z^{-n}$$

$$= a X_1(z) + b X_2(z)$$

$$\therefore \boxed{a x_1[n] + b x_2[n] \xleftrightarrow{Z.T} a X_1(z) + b X_2(z)}$$

2. Time shifting property:

If $x[n] \xleftrightarrow{Z.T} X(z)$ then $x[n-k] \leftrightarrow z^{-k} X(z)$

proof: The Z-transform of the sequence $x[n]$ is given by

$$Z[x[n]] = X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$Z[x[n-k]] = \sum_{n=-\infty}^{\infty} x[n-k] z^{-n}$$

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$$\text{put } n-k=m \quad \left| \begin{array}{l} n=\infty \Rightarrow m=\infty \\ n=-\infty \Rightarrow m=-\infty \end{array} \right.$$

$$Z[x[m]] = \sum_{m=-\infty}^{\infty} x[m] z^{-m-k} = \sum_{m=-\infty}^{\infty} x[m] z^{-m} z^{-k}$$

$$= z^{-k} \sum_{m=-\infty}^{\infty} x[m] z^{-m}$$

$$= z^{-k} X(z)$$

therefore $\boxed{Z[x[n-k]] \xleftrightarrow{z.T} z^{-k} X(z)}$

⑧ Time scaling property:

if $x[n] \xleftrightarrow{z.T} X(z)$ then $a^n x[n] \xleftrightarrow{z.T} X\left(\frac{z}{a}\right)$

proof: The z-transform of sequence $x[n]$ is given by

$$Z[x[n]] = X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$Z[a^n x[n]] = \sum_{n=-\infty}^{\infty} a^n x[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[n] (a^{-1} z)^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] \left(\frac{z}{a}\right)^{-n}$$

$$= X\left(\frac{z}{a}\right)$$

$$a^n x[n] \xleftrightarrow{z.T} X\left(\frac{z}{a}\right)$$

Inverse z-Transform:

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Ex Find the z -inverse z -transform of $X(z) = \frac{2z(z-0.25)}{(z-0.2)(z-0.3)}$

Solution:

Given $X(z) = \frac{2z(z-0.25)}{(z-0.2)(z-0.3)}$ By dividing with z & taking

partial fractions

$$\frac{X(z)}{z} = \frac{2(z-0.25)}{(z-0.2)(z-0.3)} \quad \text{--- (1)}$$

$$\frac{X(z)}{z} \frac{2(z-0.25)}{(z-0.2)(z-0.3)} = \frac{A}{(z-0.2)} + \frac{B}{(z-0.3)} \quad \text{--- (2)}$$

$$2(z-0.25) = A(z-0.3) + B(z-0.2)$$

put $z = 0.2$

$$2(0.2-0.25) = A(0.2-0.3) + 0$$

$$2 \times -0.05 = A \times -0.1$$

$$0.1 = 0.1 \times A$$

$$A = 1$$

put $z = 0.3$

$$2(0.05) = 0 + 0.1 \times B$$

$$B = 1$$

Substituting A & B values in eqn (1) & (2)

$$\frac{X(z)}{z} = \frac{1}{z-0.2} + \frac{1}{z-0.3}$$

$$X(z) = \frac{z}{z-0.2} + \frac{z}{z-0.3}$$

Applying inverse Z-transform.

$$x[n] = (0.2)^n u[n] + (0.3)^n u[n]$$

$$x[n] = [(0.2)^n + (0.3)^n] u[n]$$

$$[\because a^n u[n] \longleftrightarrow \frac{z}{z-a}]$$

Sampling Theorem

(1)

The process of converting a continuous signal to a discrete signal is called sampling; a reconstruction process is applied on the samples to get the continuous signal back.

Sampling theorem:

The sampling theorem is known as Shannon's sampling theorem. States that A band-limited signal having no frequency components higher than f_m Hz is completely described (or represented) by its sample values located at uniform intervals separated by less than or equal to $\frac{1}{2f_m}$ seconds.

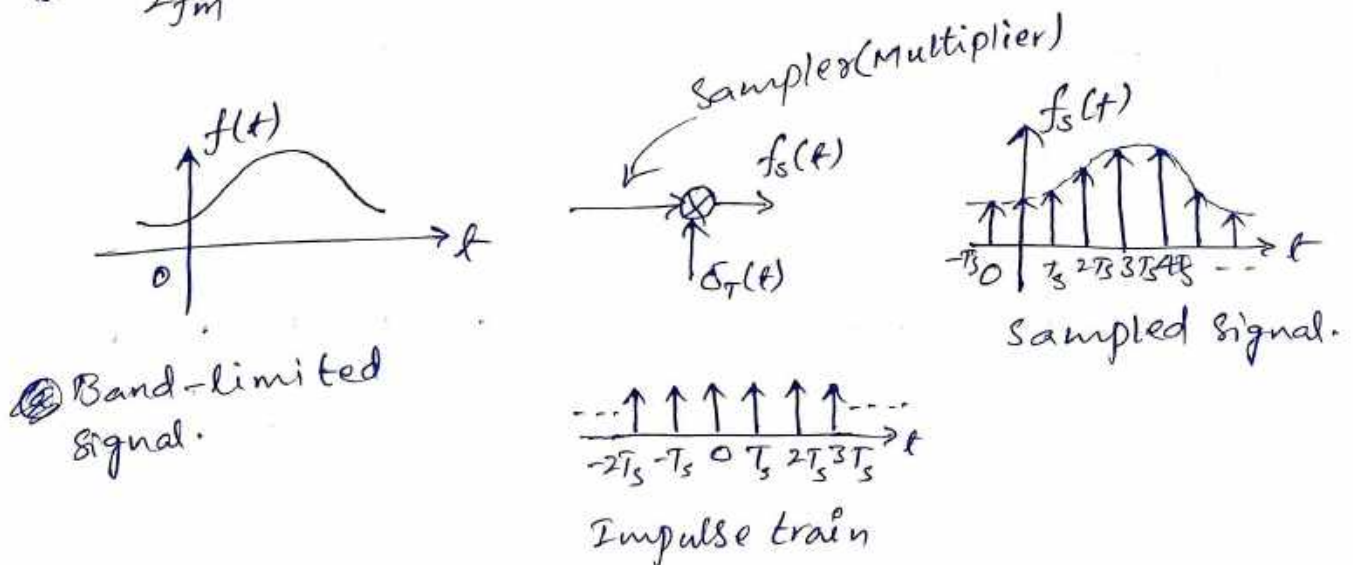
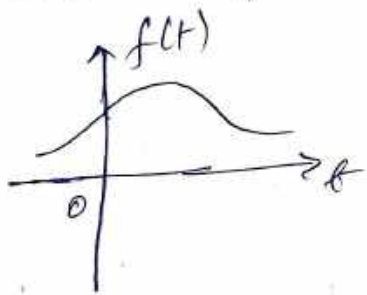


Fig 1. Sampling process of a continuous time signal.

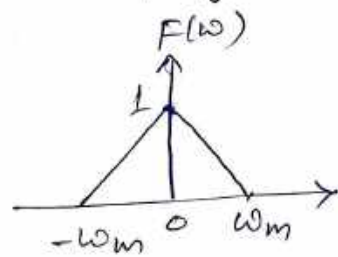
Graphical proof for band limited signals. (2)

Band-limited ^{* signal} is one whose frequency content is confined to a specific range.

Let us consider a band-limited signal $f(t)$ having no frequency components beyond f_m Hz. When this signal is multiplied by a periodic impulse function $\delta_T(t)$; the product yields a sequence of sample values located at uniform intervals of T_s seconds, shown in Fig. 2. This product of $f(t)$ and $\delta_T(t)$ represents the samples of $f(t)$ located at uniform interval of T_s . And frequency spectrum of also shown in the Fig.

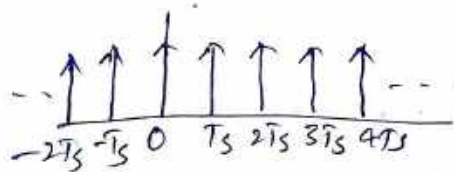


$\longleftrightarrow$

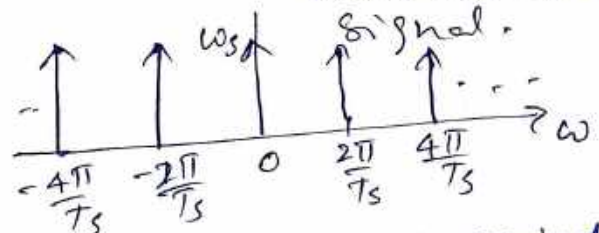


(a) Band-limited signal

(b) spectrum of band-limited signal.



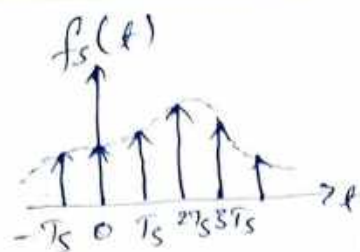
$\longleftrightarrow$



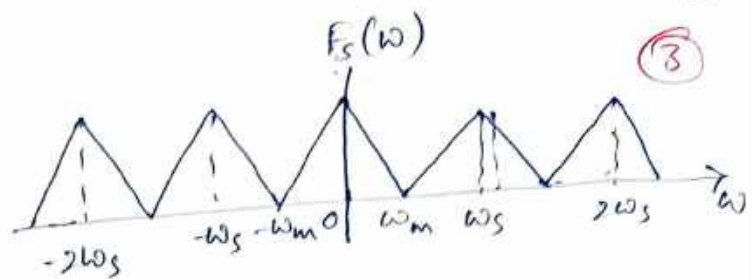
(c) Train of impulse (or periodic impulse)

(d) spectrum of periodic impulse.

continues in next page



(c) Sampled version of band-limited signal



(d) Spectrum of sampled band-limited signal.

Fig 2. Graphical representation of sampling process for band-limited signal.

It is observed that the function $F(\omega)$ repeats itself every ω_s rad/sec interval. That is, $F(\omega)$ will repeat periodically in the frequency domain without overlap, provided $\omega_s \geq 2\omega_m$. That is,

$$\frac{2\pi}{T_s} \geq 2(2\pi f_m) \quad \text{or} \quad T_s \leq \frac{1}{2f_m} \text{ Seconds.}$$

Therefore, if the samples of $f(t)$ are taken at regular intervals less than ~~$1/2f_m$~~ $1/2f_m$ seconds apart, the continuous signal can be recovered without loss of information by passing the sampled signal through a lowpass filter with cut-off frequency f_m Hz.

If the sampling frequency of $F_s(\omega)$ is less than the bandwidth of $F(\omega)$, i.e., $\omega_s < 2\omega_m$, then the spectra of

$F(\omega)$ overlaps with each other, resulting in a distorted spectrum. (4)

If the sampling frequency of $F_s(\omega)$ is equal to the bandwidth of $F(\omega)$, i.e., $\omega_s = 2\omega_m$, then this condition is called Nyquist criterion, and the sampling frequency (ω_s) is called Nyquist frequency.

→ The minimum sampling frequency that required for proper sampling is called Nyquist sampling rate or sampling frequency ($f_s = 2f_m$).

→ The maximum time interval is called (T_s = $\frac{1}{2f_m}$ seconds.) Nyquist interval.

Analytical proof of sampling theorem for band-limited signal:

proof using Frequency convolution theorem:

Let us consider a band-limited signal

$f(t)$ having no frequency components beyond f_m Hz. when this signal is multiplied by a periodic function

(signal) $\delta_T(t)$; the product yields a sequence of sample values located at uniform intervals of T sec.

Shown in Fig 1. This product of $f(t)$ and $\delta_T(t)$

represents the samples of $f(t)$ located at (5)
 uniform interval of $T_s(T)$. Let us denote this
 sampled version of $f(t)$ as $f_s(t)$, then

$$f_s(t) = f(t) \delta_T(t)$$

$$\text{where } \delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Let $F(\omega)$ be the Fourier transform of $f(t)$,

$F_s(\omega)$ is the Fourier transform of $f_s(t)$, and

$\delta_{\omega_s}(\omega)$ is the Fourier transform of $\delta_T(t)$.

using convolution theorem, the Fourier transform
 of sampled signal is the convolution of $F(\omega)$ and
 $\delta_{\omega_s}(\omega)$.

$$F[f_s(t)] = F\{f(t) \delta_T(t)\}$$

$$F_s(\omega) = \frac{1}{2\pi} [F(\omega) * \delta_{\omega_s}(\omega)] \quad \left[\because f_1(t) \cdot f_2(t) \xrightarrow{F.T} \frac{1}{2\pi} [F_1(\omega) * F_2(\omega)] \right]$$

We know that the Fourier transform of impulse
 function is

$$\delta_{\omega_s}(\omega) = \omega_s \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$$

$$\text{Then } F_s(\omega) = \frac{1}{2\pi} F(\omega) * \omega_s \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$$

$$F_s(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} F(\omega) * \delta(\omega - n\omega_s)$$

$$\left[\because \omega_s = \frac{2\pi}{T} \Rightarrow \frac{\omega_s}{2\pi} = \frac{1}{T_s} \right]$$

$$\boxed{\therefore F_s(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} F(\omega - n\omega_s)} \quad \left[\because f(t) * \delta(t - t_0) = f(t - t_0) \right]$$

or

$$F_s(\omega) = \frac{1}{T} [\dots + F(\omega + 2\omega_s) + F(\omega + \omega_s) + F(\omega) + F(\omega - \omega_s) + \dots]$$

Thus the Fourier transform of the sampled signal is equal to the infinite sum of shifted versions of the Fourier transform of continuous time signal $f(t)$. If the sampling frequency

If the sampling frequency (f_s or ω_s) is greater than or equal to the frequency of band limited signal, i.e., $\omega_s \geq 2\omega_m$ and $T \leq \frac{1}{2f_m}$, then the shifted spectra of $F(\omega)$ do not overlap with each other and $f(t)$ can be uniquely represented by its sampled values. This proves the sampling theorem.

Impulse sampling or Ideal sampling:

The process of multiplying the continuous-time signal $f(t)$ with a periodic impulse function or impulse train $\delta_T(t)$ is called ideal or impulse sampling. It is also called instantaneous sampling.

Reconstruction of a signal from its samples.

consider a band limited signal $f(t)$ that is sampled at $\omega_s \geq 4\pi f_m \geq 2\omega_m$. The process of recovering $f(t)$ from its sampled signal $f_s(t)$ is accomplished by passing $f_s(t)$ through a reconstruction filter, which is a lowpass filter (LPF) with cut-off frequency f_m (shown in Fig. 1). Mathematically, we can show this by using the time convolution theorem.

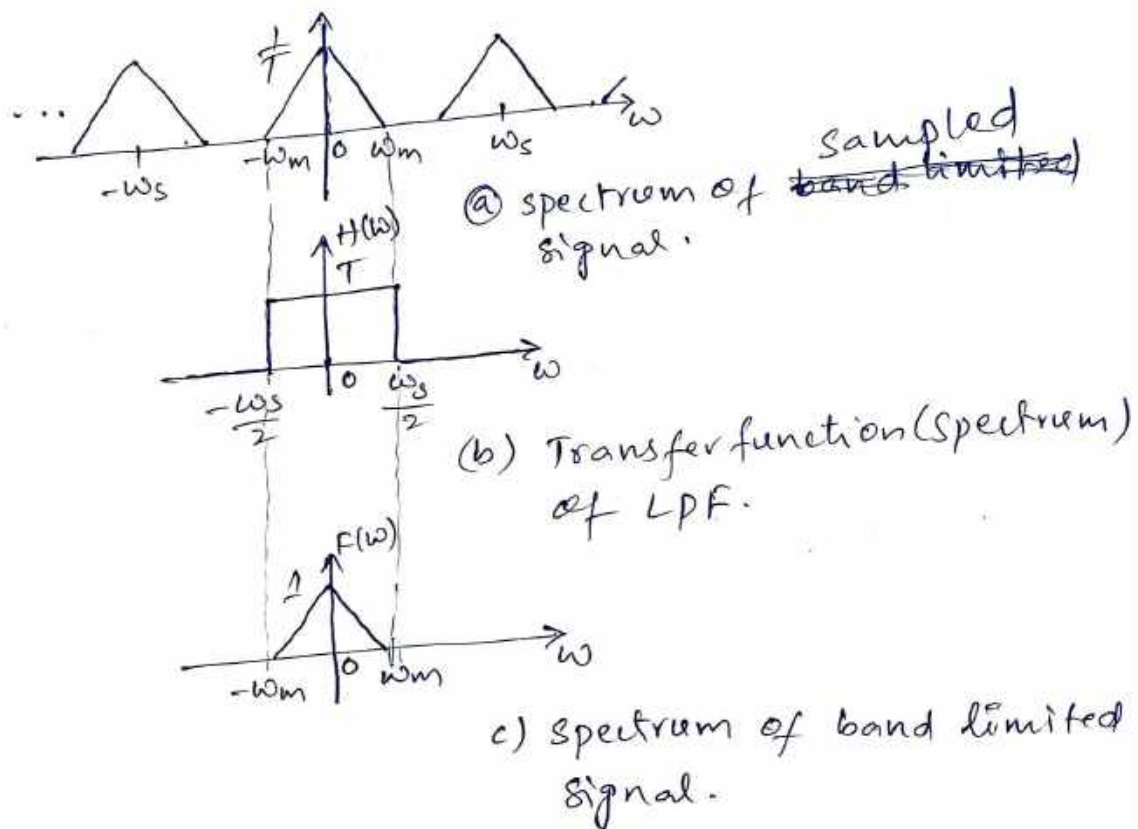


Fig. 1. The reconstruction process.

The spectrum of $f_s(t)$ is given by

$$F_s(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} F(\omega - n\omega_s) = \frac{1}{T} \sum_{n=-\infty}^{\infty} F(\omega - 2n\omega_m)$$

The process of lowpass filtering is equivalent to multiplying $F_s(\omega)$ by the transfer function $H(\omega)$ of a reconstruction filter. Thus

$$F(\omega) = F_s(\omega) H(\omega)$$

The recovered signal $f(t)$ in time domain is

$$f(t) = f_s(t) * h(t)$$

where $h(t)$ is impulse response of LPF.

$$f(t) = \sum_{n=-\infty}^{\infty} f(nT) \delta(t - nT) * h(t)$$

$$= \sum_{n=-\infty}^{\infty} f(nT) h(t - nT)$$

Thus the reconstructed signal is equal to the weighted sum of impulse responses of LPF with weights equal to the instant sample values.

Aliasing effect:

When the signal is sampled with sampling frequency $\omega_s \leq 2\omega_m$, called under sampling.

The spectrum of the adjacent sampled signal overlaps with each other and the original signal cannot be recovered from its samples. This condition is called the aliasing effect.

This condition is also called as the frequently "folding effect". Since, high frequencies in the original signal 'fold back' into lower frequencies.

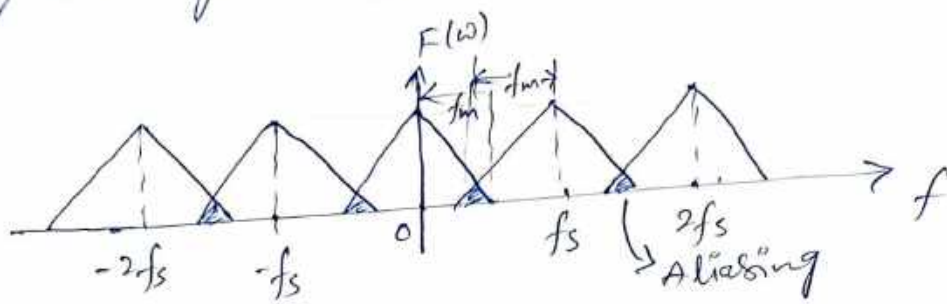


Fig. Aliasing effect due to under sampling ($f_s < 2f_m$)

Hence the signal is undersampled when $f_s < 2f_m$ and some amount of aliasing produced

Due to aliasing, it is not possible to recover the original signal $f(t)$ from the sampled version of $f(t)$ by lowpass filter. It also occurs if the signal is not band-limited.

if the signal is not band-limited, to avoid aliasing effect the signal should be ~~increased to~~ ~~greater~~ band-limited and the sampling frequency should be increased to greater than Nyquist rate, i.e., $\omega_s \geq 2\omega_m$ (or $f_s \geq 2f_m$)

Anti-aliasing Filter:

consider a continuous time band-limited signal $f(t)$ which is corrupted by high frequency

noise. Even though the signal is sampled (10) with sampling frequency $\omega_s \geq 2\omega_m$, aliasing may occur due to high frequency noise components. To avoid this condition, first the continuous-time signal is band limited to a maximum frequency ω_m by passing it through a low-pass filter whose cut-off frequency is ω_m . The lowpass filter eliminates all high frequency components greater than ω_m . This filter is called anti-aliasing filter.



Fig. sampling process with anti-aliasing filter.

correlation: The correlation between two waveforms is the measure of similarity between one waveform, and time delayed version of the other waveform. This expresses how much one waveform is related to the time delayed version of the other waveform when scanned over time axis.

In most of the applications, it is necessary to examine the similarity between the two signals. This similarity is called as correlation or coherence.

correlation can be divided into two parts

i) cross-correlation. and

ii) Auto correlation [which is special case of cross-correlation]

i) cross correlation function:

The cross correlation is used to measure the match or similarity between two different wave forms. For cross correlation second wave form is time delayed. The cross correlation is defined separately for energy signals and power signals (periodic).

cross correlation of Energy signals and its properties

Let the two signals $f_1(t)$ & $f_2(t)$ be a ^{pair of} complex valued signals of finite energy. The cross correlation

of these two signals is given as

$$R_{12}(\tau) = \int_{-\infty}^{\infty} f_1(t) f_2^*(t-\tau) dt \quad \text{or}$$

$$R_{21}(\tau) = \int_{-\infty}^{\infty} f_2(t) f_1^*(t-\tau) dt$$

↑
Searching
parameter

If $f_1(t)$ and $f_2(t)$ have ~~the~~ some similarity, then the cross correlation function $R_{12}(\tau)$ will have some finite value over the range of τ .

$$\text{If } \int_{-\infty}^{\infty} f_1(t) f_2^*(t) dt = 0 \quad \text{i.e.,}$$

$$R_{12}(0) = 0$$

then the signals $f_1(t)$ and $f_2(t)$ are called orthogonal signals. That is the two signals have no cross correlation between them.

Other mathematical relations are as

$$R_{12}(\tau) = \int_{-\infty}^{\infty} f_1(t) f_2^*(t+\tau) dt$$

$$R_{12}(\tau) = \int_{-\infty}^{\infty} f_1(t-\tau) f_2^*(t) dt \quad \text{and}$$

$$R_{12}(\tau) = \int_{-\infty}^{\infty} f_1(t+\tau) f_2^*(t) dt$$

where τ is the searching parameter or scanning parameter. All the above mathematical equations lead to the similar results.

For discrete time signals

$$R_{12}(m) = \sum_{n=-\infty}^{\infty} f_1[n] f_2^*[n-m]$$

or

$$R_{21}(m) = \sum_{n=-\infty}^{\infty} f_2[n] \cdot f_1^*[n-m]$$

(2)

properties of cross-correlation for energy signals.

property 1: The cross correlation functions ~~exhibit~~ ^{exhibit} conjugate exhibit

Symmetry i.e.,

$$R_{12}(\tau) = R_{21}^*(\tau)$$

property 2:

Cross correlation is not in general commutative i.e.,

$$R_{12}(\tau) \neq R_{21}(\tau)$$

property 3: If $R_{12}(0) = 0$ i.e., [If cross correlation is executed at origin ($\tau=0$)]

$$\int_{-\infty}^{\infty} f_1(t) f_2^*(t) dt = 0$$

Then the signals are said to be orthogonal over the entire interval.

property 4: The cross correlation of the two energy signals corresponds to the multiplication of the Fourier transform of one signal by the complex conjugate of Fourier transform of the second signal.

$$\text{i.e., } R_{12}(\tau) \longleftrightarrow F_1(\omega) \cdot F_2^*(\omega)$$

or

$$F_1(f) \cdot F_2^*(f)$$

The relation is known as correlation theorem.

Properties of cross-correlation function for periodic (or) power signals & its properties.

The cross correlation function $R_{12}(\tau)$ for two periodic signals $f_1(t)$ & $f_2(t)$ can be defined with the help of average form of correlation i.e.,

$$R_{12}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} f_1(t) f_2^*(t-\tau) dt$$

Similarly the second correlation function $R_{21}(\tau)$ is defined as (4)

$$R_{21}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} f_2(t) f_1^*(t-\tau) dt$$

If the two signals $f_1(t)$ and $f_2(t)$ have the same time period T_0 , then cross correlation is defined as

$$R_{12}(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f_1(t) f_2^*(t-\tau) dt$$

and

$$R_{21}(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f_2(t) f_1^*(t-\tau) dt$$

properties of cross correlation function for periodic signals

property 1: It exhibits conjugate symmetry

$$R_{12}(\tau) = R_{21}^*(\tau)$$

property 2: The cross correlation is not commutative

$$R_{12}(\tau) \neq R_{21}(\tau)$$

Property 3: If cross correlation is executed at origin ($\tau=0$)

$$R_{12}(0) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} f_1(t) f_2^*(t) dt = 0$$

$$R_{12}(0) = 0$$

The signals are said to be orthogonal over the entire interval.

ii) Auto correlation:

when we calculate correlation function, function of the signal with itself, then it is called autocorrelation. Thus if $f_1(t) = f_2(t)$, then correlation becomes autocorrelation.

Autocorrelation of Energy signals:

$$R(\tau) = \int_{-\infty}^{\infty} f(t) f^*(t-\tau) dt$$

or

$$R(\tau) = \int_{-\infty}^{\infty} f(t+\tau) f^*(t) dt$$

For discrete time signals

$$R(m) = \sum_{n=-\infty}^{\infty} f[n] f^*[n-m]$$

or

$$R(m) = \sum_{n=-\infty}^{\infty} f[n+m] f^*[n]$$

consider auto correlation of energy signal

$$R(\tau) = \int_{-\infty}^{\infty} f(t) f^*(t-\tau) dt \quad \text{with zero shift } (\tau=0)$$

$$R(0) = \int_{-\infty}^{\infty} f(t) f^*(t) dt = \int_{-\infty}^{\infty} |f(t)|^2 dt$$

$$R(0) = E.$$

Thus at $\tau=0$, autocorrelation function shows cor is equal to total energy of the signal.

properties: For energy signals.

property 1: The autocorrelation function shows conjugate symmetry $R(\tau) = R^*(-\tau)$

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proof: consider the autocorrelation function of energy signal

$$R(\tau) = \int_{-\infty}^{\infty} f(t) f^*(t-\tau) dt$$

By taking complex conjugate

$$\therefore R^*(\tau) = \int_{-\infty}^{\infty} f^*(t) f(t-\tau) dt$$

replacing τ by $-\tau$

$$\therefore R^*(-\tau) = \int_{-\infty}^{\infty} f^*(t) f(t+\tau) dt$$

$$= R(\tau) \quad \text{by definition}$$

$$\boxed{R(\tau) = R^*(-\tau)}$$

property 2: The value of autocorrelation function of $\tau=0$ (i.e., at origin) is equal to the energy of the signal.

$$R(0) = E = \int_{-\infty}^{\infty} |f(t)|^2 dt.$$

proof: consider the autocorrelation function of energy

signal $R(\tau) = \int_{-\infty}^{\infty} f(t) f^*(t-\tau) dt$

put $\tau = 0$, gives

$$R(0) = \int_{-\infty}^{\infty} f(t) f^*(t) dt$$

$$= \int_{-\infty}^{\infty} |f(t)|^2 dt$$

$$= E$$

$$\boxed{\therefore R(0) = E = \int_{-\infty}^{\infty} |f(t)|^2 dt}$$

property 3: If τ is increased in either direction, the autocorrelation reduces. As τ reduces, autocorrelation increases and it is maximum at $\tau=0$ i.e., at origin. (7)

$$|R(\tau)| \leq R(0) \text{ for all } \tau.$$

proof: consider the function $x(t)$ and $x(t+\tau)$.
 $[x(t) \pm x(t+\tau)]^2$ is always greater than or equal to zero
 since it is squared.

$$\text{i.e. } x^2(t) + x^2(t+\tau) \pm 2x(t)x(t+\tau) \geq 0$$

$$\text{i.e. } x^2(t) + x^2(t+\tau) \geq \mp 2x(t)x(t+\tau)$$

Integrating both the sides, from $-\infty$ to $+\infty$

$$\therefore \int_{-\infty}^{\infty} |x^2(t)|^2 dt + \int_{-\infty}^{\infty} |x(t+\tau)|^2 dt \geq \pm 2 \int_{-\infty}^{\infty} x(t) \cdot x(t+\tau) dt$$

$$E + E \geq \pm 2 R(\tau) \quad \text{if } x(t) \text{ is real valued function.}$$

$$E \geq \pm R(\tau)$$

$$R(0) \geq |R(\tau)|$$

$$\text{or } |R(\tau)| \leq R(0) \quad \because R(0) = E$$

property 4: The autocorrelation function and energy spectral density function of energy signal forms a fourier transform.

$$R(\tau) \longleftrightarrow \psi(f)$$

Auto correlation of power signals (periodic signals):

Auto correlation function of power signal is defined as

$$\text{as } R(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} f(t) f^*(t-\tau) dt$$

if sign $f(t)$ is periodic with period T_0

(8)

$$R(\tau) = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} f(t) f^*(t-\tau) dt$$

or

$$R(\tau) = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} f(t+\tau) f^*(t) dt$$

for discrete time signal

$$R(\tau) = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N f[n] f^*[n+m]$$

properties:

property 1: The autocorrelation ^{function} exhibits conjugate symmetry i.e., $R(\tau) = R^*(-\tau)$

property 2: The value of autocorrelation function at $\tau=0$ (origin) is equal to the average power of the signal.

$$\text{i.e., } R(0) = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |f(t)|^2 dt = P.$$

property 3: The autocorrelation function has maximum value at the origin i.e., $|R(\tau)| \leq R(0)$.

The value of Autocorrelation decreases as τ increases from zero.

property 4: The autocorrelation function and power spectral density form a Fourier transform pair i.e.,

$$R(\tau) \xleftrightarrow{\text{F.T.}} S(f)$$

where $S(f)$ is power spectral density.

property 5: The autocorrelation function is periodic with the same period as periodic signal itself i.e.,

$$R(\tau) = R(\tau \pm nT_0), \quad n=1, 2, 3, 4, \dots$$

(9)

compare convolution and correlation:

convolution

1. convolution is a mathematical operation used to combine two signals or functions to produce a third signal or function.
2. The primary purpose of convolution is to perform tasks, such as signal transformation and modification.
3. the output of convolution of two signals is a new function or signal which is a combination of input signals.
4.
$$g(t) = f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau) h(t-\tau) d\tau$$

or
$$g(t) = f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau.$$
5. convolution widely used in the field of digital image processing, computer vision, & signal transformation, etc.

correlation

correlation is a mathematical operation used to measure or quantify the similarities and relationship between two signals.

The main purpose of correlation is to quantify the similarities between two functions.

The output of correlation is a value that represents the degree of similarity between two signals.

$$R_{12}(\tau) = \int_{-\infty}^{\infty} f_1(t) f_2^*(t-\tau) dt$$

or
$$R(\tau) = \int_{-\infty}^{\infty} f(t) \cdot f^*(t-\tau) dt.$$

correlation is used in pattern recognition, data analysis etc.

→ The most significant difference between convolution and correlation is that the convolution combines two input signals and produces a third signal as output, while the correlation measures the similarity between two input signals.

Spectral Density functions:

(11)

The spectral density of the periodic or non periodic signal $f(t)$ represents the distribution of power or energy in the frequency domain.

For the energy and power signals two types of spectral density functions are normally defined i.e. energy spectral density (ESD) and power spectral density (PSD).

Energy Spectral Density (ESD):

The energy spectral density function gives the distribution of energy of the signal in the frequency domain. we know that, energy of the function is

$$E = \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |F(f)|^2 df$$

[$\because$ from Rayleigh's theorem]

or simply

$$E = \int_{-\infty}^{\infty} |F(f)|^2 df \quad \text{--- ①}$$

This equation gives the total energy of the signal from its frequency components in frequency domain. In this equation, $|F(f)|$ is the amplitude spectrum of the signal.

Let us denote ^{the} squared amplitude spectrum $|F(f)|^2$

of the signal by $\psi(f)$ i.e.,

(12)

Energy spectral density $\psi(f) = |F(f)|^2$

putting above value in equation (1), an energy of the signal is expressed in terms of energy spectral density

$$E = \int_{-\infty}^{\infty} \psi(f) df$$

→ This equation shows that total energy of the signal is given by total area under the curve $\psi(f)$. That is $\psi(f)$ represents the energy density of the signal $f(t)$ in joules per hertz.
properties of ESD:

property 1: The autocorrelation function $R(\tau)$ and energy spectral density function $\psi(f)$ forms a Fourier transform pair i.e.,

$$R(\tau) \xleftrightarrow{F.T.} \psi(f)$$

property 2: The total area under the energy spectral density function gives total energy of that signal.

$$\text{i.e., } E = \int_{-\infty}^{\infty} \psi(f) df$$

property 3: If $f(t)$ is input to an LTI system with transfer function $H(f)$, then input and output energy spectral density functions are related by
 $\psi_o(f) = |H(f)|^2 \psi_i(f)$.

Here $\psi_o(f)$ is output energy spectral density function
 $\psi_i(f)$ is input energy spectral density function. (13)
 $|H(f)|^2$ is energy gain at frequency f .

Ex:

- i) Determine the autocorrelation function and
- ii) Energy spectral density of the function $f(t) = e^{-at} u(t)$ and hence verify that $\int_{-\infty}^{\infty} R(\tau) d\tau = E$
- iii) $R(0) = E$.

Sol: (i) To obtain autocorrelation function

the given function is

$$f(t) = e^{-at} u(t).$$

This is an energy signal, and autocorrelation function for energy signal is given as $R(\tau) = \int_{-\infty}^{\infty} f(t) f^*(t-\tau) dt$

If $f(t)$ is real, then the above equation becomes

$$R(\tau) = \int_{-\infty}^{\infty} f(t) f(t-\tau) dt$$

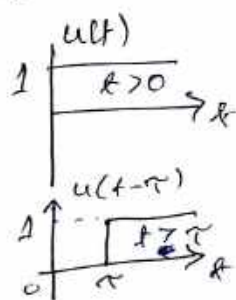
Substituting $f(t) = e^{-at} u(t)$ in above equation.

$$R(\tau) = \int_{-\infty}^{\infty} e^{-at} u(t) \cdot e^{-a(t-\tau)} u(t-\tau) dt$$

We know that $u(t) = 1; t > 0$ ($0 < t < \infty$)

and $u(t-\tau) = 1; t \geq \tau$ ($\tau < t < \infty$)

Hence limits of integration will be from τ to ∞



i.e.,

$$R(\tau) = \int_{\tau}^{\infty} e^{-at} \cdot e^{-a(t-\tau)} dt$$

$$= e^{a\tau} \int_{\tau}^{\infty} e^{-2at} dt = e^{a\tau} \left[\frac{e^{-2at}}{-2a} \right]_{\tau}^{\infty}$$

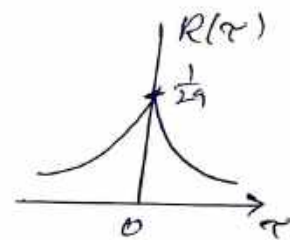
$$= e^{a\tau} \left[0 + \frac{e^{-2a\tau}}{2a} \right]$$

$$\boxed{R(\tau) = \frac{e^{-a\tau}}{2a}} \quad \text{--- (1)}$$

Since $R(\tau)$ is real, then $R(\tau) = R(-\tau)$

Thus $R(\tau)$ is an even function of τ , Hence auto correlation becomes

$$\boxed{R(\tau) = \frac{e^{-a|\tau|}}{2a}} \quad \text{--- (1)}$$



Auto correlation function.

ii) To obtain Energy spectral density consider the F.T of $f(t)$ i.e.,

$$F(\omega) = F\{f(t)\}$$

$$F(2\pi f) = F\{e^{-at} u(t)\}$$

$$F(f) = \frac{1}{a + j2\pi f}$$

$$\therefore |F(f)|^2 = \frac{1}{a^2 + (2\pi f)^2}$$

Energy spectral density is given as $\varphi(f) = |F(f)|^2$

$$\varphi(f) = \frac{1}{a^2 + (2\pi f)^2} \quad \text{--- (2)}$$

This is an expression for ESD.

iii) To show $R(0) = E$, let us consider eqn ①

from eqn ① $R(\tau) = \frac{\cancel{e^{-a|\tau|}}}{\cancel{2a}} \frac{e^{-a|\tau|}}{2a}$

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put $\tau = 0$, i.e.,

$$R(0) = \frac{1}{2a} \text{ Joules} \quad \text{--- ②}$$

And Energy of given signal $E = \int_{-\infty}^{\infty} |f(t)|^2 dt$

$$E = \int_{-\infty}^{\infty} |e^{at} u(t)|^2 dt = \int_0^{\infty} e^{-2at} dt$$

$$= \left. \frac{e^{-2at}}{-2a} \right|_0^{\infty} = 0 + \frac{1}{2a}$$

$$E = \frac{1}{2a} \text{ Joules} \quad \text{--- ③}$$

From eqn ② and eqn ③

$$\boxed{R(0) = E = \frac{1}{2a}}$$

Hence proved.

power spectral density (PSD) function:

This function gives the distribution of power of a signal at various frequencies in frequency domain.

The average power P of the function $f(t)$ is given by $P = \int_{-\infty}^{\infty} S(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) d\omega$.

where $S(\omega) = \lim_{\tau \rightarrow \infty} \frac{|F_{\tau}(\omega)|^2}{\tau} \rightarrow \text{power spectral density.}$

properties:

property 1: Auto correlation function and power spectral density form a Fourier transform pair i.e.,

$$R(\tau) \xrightarrow{F.T} S(f)$$

property 2: The area under power spectral density function gives average power of the signal i.e.,

$$P = \int_{-\infty}^{\infty} S(f) df.$$

property 3: If $|H(f)|^2$ is the power gain of the linear time invariant (LTI) system, then o/p power spectral density $S_o(f)$ and i/p PSD $S_i(f)$ are related by,

$$S_o(f) = |H(f)|^2 S_i(f)$$

↓
o/p PSD

↓
i/p PSD